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N. Hayashi · E.I. Kaikina · P.I. Naumkin
I.A Shishmarev

Asymptotics for Dissipative Nonlinear Equations

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Authors

Nakao Hayashi
Department of Mathematics
Graduate School of Science
Osaka University, Osaka
Toyonaka 560-0043
Japan
e-mail: nhayashi@math.wani.osaka-u.ac.jp

Pavel I. Naumkin
Instituto de Matemáticas
UNAM Campus Morelia
AP 61-3 (Xangari)
Morelia CP 58089, Michoacán
Mexico
e-mail: pavelni@matmor.unam.mx

Elena I. Kaikina
Instituto de Matemáticas
UNAM Campus Morelia
AP 61-3 (Xangari)
Morelia CP 58089, Michoacán
Mexico
e-mail: ekaikina@matmor.unam.mx

Ilya A. Shishmarev
Department of Computational
Mathematics and Cybernetics
Moscow State University
Moscow 119 899
Russia
e-mail: shish@voxnet.ru

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Preface

Modern mathematical physics is almost exclusively a mathematical theory of nonlinear partial differential equations describing various physical processes. Since only a few partial differential equations have succeeded in being solved explicitly, different qualitative methods play a very important role. One of the most effective ways of qualitative analysis of differential equations are asymptotic methods, which enable us to obtain an explicit approximate representation for solutions with respect to a large parameter time. Asymptotic formulas allow us to know such basic properties of solutions as how solutions grow or decay in different regions, where solutions are monotonous and where they oscillate, which information about initial data is preserved in the asymptotic representation of the solution after large time, and so on. It is interesting to study the influence of the nonlinear term in the asymptotic behavior of solutions. For example, compared with the corresponding linear case, the solutions of the nonlinear problem can obtain rapid oscillations, can converge to a self-similar profile, can grow or decay faster, and so on. It is very difficult to obtain this information via numerical experiments. Thus asymptotic methods are important not only from the theoretical point of view, but also they are widely used in practice as a complement to numerical methods. It is worth mentioning that in practice large time could be a rather bounded value, which is sufficient for all the transitional processes caused by the initial perturbations in the system to happen.

The theory of asymptotic methods for nonlinear evolutionary equations is relatively young and traditional questions of general theory are far from being answered. A description of the large time asymptotic behavior of solutions of nonlinear evolution equations requires principally new approaches and the reorientation of points of view in the asymptotic methods. For example, the requirements of the infinite differentiability and a compact support usually acceptable in the linear theory are too strong in the nonlinear theory.

Asymptotic theory is difficult even in the case of linear evolutionary equations (see books Dix [1997], Fedoryuk [1999]). The difficulty of the asymptotic methods is explained by the fact that they need not only a global existence of solutions, but also a number of additional a priori estimates of the difference between the solution and the approximate solution (usually in the weighted norms). Also the generalized solutions could not be acceptable for the asymptotic theory, so we consider classical and semiclassical (mild) solutions, belonging to some Lebesgue spaces. Moreover in the case of nonlinear equations it is necessary to prove global existence of classical solutions and to obtain some additional estimates to clarify the asymptotic expansions. Every type of nonlinearity should be studied individually, especially in the case of large initial data.

A great number of publications have dealt with asymptotic representations of solutions to the Cauchy problem for nonlinear evolution equations in the last twenty years. While not attempting to provide a complete review of these publications, we do list some known results: Amick et al. [1989], Biler [1984], Biler et al. [1998], Biler et al. [2000], Bona et al. [1999], Bona and Luo [2001], Dix [1991], Dix [1992], Escobedo et al. [1995], Escobedo et al. [1993a], Galaktionov et al. [1985], Giga and Kambe [1988], Gmira and Véron [1984], Hayashi et al. [2001], Il′in and Oleĭnik [1960], Karch [1999a], Kavian [1987], Naumkin and Shishmarev [1989], Naumkin and Shishmarev [1990], Naumkin and Shishmarev [1994b], Schonbek [1986], Schonbek [1991], Strauss [1981], Zhang [2001], Zuazua [1993], Zuazua [1994], where, in particular, the optimal time decay estimates and asymptotic formulas of solutions to different nonlinear local and nonlocal dissipative equations were obtained. In the case of dispersive equations some progress in the asymptotic methods was achieved due to the discovery of the Inverse Scattering Transform method (see books Ablowitz and Segur [1981], Novikov et al. [1984] and papers Deift et al. [1993], Deift et al. [1994]). Some other functional analytic methods were applied for the study of the large time asymptotic behavior of solutions to dispersive equations in Cazenave [2003], Christ and Weinstein [1991], Clément and Nohel [1981], Georgiev and Milani [1998], Giga and Kambe [1988], Ginibre and Ozawa [1993], Glassey [1973a], Kenig et al. [2000], Kenig et al. [1997], Kita and Ozawa [2005], Kita and Wada [2002], Klainerman [1982], Klainerman and Ponce [1983], Ozawa [1995], Ozawa [1991], Ponce and Vega [1990], Segal [1968], Shimomura and Tonegawa [2004], Strauss [1974], Strichartz [1977], Tsutsumi and Hayashi [1984], Tsutsumi [1994].

This book is the first attempt to give a systematic approach for obtaining the large time asymptotic representations of solutions to the nonlinear evolution equations with dissipation. We restrict our attention to the investigation of the Cauchy problems (initial value problems) leaving outside the wide and important class of the initial-boundary value problems (in some respects the reader can fill this gap by consulting a recent book Hayashi and Kaikina [2004]). In our book we pay much attention to typical well-known equations which have huge applications: the nonlinear heat equation, Burgers equation, Korteweg-de Vries-Burgers equation, nonlinear damped wave equation, Landau-Ginzburg equation, Sobolev type equations, systems of equations of Boussinesq, Navier-Stokes equations and others. Certainly we do not claim that we could embrace all equations and all cases. However we succeeded in selecting a sufficiently wide class of equations, which could be treated by a unified approach and which fall into the same theory. Many of the methods proposed in this book have been developed by a great number of authors. The results and proofs presented throughout the book are mainly based on the research articles of the authors.

We divide nonlinear equations into three general types: asymptotically weak nonlinearity, critical nonlinearity and strong (or subcritical) nonlinearity. Also the critical and subcritical nonlinearities are divided into convective type and nonconvective type. In many cases nonlinearity leads to the blow-up of solutions in a finite time, so to be able to study global solutions we have to restrict our attention to small initial data. However we also closely examine the large time asymptotics for initial data of arbitrary size (not small).

Let us explain our classification taking the nonlinear heat equation as an example:

$$u_t - \Delta u + u^\rho = 0.$$

We assume that initial data are most general; however, in applications, the initial data are usually considered as sufficiently rapidly decaying at infinity and smooth, for example initial data $u_0 \in \mathbf{C}_0^\infty(\mathbf{R}^n)$ are acceptable. The decay rate at infinity of the initial data u_0 appears very essential, since for example taking $u_0 = C$ with a constant C, we then obtain the solution $u(t,x) = C$ of the Cauchy problem for the linear heat equation $u_t - \Delta u = 0$, so the solution does not decay at all. Physically this situation is slightly special, since usually some physical quantities, such as energy, mass and momentum, expressed via integrals of the solution, should be finite. Therefore we are interested when the initial data are integrable $u_0 \in \mathbf{L}^1(\mathbf{R}^n)$. Another special case occurs when the total mass of the initial data vanishes: $\int_{\mathbf{R}^n} u_0(x)\,dx = 0$. Then the solution of the linear heat equation $u_t - \Delta u = 0$ obtains some more rapid decay rate with time. In this case the critical value of the order of the nonlinearity is changed.

Now let us give a heuristic classification of the nonlinearities. Consider initial data $u_0(x) \in \mathbf{L}^1(\mathbf{R}^n)$ with nonzero total mass $\int_{\mathbf{R}^n} u_0(x)\,dx \neq 0$. Then the solution $u(t,x)$ of the Cauchy problem for the linear heat equation $u_t - \Delta u = 0$ has the following time decay rate $u(t,x) \sim t^{-\frac{n}{2}}$. If we compute the decay rate of the linear part of the equation we get $u_t - \Delta u \sim t^{-\frac{n}{2}-1}$. For such behavior of the solution the nonlinearity u^ρ decays as $u^\rho \sim t^{-\frac{n}{2}\rho}$ for $t \to \infty$. Now we see that if $\rho > 1 + \frac{2}{n}$, then the nonlinear term decays more rapidly than the linear part as time goes to infinity. We can expect that in this case the large time behavior of solutions is similar to the linear one, and it is possible to apply the results of the well-developed linear theory. We call $\rho > 1+\frac{2}{n}$ supercritical and the nonlinearity asymptotically weak. Respectively, cases $\rho = 1 + \frac{2}{n}$ and $\rho < 1 + \frac{2}{n}$ we call critical and subcritical. In the critical case there is a kind of equilibrium in the large time asymptotic behavior of linear and nonlinear parts of the equation. In the subcritical case as well, the nonlinear effects win in the large time asymptotic behavior of solutions. Hence the particular form of the nonlinearity is very important to determine the large time asymptotics of solutions in the critical and subcritical cases.

We found that there are two types of nonlinearity which define different asymptotic behavior of solutions in the critical and subcritical cases. By the convective type we mean the nonlinearity $\mathcal{N}(u)$ such that $f(\mathcal{N}(u)) = 0$, where the linear functional f determines the large time asymptotics of the corresponding linear problem (see Definition 2.1 for details). For example, the Burgers equation $u_t - u_{xx} + uu_x = 0$ has this type of the nonlinearity if the total mass of the initial data is nonzero. Then in the critical case the large time asymptotics is described by the self-similar solution and in the subcritical case the asymptotics of solutions is represented as a product of a rarefaction wave and a shock wave. Another asymptotic behavior occurs for the nonlinearities of nonconvective type, as in the nonlinear heat equation $u_t - \Delta u + u^{\rho} = 0$. In the critical case $\rho = 1 + \frac{2}{n}$ solutions have some additional logarithmic decay rate compared with the corresponding linear heat equation. In the subcritical case $1 < \rho < 1 + \frac{2}{n}$ solutions asymptotically approach special self-similar solutions also. Our method for critical and subcritical nonconvective equations is based on a change of the dependent variable such that the nonlinear term $\widetilde{\mathcal{N}}(u)$ of a modified equation has the property $f\left(\widetilde{\mathcal{N}}(u)\right) = 0$. Then the solutions of this new nonlinear equation have the asymptotic properties similar to that for the supercritical or critical convective equations. In the case of subcritical convective equations this method does not work, since the functional $f(\mathcal{N}(u))$ is already zero. Thus we develop another approach representing solutions in the form of the rarefaction and shock waves.

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Nakao Hayashi
Elena I. Kaikina
Pavel I. Naumkin
Ilya A. Shishmarev

Contents

1

Preliminary results

1.1 Notations and some inequalities

1.1.1 Definitions

We denote everywhere in the book $\langle x\rangle = \sqrt{1+|x|^2}$, $\{x\} = \frac{|x|}{\langle x\rangle}$, $|x| = \left(\sum_{j=1}^n x_j^2\right)^{\frac{1}{2}}$so that the decomposition $|x| = \{x\}\langle x\rangle$is true for all $x \in \mathbf{R}^n$.

By $[\alpha]$ we denote the integer part of $\alpha \in \mathbf{R}$;more precisely $[\alpha] = \max_{k\leq\alpha} k$, where $k \in \mathbf{Z}$.

Direct Fourier transformation $\mathcal{F}_{x\to\xi}$is

$$\widehat{u}(\xi) \equiv \mathcal{F}_{x\to\xi}u = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-i\xi x} u(x)\, dx,$$

and the inverse Fourier transformation $\overline{\mathcal{F}}_{\xi\to x}$ is defined by

$$\check{u}(x) \equiv \overline{\mathcal{F}}_{\xi\to x}u = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{i\xi x} u(\xi)\, d\xi,$$

by ξx as usually we denote $\xi x = (\xi \cdot x)$ the inner product in $\mathbf{R}^n$.

Lebesgue space $\mathbf{L}^p(\mathbf{R}^n) = \{\varphi \in \mathcal{S}'; \|\varphi\|_{\mathbf{L}^p} < \infty\}$, where

$$\|\varphi\|_{\mathbf{L}^p} = \left(\int_{\mathbf{R}^n} |\varphi(x)|^p\, dx\right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$ and

$$\|\varphi\|_{\mathbf{L}^\infty} = \text{ess.} \sup_{x\in\mathbf{R}^n} |\varphi(x)|$$

for $p = \infty$.

Weighted Lebesgue space $\mathbf{L}^{p,a}(\mathbf{R}^n) = \{\varphi \in \mathcal{S}'; \|\varphi\|_{\mathbf{L}^{p,a}} < \infty\}$, where the norm

$$\|\varphi\|_{\mathbf{L}^{p,a}} \equiv \|\langle x\rangle^a \varphi(x)\|_{\mathbf{L}^p}.$$

Sobolev spaces

$$\mathbf{W}_p^k(\mathbf{R}^n) = \left\{\varphi \in \mathcal{S}'; \left\|\langle i\partial_x\rangle^k \varphi\right\|_{\mathbf{L}^p} < \infty\right\},$$

for $1 \le p \le \infty$ and $k \ge 0$. Also we denote

$$\mathbf{H}^k(\mathbf{R}^n) = \mathbf{W}_2^k(\mathbf{R}^n)$$

for $k \ge 0$ (see Besov et al. [1979]).

Weighted Sobolev space is defined by

$$\mathbf{W}_p^{s,a}(\mathbf{R}^n) = \{\varphi \in \mathcal{S}' : \|\langle x\rangle^a \langle i\partial_x\rangle^s \varphi(x)\|_{\mathbf{L}^p} < \infty\}$$

for any $s, a \in \mathbf{R}$, $1 \le p \le \infty$. In particular, we denote

$$\mathbf{H}^{s,a}(\mathbf{R}^n) = \mathbf{W}_2^{s,a}(\mathbf{R}^n) = \{\varphi \in \mathcal{S}' : \|\langle x\rangle^a \langle i\partial_x\rangle^s \varphi(x)\|_{\mathbf{L}^2} < \infty\}.$$

Obviously we have $\mathbf{W}_p^{s,0}(\mathbf{R}^n) = \mathbf{W}_p^s(\mathbf{R}^n)$. We denote the norm

$$\|\varphi\|_{\mathbf{B}^{\omega,q}} = \int_{\mathbf{R}^n} |y|^{-n-\omega} \|\varphi(x-y) - \varphi(x)\|_{\mathbf{L}^q}\, dy$$

of the homogeneous Besov (see. Besov et al. [1979]) space $\mathbf{B}^{\omega,q}(\mathbf{R}^n)$ of order $\omega \in (0,1)$.

By $\mathbf{C}(\mathbf{I};\mathbf{B})$ we denote the space of continuous functions from a time interval $\mathbf{I}$ to the Banach space $\mathbf{B}$. Different positive constants are denoted by the same letter C.

1.1.2 Asymptotic notations

We say that the function $f(x)$ is equivalent to the function $g(x)$ at some point x_0 and write $f(x) \sim g(x)$ as $x \to x_0$ if $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$.

We say that the function $f(x)$ is infinitesimal with respect to the function $g(x)$ at some point x_0 and write $f(x) = o(g(x))$ as $x \to x_0$ if $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$.

Finally we say that the function $f(x)$ is of the same order as the function $g(x)$ at some point x_0 and denote $f(x) = O(g(x))$ as $x \to x_0$ if the inequality $|f(x)| \le C|g(x)|$ is valid with a constant $C > 0$ for a neighborhood of the point x_0. Also it is common to write $f(x) = O(g(x))$ for $x \in \mathbf{D}$, when the inequality $|f(x)| \le C|g(x)|$ is true for all $x \in \mathbf{D} \subseteq \mathbf{R}^n$, where $C > 0$ is some constant.

1.1.3 Hölder inequality

Theorem 1.1. *Let* $f \in \mathbf{L}^p(\mathbf{R}^n)$ *and* $g \in \mathbf{L}^q(\mathbf{R}^n)$ *with* $1 \le p, q \le \infty$, *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $fg \in \mathbf{L}^1(\mathbf{R}^n)$ *and the Hölder inequality*

$$\int_{\mathbf{R}^n} |f(x)\, g(x)|\, dx \le \|f\|_{\mathbf{L}^p} \|g\|_{\mathbf{L}^q} \tag{1.1}$$

is true.

1.1.4 Young inequality

We now state the Young inequality for convolutions (see Zygmund [2002] for the proof).

Theorem 1.2. *Let the functions* $f \in \mathbf{L}^p(\mathbf{R}^n)$ *and* $g \in \mathbf{L}^q(\mathbf{R}^n)$*, where* $1 \leq p, q \leq \infty$*,* $\frac{1}{p} + \frac{1}{q} \geq 1$*. Then the convolution*

$$h(x) \equiv \int_{\mathbf{R}^n} f(x-y)g(y)dy$$

belongs to $\mathbf{L}^r(\mathbf{R}^n)$*, where* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$*, and the Young inequality*

$$\|h\|_{\mathbf{L}^r} \leq \|f\|_{\mathbf{L}^p} \|g\|_{\mathbf{L}^q} \tag{1.2}$$

is true.

Remark 1.3. In the case of weighted Lebesgue spaces $\mathbf{L}^{r,a}(\mathbf{R}^n)$ with $1 \leq r \leq \infty$, $a \geq 0$ we have a simple modification of the Young inequality

$$\|h\|_{\mathbf{L}^{r,a}} \leq \|f\|_{\mathbf{L}^{p_1,a}} \|g\|_{\mathbf{L}^{q_1}} + \|f\|_{\mathbf{L}^{p_2}} \|g\|_{\mathbf{L}^{q_2,a}}, \tag{1.3}$$

where $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} - 1 = \frac{1}{p_2} + \frac{1}{q_2} - 1$.

1.1.5 Sobolev imbedding inequality

Theorem 1.4. *Let* $1 \leq p, q < \infty$ *and* $\alpha \geq 0$ *be such that* $\frac{1}{p} = \frac{1}{q} - \frac{\alpha}{n}$*. Then the inequality*

$$\|\phi\|_{\mathbf{L}^p} \leq C \left\| |\nabla|^\alpha \phi \right\|_{\mathbf{L}^q}$$

is true, provided that the right-hand side is finite, where $C > 0$ *is a constant, which does not depend on* ϕ*.*

For some more properties of Sobolev spaces we refer to books Adams [1975], Besov et al. [1979], Nikol′skiĭ [1977].

1.1.6 An interpolation inequality

Lemma 1.5. *The interpolation inequality is true*

$$\|\phi\|_{\mathbf{L}^1} \leq \left\| |\cdot|^a \phi \right\|_{\mathbf{L}^1}^{1-\beta} \|\phi\|_{\mathbf{L}^p}^\beta, \tag{1.4}$$

where $\frac{1}{\beta} = 1 + \frac{n}{a}\left(1 - \frac{1}{p}\right)$ *for* $a > 0$ *and* $p > 1$*, provided that the right-hand side is finite.*

We give the proof for the convenience of the reader.

Proof. We take $b > 0$ such that $\left(1+\frac{a}{n}\frac{p}{p-1}\right)^{-1} < b < 1$. By virtue of the Hölder inequality we have for $\rho > 0$

$$\begin{aligned}\|\phi\|_{\mathbf{L}^1} &= \int_{\mathbf{R}^n} \left(\left(\rho+|x|^a\right)|\phi(x)|\right)^b \left(\rho+|x|^a\right)^{-b} |\phi(x)|^{1-b} dx \\ &\le \|\left(\rho+|\cdot|^a\right)\phi\|_{\mathbf{L}^1}^b \left\|\left(\rho+|\cdot|^a\right)^{-\frac{b}{1-b}}\phi\right\|_{\mathbf{L}^1}^{1-b}.\end{aligned}$$

Then again applying the Hölder inequality with $q = \frac{p}{p-1} > 1$

$$\begin{aligned}\|\phi\|_{\mathbf{L}^1} &\le \|\left(\rho+|\cdot|^a\right)\phi\|_{\mathbf{L}^1}^b \left\|\left(\rho+|\cdot|^a\right)^{-\frac{b}{1-b}}\right\|_{\mathbf{L}^q}^{1-b} \|\phi\|_{\mathbf{L}^p}^{1-b} \\ &\le C\rho^{\frac{n}{aq}(1-b)} \|\phi\|_{\mathbf{L}^1}^b \|\phi\|_{\mathbf{L}^p}^{1-b} + C\rho^{\frac{n}{aq}(1-b)-b} \||\cdot|^a\phi\|_{\mathbf{L}^1}^b \|\phi\|_{\mathbf{L}^p}^{1-b},\end{aligned}$$

since $\frac{abq}{1-b} > n$. Hence taking $\rho = \||\cdot|^a\phi\|_{\mathbf{L}^1}\|\phi\|_{\mathbf{L}^1}^{-1} > 0$, we obtain

$$\|\phi\|_{\mathbf{L}^1} \le \|\phi\|_{\mathbf{L}^1}^{b-\frac{n}{aq}(1-b)} \||\cdot|^a\phi\|_{\mathbf{L}^1}^{\frac{n}{aq}(1-b)} \|\phi\|_{\mathbf{L}^p}^{1-b},$$

from which inequality (1.4) follows. Lemma 1.5 is proved.

1.1.7 Contraction mapping principle

First we give a definition.

Definition 1.6. *We call a transformation $\mathcal{M}$ by a contraction mapping in a metric space $\mathbf{X}$ with a distance d if*

$$d(\mathcal{M}(u), \mathcal{M}(v)) \le \alpha d(u,v) \tag{1.5}$$

for all $u, v \in \mathbf{X}$, where $\alpha \in (0,1)$.

The following result is usually called the contraction mapping principle (see Kolmogorov and Fomin [1957] for the proof).

Theorem 1.7. *Let $\mathbf{X}$ be a complete metric space and $\mathcal{M} : \mathbf{X} \to \mathbf{X}$ be a contraction mapping. Then there exists a unique fixed point $w \in \mathbf{X}$, that is: $\mathcal{M}(w) = w$.*

1.1.8 Gronwall's Lemma

Lemma 1.8. *If $b, \phi \ge 0$ and $a > 0$ is a constant, and if*

$$\phi(t) \le a + \int_0^t b(\tau)\phi(\tau)\, d\tau, \tag{1.6}$$

then

$$\phi(t) \le a \exp\left(\int_0^t b(\tau)\, d\tau\right).$$

See Bellman [1969] for the proof.

1.2 Local existence

In this section we prove local existence of solutions to the Cauchy problem

$$\begin{cases} u_t + \mathcal{N}(u) + \mathcal{L}u = 0,\ x \in \mathbf{R}^n,\ t > 0, \\ u(0,x) = u_0(x),\ x \in \mathbf{R}^n, \end{cases} \tag{1.7}$$

where $\mathcal{L}$ is a linear pseudodifferential operator

$$\mathcal{L}\phi = \overline{\mathcal{F}}_{\xi \to x} L(\xi)\, \widehat{\phi}(\xi)$$

and $\mathcal{N}(u)$ is some nonlinear operator, which in general can depend on derivatives of the unknown function $u(t,x)$. We always assume that $\mathcal{N}(0) = 0$. Here and below the subscripts denote the differentiation with respect to the spatial and time coordinates.

By the Duhamel principle we rewrite the Cauchy problem (1.7) as the following integral equation

$$\begin{aligned} u(t) &= \mathcal{G}(t)\, u_0 - \int_0^t \mathcal{G}(t-\tau)\, \mathcal{N}(u)(\tau)\, d\tau \qquad (1.8) \\ &= \int_{\mathbf{R}^n} G(t, x-y)\, u_0(y)\, dy - \int_0^t \int_{\mathbf{R}^n} G(t-\tau, x-y)\ \mathcal{N}(u)(\tau, y)\, dy d\tau, \end{aligned}$$

where the Green operator $\mathcal{G}$ of the corresponding linear problem is defined by the inverse Fourier transformation

$$\mathcal{G}(t)\, \phi = \overline{\mathcal{F}}_{\xi \to x} e^{-tL(\xi)} \widehat{\phi}(\xi) = \int_{\mathbf{R}^n} G(t, x-y)\, \phi(y)\, dy;$$

here $L(\xi)$ is the symbol of the operator $\mathcal{L}$ in equation (1.7).

By the local solution of the Cauchy problem (1.7) we always understand the solution $u(t)$ of the corresponding integral equation (1.8) belonging to some complete metric space $\mathbf{X}_T$ of functions defined on $[0,T] \times \mathbf{R}^n$ (the so-called mild solution).

We fix a metric space $\mathbf{Z}$ of functions defined on $\mathbf{R}^n$ and a complete metric space $\mathbf{X}_T$ of functions defined on $[0,T] \times \mathbf{R}^n$. In applications the space $\mathbf{X}_T$ often has the following form $\mathbf{X}_T = \mathbf{C}([0,T]; \mathbf{Z}) \cap \mathbf{C}((0,T]; \mathbf{Z}_1)$ with some metric space $\mathbf{Z}_1$ of functions on $\mathbf{R}^n$.

First we consider the case of arbitrary initial data from some space $\mathbf{Z}$; however, the existence time $T > 0$ could be sufficiently small.

Theorem 1.9. *Let initial data $u_0 \in \mathbf{Z}$. For some time interval $T > 0$ assume that $\mathcal{G} : \mathbf{Z} \to \mathbf{X}_T$ and the estimate is valid*

$$\|\mathcal{G}\phi\|_{\mathbf{X}_T} \le C\, \|\phi\|_{\mathbf{Z}}\,. \tag{1.9}$$

Also suppose that $\int_0^t \mathcal{G}(t-\tau)\, \mathcal{N}(v(\tau))\, d\tau \in \mathbf{X}_T$ for any $v \in \mathbf{X}_T$ and the following estimate is true

$$\left\| \int_0^t \mathcal{G}(t-\tau)\left(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau))\right) d\tau \right\|_{\mathbf{X}_T}$$
$$\leq CT^{\mu} \|w-v\|_{\mathbf{X}_T} \left(1 + \|w\|_{\mathbf{X}_T} + \|v\|_{\mathbf{X}_T}\right)^{\sigma} \tag{1.10}$$

for any $v, w \in \mathbf{X}_T$, *where* $\mu \in (0,1]$, $\sigma > 0$. *Then for some time interval* $T > 0$ *there exists a unique solution* $u \in \mathbf{X}_T$ *to the Cauchy problem (1.7).*

Proof. We apply the contraction mapping principle in a ball of a radius $\rho > 0$ in a complete metric space $\mathbf{X}_T$

$$\mathbf{X}_{T,\rho} = \left\{\phi \in \mathbf{X}_T : \ \|u\|_{\mathbf{X}_T} \leq \rho\right\},$$

where
$$\rho = \frac{1}{2C} \|u_0\|_{\mathbf{Z}}$$
(the constant $C > 0$ is taken from the conditions of the theorem). For $v \in \mathbf{X}_{T,\rho}$ we define the mapping $\mathcal{M}(v)$ by

$$\mathcal{M}(v) = \mathcal{G}(t) u_0 - \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v(\tau))\, d\tau. \tag{1.11}$$

First we prove that
$$\|\mathcal{M}(v)\|_{\mathbf{X}_T} \leq \rho,$$
where $v \in \mathbf{X}_{T,\rho}$. We have by the integral formula (1.11) and the conditions of the theorem (with $w \equiv 0$)

$$\begin{aligned} \|\mathcal{M}(v)\|_{\mathbf{X}_T} &\leq \|\mathcal{G}u_0\|_{\mathbf{X}_T} + \left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v(\tau))\, d\tau \right\|_{\mathbf{X}_T} \\ &\leq C\|u_0\|_{\mathbf{Z}} + CT^{\mu}\left(1 + \|v\|_{\mathbf{X}_T}\right)^{\sigma+1} \\ &\leq \frac{\rho}{2} + CT^{\mu}(1+\rho)^{\sigma+1} \leq \rho, \end{aligned} \tag{1.12}$$

if $T > 0$ is small enough. Therefore the mapping $\mathcal{M}$ transforms a ball of a radius $\rho > 0$ into itself in the space $\mathbf{X}_T$. As in the proof of (1.12) we have for $w, v \in \mathbf{X}_{T,\rho}$

$$\|\mathcal{M}(w) - \mathcal{M}(v)\|_{\mathbf{X}_T} \leq \left\| \int_0^t \mathcal{G}(t-\tau)\left(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau))\right) d\tau \right\|_{\mathbf{X}_T}$$
$$\leq CT^{\mu}\|w-v\|_{\mathbf{X}_T}\left(1 + \|w\|_{\mathbf{X}_T} + \|v\|_{\mathbf{X}_T}\right)^{\sigma} \leq \frac{1}{2}\|w-v\|_{\mathbf{X}_T},$$

since $T > 0$ is small enough. Thus $\mathcal{M}$ is a contraction mapping in $\mathbf{X}_{T,\rho}$; therefore there exists a unique solution $u \in \mathbf{X}_T$ to the Cauchy problem (1.7). Theorem 1.9 is proved.

Remark 1.10. Note that the existence time T can be chosen as follows (see estimate (1.12))

$$T = \left(\|u_0\|_{\mathbf{Z}} \left(1 + \frac{1}{2C} \|u_0\|_{\mathbf{Z}} \right)^{-\sigma-1} \right)^{\frac{1}{\mu}}.$$

Now we consider the case of small initial data from a space $\mathbf{Z}$, then we can guarantee that the existence time T is not small: $T \geq 1$.

Theorem 1.11. *Let initial data $u_0 \in \mathbf{Z}$ be sufficiently small. Assume that $\mathcal{G} : \mathbf{Z} \to \mathbf{X}_T$ and estimate (1.9) is valid. Also suppose that*

$$\int_0^t \mathcal{G}(t-\tau) \mathcal{N}(v(\tau))\, d\tau \in \mathbf{X}_T$$

for any $v \in \mathbf{X}_T$ and the estimate is true

$$\left\| \int_0^t \mathcal{G}(t-\tau) \left(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)) \right) d\tau \right\|_{\mathbf{X}_T} \leq CT \|w - v\|_{\mathbf{X}_T} \left(\|w\|_{\mathbf{X}_T} + \|v\|_{\mathbf{X}_T} \right)^{\sigma}$$

for any $v, w \in \mathbf{X}_T$, where $\sigma > 0$. Then for some time interval $T \geq 1$ there exists a unique solution $u \in \mathbf{X}_T$ to the Cauchy problem (1.7).

Proof. As above we apply the contraction mapping principle in

$$\mathbf{X}_{T,\rho} = \left\{ \phi \in \mathbf{X}_T : \ \|u\|_{\mathbf{X}_T} \leq \rho \right\},$$

where now

$$\rho = \frac{1}{2C} \|u_0\|_{\mathbf{Z}}$$

is sufficiently small. For $v \in \mathbf{X}_{T,\rho}$ we define the mapping $\mathcal{M}(v)$ by formula (1.11). First we prove that

$$\|\mathcal{M}(v)\|_{\mathbf{X}_T} \leq \rho.$$

We have by conditions of the theorem and integral formula (1.11)

$$\|\mathcal{M}(v)\|_{\mathbf{X}_T} \leq \|\mathcal{G}u_0\|_{\mathbf{X}_T} + \left\| \int_0^t \mathcal{G}(t-\tau) \mathcal{N}(v(\tau))\, d\tau \right\|_{\mathbf{X}_T}$$
$$\leq C \|u_0\|_{\mathbf{Z}} + CT \|v\|_{\mathbf{X}_T}^{\sigma+1} \leq \frac{\rho}{2} + CT\rho^{\sigma+1} \leq \rho,$$

if $\rho > 0$ is small enough, and $T \geq 1$. Therefore $\mathcal{M}$ transforms $\mathbf{X}_{T,\rho}$ into itself. In the same way we estimate the difference of two functions $w, v \in \mathbf{X}_{T,\rho}$

$$\|\mathcal{M}(w)-\mathcal{M}(v)\|_{\mathbf{X}_T} \le \left\| \int_0^t \mathcal{G}(t-\tau)\left(\mathcal{N}(w(\tau))-\mathcal{N}(v(\tau))\right) d\tau \right\|_{\mathbf{X}_T}$$
$$\le CT\|w-v\|_{\mathbf{X}_T}\left(\|w\|_{\mathbf{X}_T}+\|v\|_{\mathbf{X}_T}\right)^{\sigma} \le \frac{1}{2}\|w-v\|_{\mathbf{X}_T},$$

since $\rho > 0$ is small enough. Thus $\mathcal{M}$ is a contraction mapping in $\mathbf{X}_{T,\rho}$, therefore there exists a unique solution $u \in \mathbf{X}_T$ to the problem (1.7). Theorem 1.11 is proved.

Example 1.12. **Local existence of solutions to the nonlinear heat equation**

Consider the Cauchy problem for the nonlinear heat equation

$$\begin{cases} u_t - \Delta u = \lambda |u|^{\sigma} u, \ x \in \mathbf{R}^n, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^n, \end{cases} \tag{1.13}$$

where $\sigma > 0$, $\lambda \in \mathbf{R}$. Here and below $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ - is a Laplacian. Note that problem (1.13) follows from (1.7) if we choose $\mathcal{L} = -\Delta$ and $\mathcal{N}(u) = -\lambda |u|^{\sigma} u$.

Define the Green operator for the linear heat equation

$$\mathcal{G}(t)\phi = \int_{\mathbf{R}^n} G(t, x-y)\phi(y)\,dy,$$

where the $G(t,x)$ is

$$G(t,x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$

Theorem 1.13. *Let the initial data $u_0 \in \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, with $a \ge 0$ and $p > \max\left(1, \frac{n}{2}\sigma\right)$. Then for some $T > 0$ there exists a unique solution $u \in \mathbf{C}\left([0,T]; \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)\right) \cap \mathbf{C}\left((0,T]; \mathbf{L}^{\infty}(\mathbf{R}^n)\right)$ to the Cauchy problem (1.13).*

Remark 1.14. The property of the solution $u \in \mathbf{C}\left((0,T]; \mathbf{L}^{\infty}(\mathbf{R}^n)\right)$ means a smoothing effect.

Proof. To apply Theorem 1.9 we choose the space $\mathbf{Z}$ as follows

$$\mathbf{Z} = \left\{\phi \in \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)\right\}$$

with $a \ge 0$ and $p > \max\left(1, \frac{n}{2}\sigma\right)$ and

$$\mathbf{X}_T = \left\{\phi \in \mathbf{C}([0,T]; \mathbf{Z}) \cap \mathbf{C}\left((0,T]; \mathbf{L}^{\infty}(\mathbf{R}^n)\right) : \ \|\phi\|_{\mathbf{X}_T} < \infty\right\},$$

where the norm

$$\|\phi\|_{\mathbf{X}_T} = \sup_{t\in[0,T]}\left(\|\phi(t)\|_{\mathbf{L}^{1,a}} + \|\phi(t)\|_{\mathbf{L}^p}\right) + \sup_{t\in(0,T]} t^{\frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^{\infty}}.$$

Also we define the $\mathbf{Y}_T$ norm as follows

$$\|\phi\|_{\mathbf{Y}_T} = \sup_{t\in(0,T]} t^{\frac{n\sigma}{2p}} \left(\|\phi(t)\|_{\mathbf{L}^{1,a}} + \|\phi(t)\|_{\mathbf{L}^p} + t^{\frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^\infty} \right).$$

By a direct calculation we have

$$\|G(t)\|_{\mathbf{L}^q} = (4\pi t)^{-\frac{n}{2}} \left(\int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4t}q} dx \right)^{\frac{1}{q}}$$
$$= (4\pi t)^{-\frac{n}{2}} \left(\frac{4t}{q} \right)^{\frac{n}{2q}} \left(\int_{\mathbf{R}^n} e^{-|y|^2} dy \right)^{\frac{1}{q}} \le C t^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}, \tag{1.14}$$

for all $t > 0$, where $1 \le q \le \infty$, and in the same manner

$$\|G(t)\|_{\mathbf{L}^{1,a}} = (4\pi t)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \langle x \rangle^a e^{-\frac{|x|^2}{4t}} dx \le C \langle t \rangle^{\frac{a}{2}} \tag{1.15}$$

for all $t > 0$. First we prove the estimate

$$\|\mathcal{G}\phi\|_{\mathbf{X}_T} \le C \|\phi\|_{\mathbf{Z}}. \tag{1.16}$$

Therefore by virtue of (1.14) and by applying the Young inequality for convolutions (1.2) we obtain

$$\sup_{t\in[0,T]} \|\mathcal{G}(t)\phi\|_{\mathbf{L}^p} \le \|\phi\|_{\mathbf{L}^p} \sup_{t\in[0,T]} \|G(t)\|_{\mathbf{L}^1} \le C \|\phi\|_{\mathbf{L}^p}$$

and

$$\sup_{t\in(0,T]} t^{\frac{n}{2p}} \|\mathcal{G}(t)\phi\|_{\mathbf{L}^\infty} \le \|\phi\|_{\mathbf{L}^p} \sup_{t\in(0,T]} t^{\frac{n}{2p}} \|G(t)\|_{\mathbf{L}^{\frac{p}{p-1}}} \le C \|\phi\|_{\mathbf{L}^p}.$$

Similarly by (1.15) and (1.2) we find the estimate

$$\sup_{t\in[0,T]} \langle t \rangle^{-\frac{a}{2}} \|\mathcal{G}(t)\phi\|_{\mathbf{L}^{1,a}} = \sup_{t\in[0,T]} \langle t \rangle^{-\frac{a}{2}} \left\| \int_{\mathbf{R}^n} \langle x \rangle^a G(t, x-y) \phi(y) dy \right\|_{\mathbf{L}^1}$$
$$\le \|\phi\|_{\mathbf{L}^{1,a}} \sup_{t\in[0,T]} \langle t \rangle^{-\frac{a}{2}} \|G(t)\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^1} \sup_{t\in[0,T]} \langle t \rangle^{-\frac{a}{2}} \|G(t)\|_{\mathbf{L}^{1,a}}$$
$$\le C \|\phi\|_{\mathbf{L}^{1,a}};$$

hence (1.16) is true. Now let us prove the estimate (1.10) with $\mu = 1 - \frac{n\sigma}{2p} \in (0,1)$. Since

$$\left| |w|^\sigma w - |v|^\sigma v \right| \le C |w - v| \left(|w|^\sigma + |v|^\sigma \right)$$

we obtain the estimates

$$\|(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)))\|_{\mathbf{L}^{1,a}} = |\lambda| \left\| |w|^\sigma w(\tau) - |v|^\sigma v(\tau) \right\|_{\mathbf{L}^{1,a}}$$
$$\le C \|w - v\|_{\mathbf{L}^{1,a}} \left(\|w\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^\infty} \right)^\sigma$$
$$\le C \tau^{-\frac{n\sigma}{2p}} \|w - v\|_{\mathbf{X}_T} \left(\|w\|_{\mathbf{X}_T} + \|v\|_{\mathbf{X}_T} \right)^\sigma,$$

$$\|(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)))\|_{\mathbf{L}^p} \le C\|w - v\|_{\mathbf{L}^p} (\|w\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^\infty})^\sigma$$
$$\le C\tau^{-\frac{n\sigma}{2p}} \|w - v\|_{\mathbf{X}_T} \left(\|w\|_{\mathbf{X}_T} + \|v\|_{\mathbf{X}_T}\right)^\sigma,$$

and

$$\|(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)))\|_{\mathbf{L}^\infty} \le C\|w - v\|_{\mathbf{L}^\infty} (\|w\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^\infty})^\sigma$$
$$\le C\tau^{-\frac{n}{2p}(\sigma+1)} \|w - v\|_{\mathbf{X}_T} \left(\|w\|_{\mathbf{X}_T} + \|v\|_{\mathbf{X}_T}\right)^\sigma$$

for all $0 \le \tau \le t \le T$. Hence we get

$$\|\mathcal{N}(w) - \mathcal{N}(v)\|_{\mathbf{Y}_T} \le C\|w - v\|_{\mathbf{X}_T} \left(\|w\|_{\mathbf{X}_T} + \|v\|_{\mathbf{X}_T}\right)^\sigma. \tag{1.17}$$

Thus by (1.15) and (1.2)

$$\left\|\int_0^t \mathcal{G}(t-\tau)\phi(\tau)\,d\tau\right\|_{\mathbf{L}^{1,a}} \le \int_0^t \|G(t-\tau)\|_{\mathbf{L}^1} \|\phi(\tau)\|_{\mathbf{L}^{1,a}}\,d\tau$$
$$+ \int_0^t \|G(t-\tau)\|_{\mathbf{L}^{1,a}} \|\phi(\tau)\|_{\mathbf{L}^1}\,d\tau \le C\|\phi\|_{\mathbf{Y}_T} \int_0^t \langle t-\tau\rangle^{\frac{a}{2}} \tau^{-\frac{n\sigma}{2p}}\,d\tau$$
$$\le CT^\mu \|\phi\|_{\mathbf{Y}_T}$$

for all $0 \le t \le T$, where $\mu = 1 - \frac{n\sigma}{2p} \in (0,1)$. In the same manner by virtue of (1.14) and (1.2) we have

$$\left\|\int_0^t \mathcal{G}(t-\tau)\phi(\tau)\,d\tau\right\|_{\mathbf{L}^p} \le \int_0^t \|G(t-\tau)\|_{\mathbf{L}^1} \|\phi(\tau)\|_{\mathbf{L}^p}\,d\tau$$
$$\le C\|\phi\|_{\mathbf{Y}_T} \int_0^t \tau^{-\frac{n\sigma}{2p}}\,d\tau \le CT^\mu \|\phi\|_{\mathbf{Y}_T},$$

for all $0 \le t \le T$. In addition taking (1.2) with $r = \frac{p}{p-1}$ or $r = 1$ in view of (1.14) we find

$$\left\|\int_0^t \mathcal{G}(t-\tau)\phi(\tau)\,d\tau\right\|_{\mathbf{L}^\infty}$$
$$\le \int_0^{\frac{t}{2}} \|G(t-\tau)\|_{\mathbf{L}^{\frac{p}{p-1}}} \|\phi(\tau)\|_{\mathbf{L}^p}\,d\tau + \int_{\frac{t}{2}}^t \|G(t-\tau)\|_{\mathbf{L}^1} \|\phi(\tau)\|_{\mathbf{L}^\infty}\,d\tau$$
$$\le C\|\phi\|_{\mathbf{Y}_T} \left(\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2p}} \tau^{-\frac{n\sigma}{2p}}\,d\tau + \int_{\frac{t}{2}}^t \tau^{-\frac{n}{2p}(\sigma+1)}d\tau\right)$$
$$\le CT^\mu t^{-\frac{n}{2p}} \|\phi\|_{\mathbf{Y}_T}$$

for all $t \in (0,T]$. In view of (1.17) estimate (1.10) is then fulfilled. Therefore by applying Theorem 1.9 we see that for some $T > 0$ there exists a unique solution $u \in \mathbf{C}\left([0,T];\mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)\right) \cap \mathbf{C}((0,T];\mathbf{L}^\infty(\mathbf{R}^n))$ to the Cauchy problem (1.13). Theorem 1.13 is proved.

Example 1.15. **Local existence of solutions to the Burgers type equations**

Consider the Cauchy problem for the Burgers type equation

$$\begin{cases} u_t - \Delta u = (\lambda \cdot \nabla) |u|^{\sigma} u, \ x \in \mathbf{R}^n, \ t > 0, \\ u(0, x) = u_0(x), \ x \in \mathbf{R}^n, \end{cases} \tag{1.18}$$

where $\sigma > 0$, $\lambda \in \mathbf{R}^n$. Here the nonlinear term $\mathcal{N}(u) = (\lambda \cdot \nabla) |u|^{\sigma} u$ has the form of the full derivative. In general (if the total mass of the initial data is nonzero) such a type of the nonlinearity behaves for large times as a convective one.

We prove here the local existence of solutions to the Cauchy problem (1.18).

Theorem 1.16. *Let the initial data* $u_0 \in \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, *with* $a \geq 0$ *and* $p > \max(1, n\sigma)$. *Then for some* $T > 0$ *there exists a unique solution* $u \in \mathbf{C}([0,T]; \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)) \cap \mathbf{C}((0,T]; \mathbf{W}^1_\infty(\mathbf{R}^n))$ *to the Cauchy problem (1.18).*

Proof. To apply Theorem 1.9 we choose the space $\mathbf{Z}$ as follows

$$\mathbf{Z} = \left\{ \phi \in \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n) \right\}$$

with $a \geq 0$ and $p > \max(1, n\sigma)$ and

$$\mathbf{X}_T = \left\{ \phi \in \mathbf{C}([0,T]; \mathbf{Z}) \cap \mathbf{C}((0,T]; \mathbf{W}^1_\infty(\mathbf{R}^n)) : \ \|\phi\|_{\mathbf{X}_T} < \infty \right\},$$

where the norm

$$\begin{aligned} \|\phi\|_{\mathbf{X}_T} &= \sup_{t \in [0,T]} \left(\|\phi(t)\|_{\mathbf{L}^{1,a}} + \|\phi(t)\|_{\mathbf{L}^p} \right) \\ &+ \sup_{t \in (0,T]} \left(t^{\frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^\infty} + t^{\frac{n}{2p} + \frac{1}{2}} \|\nabla \phi(t)\|_{\mathbf{L}^\infty} \right). \end{aligned}$$

Also we define the $\mathbf{Y}_T$ norm

$$\|\phi\|_{\mathbf{Y}_T} = \sup_{t \in (0,T]} t^{\frac{n\sigma}{2p} + \frac{1}{2}} \left(\|\phi(t)\|_{\mathbf{L}^{1,a}} + \|\phi(t)\|_{\mathbf{L}^p} + t^{\frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^\infty} \right).$$

By a direct calculation we have

$$\begin{aligned} \|\nabla G(t)\|_{\mathbf{L}^q} &= (4\pi t)^{-\frac{n}{2}} \left(\int_{\mathbf{R}^n} \left| \frac{x}{2t} \right|^q e^{-\frac{|x|^2}{4t} q} dx \right)^{\frac{1}{q}} \\ &\leq C t^{-\frac{n}{2}\left(1 - \frac{1}{q}\right) - \frac{1}{2}} \left(\int_{\mathbf{R}^n} e^{-|y|^2} dy \right)^{\frac{1}{q}} \leq C t^{-\frac{n}{2}\left(1 - \frac{1}{q}\right) - \frac{1}{2}}, \end{aligned} \tag{1.19}$$

for all $t > 0$, where $1 \leq q \leq \infty$. Therefore applying the Young inequality (1.2) we get

$$\sup_{t\in(0,T]} t^{\frac{n}{2p}+\frac{1}{2}} \|\nabla\mathcal{G}(t)\phi\|_{\mathbf{L}^\infty} \le \|\phi\|_{\mathbf{L}^p} \sup_{t\in(0,T]} t^{\frac{n}{2p}+\frac{1}{2}} \|\nabla G(t)\|_{\mathbf{L}^{\frac{p}{p-1}}} \le C\|\phi\|_{\mathbf{L}^p}.$$

Hence as in the proof of Theorem 1.13 we have estimate (1.9).

Next we prove estimate (1.17) for the nonlinearity. We obtain the estimates

$$\begin{aligned}
&\|(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)))\|_{\mathbf{L}^{1,a}} = \|(\lambda\cdot\nabla)(|w|^\sigma w(\tau) - |v|^\sigma v(\tau))\|_{\mathbf{L}^{1,a}}\\
&\le C\|w-v\|_{\mathbf{L}^{1,a}}(\|w\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^\infty})^{\sigma-1}(\|\nabla w\|_{\mathbf{L}^\infty} + \|\nabla v\|_{\mathbf{L}^\infty})\\
&+ C\|\nabla(w-v)\|_{\mathbf{L}^\infty}(\|w\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^\infty})^{\sigma-1}(\|w\|_{\mathbf{L}^{1,a}} + \|v\|_{\mathbf{L}^{1,a}})\\
&\le C\tau^{-\frac{n}{2p}\sigma-\frac{1}{2}}\|w-v\|_{\mathbf{X}_T}\left(\|w\|_{\mathbf{X}_T} + \|v\|_{\mathbf{X}_T}\right)^\sigma,
\end{aligned}$$

$$\begin{aligned}
&\|(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)))\|_{\mathbf{L}^p}\\
&\le C\|w-v\|_{\mathbf{L}^p}(\|w\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^\infty})^{\sigma-1}(\|\nabla w\|_{\mathbf{L}^\infty} + \|\nabla v\|_{\mathbf{L}^\infty})\\
&+ C\|\nabla(w-v)\|_{\mathbf{L}^\infty}(\|w\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^\infty})^{\sigma-1}(\|w\|_{\mathbf{L}^p} + \|v\|_{\mathbf{L}^p})\\
&\le C\tau^{-\frac{n}{2p}\sigma-\frac{1}{2}}\|w-v\|_{\mathbf{X}_T}\left(\|w\|_{\mathbf{X}_T} + \|v\|_{\mathbf{X}_T}\right)^\sigma,
\end{aligned}$$

and

$$\begin{aligned}
&\|(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)))\|_{\mathbf{L}^\infty}\\
&\le C\|w-v\|_{\mathbf{L}^\infty}(\|w\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^\infty})^{\sigma-1}(\|\nabla w\|_{\mathbf{L}^\infty} + \|\nabla v\|_{\mathbf{L}^\infty})\\
&+ C\|\nabla(w-v)\|_{\mathbf{L}^\infty}(\|w\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^\infty})^{\sigma}\\
&\le C\tau^{-\frac{n}{2p}(\sigma+1)-\frac{1}{2}}\|w-v\|_{\mathbf{X}_T}\left(\|w\|_{\mathbf{X}_T} + \|v\|_{\mathbf{X}_T}\right)^\sigma
\end{aligned}$$

for all $0 \le \tau \le t \le T$. Consequently estimate (1.17) is true. Now by (1.15) and (1.2) we get

$$\begin{aligned}
&\left\|\int_0^t \mathcal{G}(t-\tau)\phi(\tau)\,d\tau\right\|_{\mathbf{L}^{1,a}} \le \int_0^t \|G(t-\tau)\|_{\mathbf{L}^1}\|\phi(\tau)\|_{\mathbf{L}^{1,a}}\,d\tau\\
&+ \int_0^t \|G(t-\tau)\|_{\mathbf{L}^{1,a}}\|\phi(\tau)\|_{\mathbf{L}^{1,a}}\,d\tau \le C\|\phi\|_{\mathbf{Y}_T}\int_0^t \langle t-\tau\rangle^{\frac{a}{2}}\tau^{-\frac{n\sigma}{2p}-\frac{1}{2}}d\tau\\
&\le CT^\mu\|\phi\|_{\mathbf{Y}_T}
\end{aligned}$$

for all $0 \le t \le T$, where $\mu = \frac{1}{2} - \frac{n\sigma}{2p} \in (0,1)$. In the same manner by virtue of (1.14) and (1.2) we have

$$\begin{aligned}
&\left\|\int_0^t \mathcal{G}(t-\tau)\phi(\tau)\,d\tau\right\|_{\mathbf{L}^p} \le \int_0^t \|G(t-\tau)\|_{\mathbf{L}^1}\|\phi(\tau)\|_{\mathbf{L}^p}\,d\tau\\
&\le C\|\phi\|_{\mathbf{Y}_T}\int_0^t \tau^{-\frac{n\sigma}{2p}-\frac{1}{2}}d\tau \le CT^\mu\|\phi\|_{\mathbf{Y}_T},
\end{aligned}$$

for all $0 \le t \le T$. By taking (1.2) with $r = \frac{p}{p-1}$ or $r = 1$ in view of (1.14) we find

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}(t-\tau)\,\phi(\tau)\,d\tau \right\|_{\mathbf{L}^\infty} \\
&\le \int_0^{\frac{t}{2}} \|G(t-\tau)\|_{\mathbf{L}^{\frac{p}{p-1}}} \|\phi(\tau)\|_{\mathbf{L}^p}\,d\tau + \int_{\frac{t}{2}}^t \|G(t-\tau)\|_{\mathbf{L}^1} \|\phi(\tau)\|_{\mathbf{L}^\infty}\,d\tau \\
&\le C\|\phi\|_{\mathbf{Y}_T} \left(\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2p}} \tau^{-\frac{n\sigma}{2p}-\frac{1}{2}}\,d\tau + \int_{\frac{t}{2}}^t \tau^{-\frac{n}{2p}(\sigma+1)-\frac{1}{2}}\,d\tau \right) \\
&\le CT^\mu t^{-\frac{n}{2p}} \|\phi\|_{\mathbf{Y}_T}
\end{aligned}$$

for all $t \in (0,T]$. Finally, taking (1.2) with $r = \frac{p}{p-1}$ or $r = 1$ in view of (1.19) we find

$$\begin{aligned}
&\left\| \nabla \int_0^t \mathcal{G}(t-\tau)\,\phi(\tau)\,d\tau \right\|_{\mathbf{L}^\infty} \\
&\le \int_0^{\frac{t}{2}} \|\nabla G(t-\tau)\|_{\mathbf{L}^{\frac{p}{p-1}}} \|\phi(\tau)\|_{\mathbf{L}^p}\,d\tau + \int_{\frac{t}{2}}^t \|\nabla G(t-\tau)\|_{\mathbf{L}^1} \|\phi(\tau)\|_{\mathbf{L}^\infty}\,d\tau \\
&\le C\|\phi\|_{\mathbf{Y}_T} \left(\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2p}-\frac{1}{2}} \tau^{-\frac{n\sigma}{2p}-\frac{1}{2}}\,d\tau + \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}} \tau^{-\frac{n}{2p}(\sigma+1)-\frac{1}{2}}\,d\tau \right) \\
&\le CT^\mu t^{-\frac{n}{2p}-\frac{1}{2}} \|\phi\|_{\mathbf{Y}_T}
\end{aligned}$$

for all $t \in (0,T]$. Thus in view of (1.17) estimate (1.10) with $\mu = \frac{1}{2} - \frac{n\sigma}{2p} \in (0,1)$ is fulfilled. Therefore applying Theorem 1.9 we see that for some $T > 0$ there exists a unique solution $u \in \mathbf{C}\left([0,T];\mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)\right) \cap \mathbf{C}\left((0,T];\mathbf{W}^1_\infty(\mathbf{R}^n)\right)$ to the Cauchy problem (1.18). Theorem 1.16 is proved.

1.3 Global existence for small initial data

In this section we prove the global in time existence of solutions to the Cauchy problem (1.7). We fix a metric space $\mathbf{Z}$ of functions defined on $\mathbf{R}^n$ and a complete metric space $\mathbf{X}$ of functions defined on $[0,\infty) \times \mathbf{R}^n$.

Theorem 1.17. *Let the initial data $u_0 \in \mathbf{Z}$ be sufficiently small. Assume that the operator $\mathcal{G} : \mathbf{Z} \to \mathbf{X}$ and the estimate is valid*

$$\|\mathcal{G}\phi\|_{\mathbf{X}} \le C\|\phi\|_{\mathbf{Z}}. \tag{1.20}$$

Also suppose that $\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v(\tau))\,d\tau \in \mathbf{X}$ for any $v \in \mathbf{X}$, and that the estimate is true

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}(t-\tau)\left(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau))\right)d\tau \right\|_{\mathbf{X}} \\
&\le C\|w-v\|_{\mathbf{X}} \left(\|w\|_{\mathbf{X}} + \|v\|_{\mathbf{X}}\right)^\sigma,
\end{aligned} \tag{1.21}$$

for all $v, w \in \mathbf{X}$, *where* $\sigma > 0$. *Then there exists a unique solution* $u \in \mathbf{X}$ *to the Cauchy problem (1.7) and the estimate*

$$\|u\|_{\mathbf{X}} \leq C \|u_0\|_{\mathbf{Z}} \tag{1.22}$$

is fulfilled.

Proof. The proof almost repeats the proof of Theorem 1.11. We apply the contraction mapping principle in a ball $\mathbf{X}_\rho = \{\phi \in \mathbf{X} : \|\phi\|_{\mathbf{X}} \leq \rho\}$ in the space $\mathbf{X}$ of a radius

$$\rho = \frac{1}{2C} \|u_0\|_{\mathbf{Z}} > 0.$$

For $v \in \mathbf{X}_\rho$ we define the mapping $\mathcal{M}(v)$ by formula (1.11). First we prove that

$$\|\mathcal{M}(v)\|_{\mathbf{X}} \leq \rho,$$

where $\rho > 0$ is sufficiently small. We have by the conditions of the theorem and the integral formula (1.11)

$$\begin{aligned} \|\mathcal{M}(v)\|_{\mathbf{X}} &\leq \|\mathcal{G}u_0\|_{\mathbf{X}} + \left\| \int_0^t \mathcal{G}(t-\tau) \mathcal{N}(v(\tau))\, d\tau \right\|_{\mathbf{X}} \\ &\leq C \|u_0\|_{\mathbf{Z}} + C \|v\|_{\mathbf{X}}^{\sigma+1} \leq \frac{\rho}{2} + C\rho^{\sigma+1} < \rho, \end{aligned}$$

since $\rho > 0$ is sufficiently small. Hence the mapping $\mathcal{M}$ transforms a ball $\mathbf{X}_\rho$ into itself. In the same manner we estimate the difference

$$\|\mathcal{M}(w) - \mathcal{M}(v)\|_{\mathbf{X}} \leq \frac{1}{2} \|w - v\|_{\mathbf{X}},$$

which shows that $\mathcal{M}$ is a contraction mapping. Therefore there exists a unique solution $u \in \mathbf{X}$ to the Cauchy problem (1.7). Theorem 1.17 is proved.

Example 1.18. **Global existence for the nonlinear heat equation with small initial data**

Consider the Cauchy problem (1.13) with $\sigma > \frac{2}{n}$ and $\lambda \in \mathbf{R}$.

Theorem 1.19. *Let* $\sigma > \frac{2}{n}$, $\lambda \in \mathbf{R}$. *Let the initial data* $u_0 \in \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, *with* $a \in (0, 1]$ *and* $p > \max\left(1, \frac{n}{2}\sigma\right)$, *and the norm* $\|u_0\|_{\mathbf{L}^{1,a}} + \|u_0\|_{\mathbf{L}^p}$ *be sufficiently small. Then there exists a unique solution*

$$u \in \mathbf{C}\left([0, \infty); \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)\right) \cap \mathbf{C}\left((0, \infty); \mathbf{L}^\infty(\mathbf{R}^n)\right)$$

to the Cauchy problem (1.13). Moreover the optimal time decay estimate

$$\begin{aligned} &t^{-\frac{a}{2}} \|u(t)\|_{\mathbf{L}^{1,a}} + t^{\frac{n}{2}\left(1-\frac{1}{p}\right)} \|u(t)\|_{\mathbf{L}^p} + t^{\frac{n}{2}} \|u(t)\|_{\mathbf{L}^\infty} \\ &\leq C \|u_0\|_{\mathbf{L}^p} + C \|u_0\|_{\mathbf{L}^{1,a}} \end{aligned} \tag{1.23}$$

is true for all $t \geq 1$.

Proof. By the local existence Theorem 1.13, it follows that the global solution (if it exists) is unique. Indeed, on the contrary, we suppose that there exist two global solutions with the same initial data. And these solutions are different at some time $t > 0$. By virtue of the continuity of solutions with respect to time, we can find a maximal time segment $[0, T]$, where the solutions are equal, but for $t > T$ they are different. Now we apply the local existence theorem taking the initial time T and obtain that these solutions coincide on some interval $[T, T_1]$, which gives us a contradiction with the fact that T is the maximal time of coincidence. So our main purpose in the proof of Theorem 1.19 is to show the global in time existence of solutions.

To apply Theorem 1.17 we choose as above the space

$$\mathbf{Z} = \left\{ \phi \in \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n) \right\}$$

with $a \in (0,1]$ and $p > \max\left(1, \frac{n}{2}\sigma\right)$ and the space

$$\mathbf{X} = \{ \phi \in \mathbf{C}([0,\infty);\mathbf{Z}) \cap \mathbf{C}((0,\infty);\mathbf{L}^\infty(\mathbf{R}^n)) : \ \|\phi\|_{\mathbf{X}} < \infty \},$$

where now the norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t \geq 0} \left(\langle t \rangle^{-\frac{a}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} + \langle t \rangle^{\frac{n}{2}\left(1-\frac{1}{p}\right)} \|\phi(t)\|_{\mathbf{L}^p} \right) + \sup_{t > 0} \{t\}^{\frac{n}{2p}} \langle t \rangle^{\frac{n}{2}} \|\phi(t)\|_{\mathbf{L}^\infty}$$

reflects the optimal time decay properties of the solution. Also we define the norm

$$\|\phi\|_{\mathbf{Y}} = \sup_{t > 0} \{t\}^{\frac{n\sigma}{2p}} \langle t \rangle^{\frac{n\sigma}{2}} \left(\langle t \rangle^{-\frac{a}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} + \langle t \rangle^{\frac{n}{2}\left(1-\frac{1}{p}\right)} \|\phi(t)\|_{\mathbf{L}^p} + \{t\}^{\frac{n}{2p}} \langle t \rangle^{\frac{n}{2}} \|\phi(t)\|_{\mathbf{L}^\infty} \right)$$

in order to estimate the nonlinearity. As in the proof of Theorem 1.13 by virtue of (1.14) and (1.2) we obtain

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^p} \leq \|\phi\|_{\mathbf{L}^p} \ \|G(t)\|_{\mathbf{L}^1} \leq C\|\phi\|_{\mathbf{L}^p}$$

for all $t \in [0,1]$ and

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^p} \leq \|\phi\|_{\mathbf{L}^1} \ \|G(t)\|_{\mathbf{L}^p} \leq C t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)} \|\phi\|_{\mathbf{L}^1}$$

for all $t \geq 1$. In the same manner we find

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^\infty} \leq \|\phi\|_{\mathbf{L}^p} \ \|G(t)\|_{\mathbf{L}^{\frac{p}{p-1}}} \leq C t^{-\frac{n}{2p}} \|\phi\|_{\mathbf{L}^p}$$

for all $t \in (0,1]$ and

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^\infty} \leq \|\phi\|_{\mathbf{L}^1} \ \|G(t)\|_{\mathbf{L}^\infty} \leq C t^{-\frac{n}{2}} \|\phi\|_{\mathbf{L}^p}$$

for all $t \geq 1$. Similarly by (1.15) and (1.2) we find

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^{1,a}} \leq \|\phi\|_{\mathbf{L}^{1,a}} \|G(t)\|_{\mathbf{L}^{1,a}} \leq C\langle t\rangle^{\frac{a}{2}} \|\phi\|_{\mathbf{L}^{1,a}}$$

for all $t \geq 0$. Hence we obtain the estimate (1.20). As in the proof of Theorem 1.13 we obtain the estimates

$$\begin{aligned}&\|(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)))\|_{\mathbf{L}^{1,a}} \leq C\|w-v\|_{\mathbf{L}^{1,a}} (\|w\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^\infty})^\sigma \\ &\leq C\{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{\frac{a}{2}-\frac{n\sigma}{2}} \|w-v\|_{\mathbf{X}} (\|w\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma,\end{aligned}$$

$$\begin{aligned}&\|(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)))\|_{\mathbf{L}^p} \leq C\|w-v\|_{\mathbf{L}^p} (\|w\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^\infty})^\sigma \\ &\leq C\{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{\frac{n}{2p}-\frac{n}{2}(\sigma+1)} \|w-v\|_{\mathbf{X}} (\|w\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma,\end{aligned}$$

and

$$\begin{aligned}&\|(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)))\|_{\mathbf{L}^\infty} \leq C\|w-v\|_{\mathbf{L}^\infty} (\|w\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^\infty})^\sigma \\ &\leq C\{\tau\}^{-\frac{n}{2p}(\sigma+1)} \langle\tau\rangle^{-\frac{n}{2}(\sigma+1)} \|w-v\|_{\mathbf{X}} (\|w\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma\end{aligned}$$

for all $\tau > 0$. Thus we get

$$\|\mathcal{N}(w) - \mathcal{N}(v)\|_{\mathbf{Y}} \leq C\|w-v\|_{\mathbf{X}} (\|w\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^\sigma. \tag{1.24}$$

By (1.4) we see that

$$\|\phi(\tau)\|_{\mathbf{L}^1} \leq C\{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{-\frac{n\sigma}{2}} \|\phi\|_{\mathbf{Y}}. \tag{1.25}$$

Now by (1.15) and (1.2) we get

$$\begin{aligned}&\left\|\int_0^t \mathcal{G}(t-\tau)\phi(\tau)\,d\tau\right\|_{\mathbf{L}^{1,a}} \\ &\leq \int_0^t \|G(t-\tau)\|_{\mathbf{L}^1} \|\phi(\tau)\|_{\mathbf{L}^{1,a}}\,d\tau + \int_0^t \|G(t-\tau)\|_{\mathbf{L}^{1,a}} \|\phi(\tau)\|_{\mathbf{L}^1}\,d\tau \\ &\leq C\|\phi\|_{\mathbf{Y}} \left(\int_0^t \langle t-\tau\rangle^{\frac{a}{2}} \{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{-\frac{n\sigma}{2}}\,d\tau + \int_0^t \{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{\frac{a}{2}-\frac{n\sigma}{2}}\,d\tau\right) \\ &\leq C\langle t\rangle^{\frac{a}{2}} \|\phi\|_{\mathbf{Y}}\end{aligned} \tag{1.26}$$

for all $t \geq 0$. In the same manner by virtue of (1.14), (1.25) and (1.2) we have

$$\begin{aligned}&\left\|\int_0^t \mathcal{G}(t-\tau)\phi(\tau)\,d\tau\right\|_{\mathbf{L}^p} \\ &\leq \int_0^{\frac{t}{2}} \|G(t-\tau)\|_{\mathbf{L}^p} \|\phi(\tau)\|_{\mathbf{L}^1}\,d\tau + \int_{\frac{t}{2}}^t \|G(t-\tau)\|_{\mathbf{L}^1} \|\phi(\tau)\|_{\mathbf{L}^p}\,d\tau \\ &\leq C\|\phi\|_{\mathbf{Y}} \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}\left(1-\frac{1}{p}\right)} \{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{-\frac{n\sigma}{2}}\,d\tau \\ &+ C\|\phi\|_{\mathbf{Y}} \int_{\frac{t}{2}}^t \{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{\frac{n}{2p}-\frac{n}{2}(\sigma+1)}\,d\tau \\ &\leq C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)} \|\phi\|_{\mathbf{Y}}\end{aligned} \tag{1.27}$$

for all $t \geq 0$. Also taking (1.2) with $r = \frac{p}{p-1}$ or $r = 1$ in view of (1.14) we find

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}(t-\tau)\phi(\tau)\,d\tau \right\|_{\mathbf{L}^\infty} \\
&\leq \int_0^{\frac{t}{2}} \|G(t-\tau)\|_{\mathbf{L}^{\frac{p}{p-1}}} \|\phi(\tau)\|_{\mathbf{L}^p}\,d\tau + \int_{\frac{t}{2}}^t \|G(t-\tau)\|_{\mathbf{L}^1} \|\phi(\tau)\|_{\mathbf{L}^\infty}\,d\tau \\
&\leq C\|\phi\|_{\mathbf{Y}} \left(\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2p}} \tau^{-\frac{n\sigma}{2p}}\,d\tau + \int_{\frac{t}{2}}^t \tau^{-\frac{n}{2p}(\sigma+1)} d\tau \right) \leq Ct^{-\frac{n}{2p}} \|\phi\|_{\mathbf{Y}}
\end{aligned}$$

for $t \in (0,1]$, and similarly, in view of (1.25) we have

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}(t-\tau)\phi(\tau)\,d\tau \right\|_{\mathbf{L}^\infty} \\
&\leq \int_0^{\frac{t}{2}} \|G(t-\tau)\|_{\mathbf{L}^\infty} \|\phi(\tau)\|_{\mathbf{L}^1}\,d\tau + \int_{\frac{t}{2}}^t \|G(t-\tau)\|_{\mathbf{L}^1} \|\phi(\tau)\|_{\mathbf{L}^\infty}\,d\tau \\
&\leq C\|\phi\|_{\mathbf{Y}} \left(\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}} \{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{-\frac{n\sigma}{2}}\,d\tau + \int_{\frac{t}{2}}^t \langle\tau\rangle^{-\frac{n}{2}(\sigma+1)}\,d\tau \right) \\
&\leq C\,\langle t\rangle^{-\frac{n}{2}} \|\phi\|_{\mathbf{Y}}
\end{aligned} \tag{1.28}$$

for all $t \geq 1$. Thus by virtue of (1.26), (1.27) and (1.28) we get

$$\left\| \int_0^t \mathcal{G}(t-\tau)\phi(\tau)\,d\tau \right\|_{\mathbf{X}} \leq C\|\phi(\tau)\|_{\mathbf{Y}};$$

hence in view of (1.24) estimate (1.21) follows. Therefore applying Theorem 1.17 we see that there exists a unique solution $u \in \mathbf{C}\left([0,\infty);\mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)\right) \cap \mathbf{C}((0,\infty);\mathbf{L}^\infty(\mathbf{R}^n))$ to the Cauchy problem (1.13). Moreover by (1.22) we have $\|u\|_{\mathbf{X}} \leq C\|u_0\|_{\mathbf{Z}}$ from which the optimal time decay estimate (1.23) follows. Theorem 1.19 is proved.

1.4 Global existence for large initial data

Now we define the space $\mathbf{X}[T_1,T_2] = \mathbf{C}([T_1,T_2];\mathbf{Z}) \cap \mathbf{C}((T_1,T_2];\mathbf{Z}_1)$ where $\mathbf{Z}$ and $\mathbf{Z}_1$ are some metric spaces of functions on $\mathbf{R}^n$. The norm

$$\|\psi\|_{\mathbf{X}[T_1,T_2]} = \sup_{t\in[T_1,T_2]} \|\psi(t)\|_{\mathbf{Z}} + \sup_{t\in(T_1,T_2]} \|\psi(t)\|_{\mathbf{Z}_1}.$$

Theorem 1.20. *Let the initial data* $u_0 \in \mathbf{Z}$. *Assume that* $\mathcal{G}(t-T_1) : \mathbf{Z} \to \mathbf{X}[T_1,T_2]$ *for any* $T_2 > T_1 \geq 0$, *and the estimate is valid*

$$\|\mathcal{G}(t-T_1)\phi\|_{\mathbf{X}[T_1,T_2]} \leq C\|\phi\|_{\mathbf{Z}}.$$

Also suppose that $\int_{T_1}^{t} \mathcal{G}(t-\tau)\mathcal{N}(v(\tau))\,d\tau \in \mathbf{X}[T_1,T_2]$ *for any* $v \in \mathbf{X}[T_1,T_2]$, $T_2 > T_1 \geq 0$, *and the estimate is true*

$$\left\| \int_{T_1}^{t} \mathcal{G}(t-\tau)\left(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau))\right) d\tau \right\|_{\mathbf{X}[T_1,T_2]}$$
$$\leq C\|w-v\|_{\mathbf{X}[T_1,T_2]}\left(1+\|w\|_{\mathbf{X}[T_1,T_2]}+\|v\|_{\mathbf{X}[T_1,T_2]}\right)^{\sigma},$$

for all $v, w \in \mathbf{X}[T_1,T_2]$, *where* $\sigma > 0$. *Let the following a priori estimate be valid*

$$||u||_{\mathbf{X}[0,T)} \leq C(T)\|u_0\|_{\mathbf{Z}}, \tag{1.29}$$

provided that there exists a solution $u \in \mathbf{X}[0,T)$ *for some* $T > 0$. *Then there exists a unique global solution* $u \in \mathbf{X}[0,\infty)$ *to the Cauchy problem (1.7).*

Proof. Using a priori estimates (1.29) we can prolong the local solution given by Theorem 1.9 for all times $t > 0$. Indeed, on the contrary we can suppose that there exists a maximal existence time $T > 0$ such that $u \in \mathbf{X}[0,T)$. If we choose a new initial time $T_1 \in [0,T)$ and consider the Cauchy problem (1.7) with initial data $u(T_1)$, then via a priori estimate (1.29) the norm $||u(T_1)||_{\mathbf{Z}}$ is bounded uniformly with respect to $T_1 \in [0,T)$. Then the existence time given by the local existence Theorem 1.9 is bounded from below uniformly with respect to $T_1 \in [0,T)$. Therefore if a new initial time $T_1 > 0$ is chosen to be sufficiently close to the maximal time T, then by virtue of the local existence Theorem 1.9 we can guarantee that there exists a unique solution $u \in \mathbf{X}[0,T]$. Now putting $u(T)$ as a new initial data at time T we can apply the local existence Theorem 1.9 and prolong the solution $u(t)$ on some bigger time interval $[0,T+T_2]$. This contradicts the fact that T is a maximal existence time. Hence there exists a unique solution $u \in \mathbf{X}[0,\infty)$ to the Cauchy problem (1.7). Theorem 1.20 is proved.

Example 1.21. **Global positive solutions to nonlinear heat equation with large initial data**

We consider problem (1.13) in the case $\lambda < 0$. Using the positivity of the heat kernel we can obtain a priori estimate (1.29) for positive solutions and hence remove the smallness condition on the initial data $u_0(x)$ in Theorem 1.19.

Theorem 1.22. *Let* $\lambda < 0$. *Let the initial data* $u_0 \in \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, *with* $a \in (0,1]$ *and* $p > \max\left(1, \frac{n}{2}\sigma\right)$. *Also we assume that* $u_0(x) \geq 0$ *almost everywhere on* $x \in \mathbf{R}^n$. *Then there exists a unique solution* $u \in \mathbf{C}\left([0,\infty); \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)\right) \cap \mathbf{C}\left((0,\infty); \mathbf{L}^\infty(\mathbf{R}^n)\right)$ *to the Cauchy problem (1.13). Moreover estimate (1.23) is true.*

Remark 1.23. Note that time decay estimate (1.23) is optimal in the supercritical case $\sigma > \frac{2}{n}$ as we will see below considering the large time asymptotics of solutions. However, in the critical $\sigma = \frac{2}{n}$ and subcritical $\sigma < \frac{2}{n}$ cases estimate (1.23) does not describe an optimal time decay of solutions.

Proof. By the maximum principle we can see that $u(t,x) \geq 0$ for all $t > 0$. Since the heat kernel is positive $G(t,x) > 0$ we can observe from the integral equation (1.8) that

$$\begin{aligned} 0 \leq u(t,x) &= \int_{\mathbf{R}^n} G(t, x-y)\, u_0(y) dy \\ &+ \lambda \int_0^t d\tau \int_{\mathbf{R}^n} G(t-\tau, x-y) \left| u(\tau, y) \right|^{\sigma} u(\tau, y)\, dy \\ &\leq \int_{\mathbf{R}^n} G(t, x-y)\, u_0(y) dy \end{aligned} \tag{1.30}$$

for all $t > 0$. By (1.30) applying the Young inequality (1.2) we obtain

$$\|u(t)\|_{\mathbf{L}^p} \leq \|\mathcal{G}(t)\, u_0\|_{\mathbf{L}^p} \leq \|u_0\|_{\mathbf{L}^p} \|G(t)\|_{\mathbf{L}^1} \leq C \|u_0\|_{\mathbf{L}^p}$$

for all $t \geq 0$, and

$$\|u(t)\|_{\mathbf{L}^\infty} \leq \|\mathcal{G}(t)\, u_0\|_{\mathbf{L}^\infty} \leq \|u_0\|_{\mathbf{L}^p} \|G(t)\|_{\mathbf{L}^{\frac{p}{p-1}}} \leq C t^{-\frac{n}{2p}} \|u_0\|_{\mathbf{L}^p}$$

for all $t > 0$. In the same manner we can estimate the $\mathbf{L}^{1,a}$ norm of the solution

$$\|u(t)\|_{\mathbf{L}^{1,a}} \leq \|\mathcal{G}(t)\, u_0\|_{\mathbf{L}^{1,a}} \leq C \langle t \rangle^{\frac{a}{2}} \|u_0\|_{\mathbf{L}^{1,a}}$$

for all $t > 0$. To apply Theorem 1.20 we define $\mathbf{Z} = \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, $\mathbf{Z}_1 = \mathbf{L}^\infty(\mathbf{R}^n)$ and the space $\mathbf{X}[T_1,T_2] = \mathbf{C}([T_1,T_2];\mathbf{Z}) \cap \mathbf{C}((T_1,T_2];\mathbf{L}^\infty(\mathbf{R}^n))$ with the norm

$$\begin{aligned} \|u\|_{\mathbf{X}[T_1,T_2]} &= \sup_{t \in [T_1,T_2]} \left(\|u(t)\|_{\mathbf{L}^p} + \|u(t)\|_{\mathbf{L}^{1,a}} \right) \\ &+ \sup_{t \in (T_1,T_2]} (t - T_1)^{\frac{n}{2p}} \|u(t)\|_{\mathbf{L}^\infty} \end{aligned}$$

for any $T_2 > T_1 \geq 0$. Hence a priori estimate (1.29) follows and by Theorem 1.20 we see that there exists a unique solution

$$u \in \mathbf{C}\left([0,\infty);\mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)\right) \cap \mathbf{C}((0,\infty);\mathbf{L}^\infty(\mathbf{R}^n))$$

to the Cauchy problem (1.13). By virtue of the above estimates we also can write

$$\|u\|_{\mathbf{X}} \leq \|\mathcal{G} u_0\|_{\mathbf{X}},$$

where

$$\begin{aligned} \|\phi\|_{\mathbf{X}} &= \sup_{t \geq 0} \left(\langle t \rangle^{-\frac{a}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} + \langle t \rangle^{\frac{n}{2}\left(1 - \frac{1}{p}\right)} \|\phi(t)\|_{\mathbf{L}^p} \right) \\ &+ \sup_{t > 0} \{t\}^{\frac{n}{2p}} \langle t \rangle^{\frac{n}{2}} \|\phi(t)\|_{\mathbf{L}^\infty}. \end{aligned}$$

Hence time decay estimate (1.23) follows. Theorem 1.22 is proved.

Example 1.24. **Global solutions to the nonlinear heat equation with large initial data of any sign**

Consider the Cauchy problem for the nonlinear heat equation (1.13) with $\lambda < 0$ in the case $\sigma > \frac{2}{n}$. In the following theorem we remove the requirement of the positivity of the initial data and the solution.

Theorem 1.25. *Let $\sigma > \frac{2}{n}$, $\lambda < 0$. Let the initial data $u_0 \in \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, with $a \in (0,1]$ and $p > 2k > \frac{n}{2}(\sigma+1)$ with some integer k. Then there exists a unique solution $u \in \mathbf{C}\left([0,\infty);\mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)\right) \cap \mathbf{C}((0,\infty);\mathbf{L}^\infty(\mathbf{R}^n))$ to the Cauchy problem (1.13). Moreover the optimal time decay estimate (1.23) is true.*

Proof. First let us estimate the $\mathbf{L}^1$ norm of the solution. Denote $S(t,x) = 1$ for all $u(t,x) > 0$ and $S(t,x) = -1$ for all $u(t,x) < 0$; $S(t,x) = 0$ for $u(t,x) = 0$. We multiply equation (1.13) by $S(t,x)$ and integrate with respect to x over $\mathbf{R}^n$ to get

$$\int_{\mathbf{R}^n} u_t(t,x) S(t,x)\,dx = \lambda \int_{\mathbf{R}^n} |u(t,x)|^{\sigma+1}\,dx + \int_{\mathbf{R}^n} \Delta u(t,x) S(t,x)\,dx.$$

(To justify our calculations we note that by the smoothing effect, see Naumkin and Shishmarev [1994b] for the nonlinear heat equation (1.13) the solutions $u \in \mathbf{C}((0,\infty);\mathbf{C}^\infty(\mathbf{R}^n)) \cap \mathbf{C}^1((0,\infty)\times\mathbf{R}^n)$.) Applying the estimates

$$\begin{aligned} &\int_{\mathbf{R}^n} u_t(t,x) S(t,x)\,dx = \frac{d}{dt}\|u(t)\|_{\mathbf{L}^1}, \\ &\lambda \int_{\mathbf{R}^n} |u(t,x)|^{\sigma+1}\,dx \le 0, \\ &\int_{\mathbf{R}^n} \Delta u(t,x) S(t,x)\,dx = -2 \sum_{x_k : u(t,x_k)=0} |\nabla u(t,x_k)| \le 0, \end{aligned} \tag{1.31}$$

we get

$$\frac{d}{dt}\|u(t)\|_{\mathbf{L}^1} \le 0.$$

Integration of the last inequality yields

$$\|u(t)\|_{\mathbf{L}^1} \le \|u_0\|_{\mathbf{L}^1}. \tag{1.32}$$

We now multiply equation (1.13) by $2u$, then integrating with respect to $x \in \mathbf{R}^n$ we get

$$\frac{d}{dt}\|u(t)\|_{\mathbf{L}^2}^2 = -2\|\nabla u(t)\|_{\mathbf{L}^2}^2 + \lambda \|u(t)\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2}. \tag{1.33}$$

By the Plancherel theorem using the Fourier splitting method due to Schonbek [1991], we have

$$\begin{aligned}\|\nabla u(t)\|_{\mathbf{L}^2}^2 = \|\xi \widehat{u}(t)\|_{\mathbf{L}^2}^2 &= \int_{|\xi|\le\delta} |\widehat{u}(t,\xi)|^2 |\xi|^2 \, d\xi + \int_{|\xi|\ge\delta} |\widehat{u}(t,\xi)|^2 |\xi|^2 \, d\xi \\ &\ge \delta^2 \|u(t)\|_{\mathbf{L}^2}^2 - C\delta^{2+n} \sup_{|\xi|\le\delta} |\widehat{u}(t,\xi)|^2 ,\end{aligned}$$

where $\delta > 0$. Thus from (1.33) we have the inequality

$$\frac{d}{dt} \|u(t)\|_{\mathbf{L}^2}^2 \le -2\delta^2 \|u(t)\|_{\mathbf{L}^2}^2 + C\delta^{2+n} \sup_{|\xi|\le\delta} |\widehat{u}(t,\xi)|^2 . \tag{1.34}$$

We choose $2\delta^2 = (1+n)(1+t)^{-1}$ and change $\|u(t)\|_{\mathbf{L}^2}^2 = (1+t)^{-1-n} W(t)$. Then in view of inequality $\sup_{\xi\in\mathbf{R}^n} |\widehat{u}(t,\xi)| \le (2\pi)^{-\frac{n}{2}} \|u(t)\|_{\mathbf{L}^1}$, via (1.32) we get from (1.34)

$$\frac{d}{dt} W(t) \le C(1+t)^{\frac{n}{2}} . \tag{1.35}$$

Integration of (1.35) with respect to time yields

$$W(t) \le \|u_0\|_{\mathbf{L}^2}^2 + C\left((1+t)^{\frac{n}{2}+1} - 1\right).$$

Therefore we obtain an optimal time decay estimate of the $\mathbf{L}^2$ norm

$$\|u(t)\|_{\mathbf{L}^2} \le C(1+t)^{-\frac{n}{4}} \tag{1.36}$$

for all $t > 0$. We now multiply equation (1.13) by $4u^3$, then integrating with respect to $x \in \mathbf{R}^n$ we get

$$\frac{d}{dt} \|u(t)\|_{\mathbf{L}^4}^4 = -3 \left\|\nabla\left(u^2(t)\right)\right\|_{\mathbf{L}^2}^2 + 4\lambda \|u(t)\|_{\mathbf{L}^{\sigma+4}}^{\sigma+4} . \tag{1.37}$$

As above, we have in view of (1.36)

$$\begin{aligned}&\left\|\nabla\left(u^2(t)\right)\right\|_{\mathbf{L}^2}^2 = \left\|\xi \widehat{u^2}(t)\right\|_{\mathbf{L}^2}^2 = \int_{|\xi|\le\delta} \left|\widehat{u^2}(t,\xi)\right|^2 |\xi|^2 \, d\xi \\ &+ \int_{|\xi|\ge\delta} \left|\widehat{u^2}(t,\xi)\right|^2 |\xi|^2 \, d\xi \ge \delta^2 \|u(t)\|_{\mathbf{L}^4}^4 - C\delta^{2+n} \sup_{|\xi|\le\delta} \left|\widehat{u^2}(t,\xi)\right|^2 \\ &\ge \delta^2 \|u(t)\|_{\mathbf{L}^4}^4 - C\delta^{2+n} \|u(t)\|_{\mathbf{L}^2}^4 ,\end{aligned}$$

where $\delta > 0$. Thus from (1.37) we have the inequality

$$\frac{d}{dt} \|u(t)\|_{\mathbf{L}^4}^4 \le -3\delta^2 \|u(t)\|_{\mathbf{L}^4}^4 + C\delta^{2+n} (1+t)^{-n} . \tag{1.38}$$

We choose $3\delta^2 = (1+2n)(1+t)^{-1}$ and change

$$\|u(t)\|_{\mathbf{L}^4}^4 = (1+t)^{-1-2n} W_1(t).$$

Then we get from (1.38)

$$\frac{d}{dt} W_1(t) \le C(1+t)^{\frac{n}{2}}. \tag{1.39}$$

Integration of (1.39) with respect to time yields

$$W_1(t) \le \|u_0\|_{\mathbf{L}^4}^4 + C\left((1+t)^{\frac{n}{2}+1} - 1\right).$$

Therefore we obtain an optimal time decay estimate of the $\mathbf{L}^4$ norm

$$\|u(t)\|_{\mathbf{L}^4} \le C(1+t)^{-\frac{3n}{8}} \tag{1.40}$$

for all $t > 0$.

Arguing in this way by the Hölder inequality we will have the following optimal time decay estimates

$$\|u(t)\|_{\mathbf{L}^q} \le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{q}\right)} \tag{1.41}$$

for all $t > 0$, where $1 \le q \le 2k$, with any integer k such that $p > 2k > \frac{n}{2}(\sigma+1)$.

Let us prove estimate (1.41) with $q = \infty$. To get an optimal time decay estimate for the $\mathbf{L}^\infty$ norm we use the integral equation (1.8) taking $r = \infty$ for $n = 1$ and $1 < r < \frac{n}{n-2}$ for $n \ge 2$ so that the inequalities $\frac{n}{2}\left(1 - \frac{1}{r}\right) < 1$ and $\frac{n}{2}(\sigma+1) < (\sigma+1)\frac{r}{r-1} < 2k < p$ are true

$$\begin{aligned}
\|u(t)\|_{\mathbf{L}^\infty} &\le \|\mathcal{G}(t)u_0\|_{\mathbf{L}^\infty} + |\lambda| \int_{\frac{t}{2}}^{t} \|G(t-\tau)\|_{\mathbf{L}^r} \left\||u|^\sigma u(\tau)\right\|_{\mathbf{L}^{\frac{r}{r-1}}} d\tau \\
&+ |\lambda| \int_0^{\frac{t}{2}} \|G(t-\tau)\|_{\mathbf{L}^\infty} \left\||u|^\sigma u(\tau)\right\|_{\mathbf{L}^1} d\tau \\
&\le Ct^{-\frac{n}{2}} + C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{n}{2}\left(1-\frac{1}{r}\right)} \langle\tau\rangle^{\frac{n}{2}\left(1-\frac{1}{r}\right)-\frac{n}{2}(\sigma+1)} d\tau \\
&+ C\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}} \langle\tau\rangle^{-\frac{n}{2}(\sigma+1)} d\tau \le Ct^{-\frac{n}{2}}
\end{aligned}$$

for all $t > 0$. Now estimate

$$\|u(t)\|_{\mathbf{L}^q} \le Ct^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}$$

for all $t > 0$ and $1 \le q \le \infty$ follows via the Hölder inequality.

To estimate the $\mathbf{L}^{1,a}(\mathbf{R}^n)$ norm we multiply equation (1.13) by

$$\left(\sqrt{t} + |x|\right)^a S(t,x)$$

and integrate with respect to x over $\mathbf{R}^n$ to get

$$\int_{\mathbf{R}^n} u_t(t,x)\, S(t,x) \left(\sqrt{t}+|x|\right)^a dx = \lambda \int_{\mathbf{R}^n} |u(t,x)|^{\sigma+1} \left(\sqrt{t}+|x|\right)^a dx$$
$$+ \int_{\mathbf{R}^n} \Delta u(t,x)\, S(t,x) \left(\sqrt{t}+|x|\right)^a dx.$$

We have

$$\int_{\mathbf{R}^n} u_t(t,x)\, S(t,x) \left(\sqrt{t}+|x|\right)^a dx$$
$$= \frac{d}{dt} \left\| \left(\sqrt{t}+|x|\right)^a u(t) \right\|_{\mathbf{L}^1} - \frac{a}{2} t^{-\frac{1}{2}} \left\| \left(\sqrt{t}+|x|\right)^{a-1} u(t) \right\|_{\mathbf{L}^1}$$

and

$$\lambda \int_{\mathbf{R}^n} |u(t,x)|^{\sigma+1} \left(\sqrt{t}+|x|\right)^a dx \leq 0.$$

Integrating by parts and taking into account the estimate

$$\|u(t)\|_{\mathbf{L}^1} \leq C$$

we get

$$\int_{\mathbf{R}^n} \Delta u(t,x)\, S(t,x) \left(\sqrt{t}+|x|\right)^a dx$$
$$= -2 \sum_{x_k : u(t,x_k)=0} |\nabla u(t,x_k)| \left(\sqrt{t}+|x_k|\right)^a$$
$$- a \int_{\mathbf{R}^n} S(t,x) \left(\sqrt{t}+|x|\right)^{a-1} \left(\frac{x}{|x|}\cdot\nabla\right) u(t,x)\, dx$$
$$\leq C \int_{\mathbf{R}^n} |u(t,x)| \left(\sqrt{t}+|x|\right)^{a-2} dx \leq C t^{\frac{a}{2}-1} \|u(t)\|_{\mathbf{L}^1} \leq C t^{\frac{a}{2}-1}.$$

Therefore

$$\frac{d}{dt} \left\| \left(\sqrt{t}+|x|\right)^a u(t) \right\|_{\mathbf{L}^1} \leq C t^{\frac{a}{2}-1},$$

integration with respect to time yields

$$\left\| |x|^a u(t) \right\|_{\mathbf{L}^1} \leq \left\| \left(\sqrt{t}+|x|\right)^a u(t) \right\|_{\mathbf{L}^1} \leq \left\| |x|^a u_0 \right\|_{\mathbf{L}^1} + C t^{\frac{a}{2}}$$

for all $t > 0$. Thus we get the estimate

$$\|u(t)\|_{\mathbf{L}^{1,a}} \leq C \langle t \rangle^{\frac{a}{2}}$$

for all $t > 0$. Therefore again we arrive to the a priori estimate (1.29) if we define the spaces $\mathbf{Z} = \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, $\mathbf{Z}_1 = \mathbf{L}^\infty(\mathbf{R}^n)$ and the space

$$\mathbf{X}[T_1,T_2] = \mathbf{C}([T_1,T_2];\mathbf{Z}) \cap \mathbf{C}((T_1,T_2];\mathbf{L}^\infty(\mathbf{R}^n))$$

with the norm

$$\|u\|_{\mathbf{X}[T_1,T_2]} = \sup_{t\in[T_1,T_2]} \left(\|u(t)\|_{\mathbf{L}^p} + \|u(t)\|_{\mathbf{L}^{1,a}}\right) + \sup_{t\in(T_1,T_2]} (t-T_1)^{\frac{n}{2p}} \|u(t)\|_{\mathbf{L}^\infty}$$

for any $T_2 > T_1 \geq 0$. Then by Theorem 1.20 we see that there exists a unique solution $u \in \mathbf{C}\left([0,\infty);\mathbf{L}^p(\mathbf{R}^n)\cap\mathbf{L}^{1,a}(\mathbf{R}^n)\right) \cap \mathbf{C}\left((0,\infty);\mathbf{L}^\infty(\mathbf{R}^n)\right)$ to the Cauchy problem (1.13), which satisfies the optimal time decay estimates of Theorem 1.19. Theorem 1.25 is proved.

Example 1.26. **Global existence of solutions to the Burgers type equations**

Due to the special form of the nonlinearity we can prove global existence for the Cauchy problem to the Burgers type equation (1.18) without any restriction on the size of the initial data.

Theorem 1.27. *Let $\sigma > \frac{1}{n}$. Let the initial data $u_0 \in \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, with $a \in (0,1]$ and $p > 2k > n(\sigma+1)$ for some integer k. Then there exists a unique solution $u \in \mathbf{C}\left([0,\infty);\mathbf{L}^{1,a}(\mathbf{R}^n)\cap\mathbf{L}^p(\mathbf{R}^n)\right) \cap \mathbf{C}\left((0,\infty);\mathbf{W}^1_\infty(\mathbf{R}^n)\right)$ to the Cauchy problem for the Burgers type equation (1.18). Moreover the optimal time decay estimate (1.23) is true.*

Proof. Since the nonlinear term $\mathcal{N}(u) = (\lambda\cdot\nabla)|u|^\sigma u$ has the form of the full derivative we have

$$\int_{\mathbf{R}^n} u^{2k+1}(\lambda\cdot\nabla)|u|^\sigma u dx = 0;$$

therefore applying the energy method as in the proof of Theorem 1.25 we obtain the optimal time decay estimate (1.41). Taking $1 < r < \frac{n}{n+1}$, so that the inequalities $\frac{1}{2} + \frac{n}{2}\left(1-\frac{1}{r}\right) < 1$ and $n(\sigma+1) < (\sigma+1)\frac{r}{r-1} < 2k < p$ are true, then using the integral equation (1.8) we find

$$\begin{aligned}\|u(t)\|_{\mathbf{L}^\infty} &\leq \|\mathcal{G}(t)u_0\|_{\mathbf{L}^\infty} + |\lambda|\int_{\frac{t}{2}}^{t} \|\nabla G(t-\tau)\|_{\mathbf{L}^r} \left\||u|^\sigma u(\tau)\right\|_{\mathbf{L}^{\frac{r}{r-1}}} d\tau \\ &+ |\lambda|\int_0^{\frac{t}{2}} \|\nabla G(t-\tau)\|_{\mathbf{L}^\infty} \left\||u|^\sigma u(\tau)\right\|_{\mathbf{L}^1} d\tau \\ &\leq Ct^{-\frac{n}{2}} + C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{r}\right)} \langle\tau\rangle^{-\frac{n}{2r}-\frac{n}{2}\sigma} d\tau \\ &+ C\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{2}-\frac{n}{2}} \langle\tau\rangle^{-\frac{n}{2}\sigma} d\tau \leq Ct^{-\frac{n}{2}}\end{aligned}$$

for all $t \geq 1$. Taking into account the estimates of the local existence Theorem 1.16 we get

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\{t\}^{-\frac{n}{2p}} \langle t\rangle^{-\frac{n}{2}}$$

for all $t > 0$. Now using the integral equation (1.8) we estimate the $\mathbf{L}^{1,a}$ norm

$$\begin{aligned}\|u(t)\|_{\mathbf{L}^{1,a}} &\le \|u_0\|_{\mathbf{L}^1} \|G(t)\|_{\mathbf{L}^{1,a}} + \|u_0\|_{\mathbf{L}^{1,a}} \|G(t)\|_{\mathbf{L}^1} \\ &+ C\int_0^t \left\||u(\tau)|^{\sigma+1}\right\|_{\mathbf{L}^1} \|\nabla G(t-\tau)\|_{\mathbf{L}^{1,a}} d\tau \\ &+ C\int_0^t \left\||u(\tau)|^{\sigma+1}\right\|_{\mathbf{L}^{1,a}} \|\nabla G(t-\tau)\|_{\mathbf{L}^1} d\tau;\end{aligned}$$

hence

$$\begin{aligned}\|u(t)\|_{\mathbf{L}^{1,a}} &\le C\langle t\rangle^{\frac{a}{2}} + C\int_0^t \|u(\tau)\|_{\mathbf{L}^\infty}^\sigma \|u(\tau)\|_{\mathbf{L}^1} \langle t-\tau\rangle^{\frac{a}{2}} (t-\tau)^{-\frac{1}{2}} d\tau \\ &+ C\int_0^t \|u(\tau)\|_{\mathbf{L}^\infty}^\sigma \|u(\tau)\|_{\mathbf{L}^{1,a}} (t-\tau)^{-\frac{1}{2}} d\tau \\ &\le C\langle t\rangle^{\frac{a}{2}} + C\int_0^t \{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{-\frac{n}{2}\sigma} \|u(\tau)\|_{\mathbf{L}^{1,a}} (t-\tau)^{-\frac{1}{2}} d\tau.\end{aligned}$$

Thus for the function $h(t) = \sup_{0\le\tau\le t} \|u(\tau)\|_{\mathbf{L}^{1,a}}$ we get the inequality

$$\begin{aligned}h(t) &\le C\langle t\rangle^{\frac{a}{2}} + Ch(t)\int_{t-\varepsilon t}^t \{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{-\frac{n}{2}\sigma} (t-\tau)^{-\frac{1}{2}} d\tau \\ &+ C\int_0^{t-\varepsilon t} \{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{-\frac{n}{2}\sigma} h(\tau)(t-\tau)^{-\frac{1}{2}} d\tau \\ &\le C\langle t\rangle^{\frac{a}{2}} + C\sqrt{\varepsilon} h(t) + \frac{C}{\sqrt{\varepsilon t}} \int_0^{t-\varepsilon t} \{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{-\frac{n}{2}\sigma} h(\tau) d\tau.\end{aligned}$$

Hence denoting $\phi(t) = h(t)\langle t\rangle^{-\frac{a}{2}}$ we find

$$\phi(t) \le C + C\langle t\rangle^{-\frac{a}{2}} \int_0^t \{\tau\}^{-\frac{n\sigma}{2p}-\frac{1}{2}} \langle\tau\rangle^{-\frac{n}{2}\sigma-\frac{1}{2}+\frac{a}{2}} \phi(\tau) d\tau,$$

where $\frac{n\sigma}{2p} + \frac{1}{2} < 1$ and $\frac{n}{2}\sigma - \frac{1}{2} > 1$. Thus applying the Gronwall Lemma 1.8 we obtain

$$\|u(t)\|_{\mathbf{L}^{1,a}} \le C\langle t\rangle^{\frac{a}{2}}$$

for all $t \ge 0$. Therefore we again arrive at the a priori estimate (1.29), if we define the spaces $\mathbf{Z} = \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, $\mathbf{Z}_1 = \mathbf{W}_\infty^1(\mathbf{R}^n)$ and the space $\mathbf{X}[T_1,T_2] = \mathbf{C}([T_1,T_2];\mathbf{Z}) \cap \mathbf{C}((T_1,T_2];\mathbf{W}_\infty^1(\mathbf{R}^n))$ with the norm

$$\begin{aligned}\|u\|_{\mathbf{X}[T_1,T_2]} &= \sup_{t\in[T_1,T_2]} (\|u(t)\|_{\mathbf{L}^p} + \|u(t)\|_{\mathbf{L}^{1,a}}) \\ &+ \sup_{t\in(T_1,T_2]} \left((t-T_1)^{\frac{n}{2p}} \|u(t)\|_{\mathbf{L}^\infty} + (t-T_1)^{\frac{n}{2p}+\frac{1}{2}} \|\nabla u(t)\|_{\mathbf{L}^\infty}\right)\end{aligned}$$

for any $T_2 > T_1 \geq 0$. Then by Theorem 1.20 we see that there exists a unique solution $u \in \mathbf{C}\left([0,\infty);\mathbf{L}^p(\mathbf{R}^n)\cap\mathbf{L}^{1,a}(\mathbf{R}^n)\right)\cap\mathbf{C}\left((0,\infty);\mathbf{W}^1_\infty(\mathbf{R}^n)\right)$ to the Cauchy problem (1.13), which satisfies the optimal time decay estimate (1.23). Theorem 1.27 is proved.

1.5 Some estimates for linear semigroups

Consider the linear Cauchy problem

$$\begin{cases} u_t + \mathcal{L}u = f, \ x \in \mathbf{R}^n, t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^n, \end{cases} \tag{1.42}$$

where $\mathcal{L}$ is a linear pseudodifferential operator defined by the symbol $L(\xi)$ via the inverse Fourier transformation

$$\mathcal{L}u = \overline{\mathcal{F}}_{\xi\to x}\left(L(\xi)\,\widehat{u}(\xi)\right).$$

By virtue of the Duhamel formula the solution of problem (1.42) can be written in the form

$$u(t) = \mathcal{G}(t)\,u_0 + \int_0^t \mathcal{G}(t-\tau)\,f(\tau)\,d\tau,$$

where the Green operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t)\,\phi = \overline{\mathcal{F}}_{\xi\to x}e^{-tL(\xi)}\hat{\phi}(\xi) = \int_{\mathbf{R}^n} G(t,x-y)\,\phi(y)dy$$

with a kernel $G(t,x) = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-tL(\xi)}\right)$.

First we obtain estimates in the weighted Lebesgue norms, which are the most suitable for the nonlinear theory. After that we find the estimates in the frames of the $\mathbf{L}^2$ - theory. Finally we obtain the estimates in the Fourier spaces.

1.5.1 Estimates in weighted Lebesgue spaces

Suppose that the linear operator $\mathcal{L}$ satisfies the dissipation condition which in terms of the symbol $L(\xi)$ has the form

$$\operatorname{Re} L(\xi) \geq \alpha\,\{\xi\}^{\delta}\,\langle\xi\rangle^{\nu} \tag{1.43}$$

for all $\xi \in \mathbf{R}^n$, where $\alpha > 0$, $\delta > 0$, $\nu \geq 0$. Also we suppose that the symbol is sufficiently smooth except the origin: $L(\xi) \in \mathbf{C}^N(\mathbf{R}^n \setminus \{0\})$ and obeys the estimate

$$\left|\partial^l_{\xi_j} L(\xi)\right| \leq C\,\{\xi\}^{\delta-l}\,\langle\xi\rangle^{\nu-l} \tag{1.44}$$

for all $\xi \in \mathbf{R}^n \setminus \{0\}$, $l = 0, 1, ..., N$, $j = 1, 2, ..., n$, with some $N \geq n + 2$.

We denote the fractional partial derivative $\partial_{x_j}^{\beta}$ for $\beta \geq 0$, $j = 1, 2, ..., n$, as follows

$$\partial_{x_j}^{\beta} \phi(x) = \mathcal{F}_{\xi_j \to x_j}^{-1} \left((i\xi_j)^{\varrho - 1} (i\xi_j)^{k+1} \widehat{\phi}(\xi) \right),$$

where $k = [\beta]$, $\varrho = \beta - k \in (0, 1)$ and $(i\xi_j)^{\varrho - 1}$ is the main value of the complex analytic function: $(i\xi_j)^{\varrho-1} = |\xi_j|^{\varrho - 1} \exp(i(\varrho - 1)\arg(i\xi_j))$ (see Stein [1970]). We have by Erdélyi et al. [1954]

$$\mathcal{F}_{\xi_j \to y}^{-1} \left((i\xi_j)^{\varrho - 1} \right) = \begin{cases} \frac{\pi}{\sqrt{2\pi}\Gamma(1-\varrho)\cos\left(\frac{\pi}{2}(\varrho - 1)\right)} y^{-\varrho} & \text{for } y > 0 \\ 0 & \text{for } y \leq 0 \end{cases}$$

if $\varrho \in (0, 1)$, where Γ is the Euler gamma function. Then integration by parts yields

$$\begin{aligned} \partial_{x_j}^{\beta} \phi(x) &= \frac{2\pi}{\Gamma(1-\varrho)} \int_0^{\infty} \partial_{x_j}^{k+1} \phi(\breve{x}(y)) y^{-\varrho} dy \\ &= \frac{2\pi}{\Gamma(1-\varrho)} \int_0^{\infty} \left(\partial_{x_j}^{k} \phi(\breve{x}(y)) - \partial_{x_j}^{k} \phi(x) \right) y^{-1-\varrho} dy, \end{aligned}$$

where $x = (x_1, ..., x_n)$, $\breve{x}(y) = (x_1, ..., x_j + y, ..., x_n)$.

First we collect some preliminary estimates of the Green operator $\mathcal{G}(t)$ in the weighted Lebesgue norms $\|\phi\|_{\mathbf{L}^{p,a}}$. Denote $\widetilde{t} = \{t\}^{-\frac{1}{\nu}} \langle t \rangle^{-\frac{1}{\delta}}$.

Lemma 1.28. *Let the symbol $L(\xi)$ satisfy (1.43) and (1.44) with $\nu > 0$. Then the estimates are true*

$$\left\| |\cdot|^b \partial_{x_j}^{\beta} \mathcal{G}(t) \phi \right\|_{\mathbf{L}^p} \leq C\widetilde{t}^{n\left(\frac{1}{r} - \frac{1}{p}\right) + \beta - b} \|\phi\|_{\mathbf{L}^r} + C\widetilde{t}^{n\left(\frac{1}{r} - \frac{1}{p}\right) + \beta} \|\phi\|_{\mathbf{L}^{r,b}}$$

$$\left\| |\cdot|^b \left(\mathcal{G}(t)\phi - \vartheta G(t) \right) \right\|_{\mathbf{L}^p} \leq C\widetilde{t}^{n\left(1 - \frac{1}{p}\right) + a - b} \left\| |\cdot|^a \phi \right\|_{\mathbf{L}^1}$$

for all $t > 0$, where $\vartheta = \int_{\mathbf{R}^n} \phi(x)\, dx$, $1 \leq r \leq p \leq \infty$, $0 \leq \beta \leq N - n - 2$, $0 \leq b \leq a \leq 1$, provided that the right-hand sides are finite.

Remark 1.29. Conditions of Lemma 1.28 are fulfilled for the following examples of linear equations: 1) the heat equation $\mathcal{L}u = -\Delta u$, $L(\xi) = |\xi|^2$; 2) the fractional heat equation $\mathcal{L}u = (-\Delta)^{\alpha} u$, $L(\xi) = |\xi|^{\alpha}$; 3) the linearized Korteweg-de Vries-Burgers equation $\mathcal{L}u = -u_{xx} + u_{xxx}$, $L(\xi) = \xi^2 + i\xi^3$; and 4) the Landau-Ginzburg equation $\mathcal{L}u = -\alpha\Delta u$, $L(\xi) = \alpha|\xi|^2$, $\alpha \in \mathbf{C}$, $\operatorname{Re}\alpha > 0$.

Proof. Note that the kernel $G(t, x) = \overline{\mathcal{F}}_{\xi \to x}\left(e^{-tL(\xi)}\right)$ is a smooth function $G(t, x) \in \mathbf{C}^{\infty}(\mathbf{R}^+ \times \mathbf{R}^n)$. First let us prove the estimate

$$\sup_{x \in \mathbf{R}^n} \langle x\widetilde{t} \rangle^{\mu + n + k} \left| \partial_{x_j}^{k} G(t, x) \right| \leq C\widetilde{t}^{k+n}, \tag{1.45}$$

for all $t > 0$, $1 \le j \le n$, where $0 \le k \le N - n - 1$, $0 < \mu < \min(1, \delta, \nu)$.

Indeed we have

$$\left|\partial_{x_j}^k G(t,x)\right| \le C \int_{|\xi|\le 1} |\xi|^k e^{-\alpha t|\xi|^\delta} d\xi + C \int_{|\xi|>1} |\xi|^k e^{-\alpha t |\xi|^\nu} d\xi$$
$$\le C \langle t\rangle^{-\frac{k+n}{\delta}} + C\{t\}^{-\frac{k+n}{\nu}} e^{-\alpha t} \le C\widetilde{t}^{k+n}.$$

On the other hand

$$\left|x_l^{\mu+n+k} \partial_{x_j}^k G(t,x)\right| = \left|\overline{\mathcal{F}}_{\xi\to x}\left(\partial_{\xi_l}^{\mu+n+k}\left(\xi_j^k e^{-tL(\xi)}\right)\right)\right|$$
$$\le C\left\|\partial_{\xi_l}^{\mu+n+k}\left(\xi_j^k e^{-tL(\xi)}\right)\right\|_{\mathbf{L}^1_\xi}.$$

We have denoting $\psi(t,\xi) = \partial_{\xi_l}^{n+k}\left(\xi_j^k e^{-tL(\xi)}\right)$

$$\partial_{\xi_l}^{\mu+n+k}\left(\xi_j^k e^{-tL(\xi)}\right) = C\int_0^\infty \left(\psi\left(t,\breve{\xi}(y)\right) - \psi(t,\xi)\right) y^{-1-\mu} dy = I_1 + I_2,$$

where $\breve{\xi}(y) = (\xi_1, ..., \xi_l + y, ..., \xi_n)$ and

$$I_1 = \int_{\frac{|\xi|}{2}}^\infty \left(\psi\left(t,\breve{\xi}(y)\right) - \psi(t,\xi)\right) y^{-1-\mu} dy,$$
$$I_2 = \int_0^{\frac{|\xi|}{2}} \left(\psi\left(t,\breve{\xi}(y)\right) - \psi(t,\xi)\right) y^{-1-\mu} dy.$$

Since

$$|\psi(t,\xi)| \le Ct\{\xi\}^{\delta-n}\langle\xi\rangle^{\nu-n} e^{-\frac{\alpha}{2}t\{\xi\}^\delta\langle\xi\rangle^\nu}$$

for all $\xi \in \mathbf{R}^n$, $t > 0$ we estimate the first summand I_1 by the Young inequality

$$\|I_1\|_{\mathbf{L}^1_\xi} \le C\int_0^\infty \frac{dy}{y^{1+\mu}} \int_{|\xi|\le 2|y|} \left(t\{\xi - y\}^{\delta-n}\langle\xi - y\rangle^{\nu-n} e^{-\frac{\alpha}{2}t\{\xi-y\}^\delta\langle\xi-y\rangle^\nu}\right.$$
$$\left.+t\{\xi\}^{\delta-n}\langle\xi\rangle^{\nu-n} e^{-\frac{\alpha}{2}t\{\xi\}^\delta\langle\xi\rangle^\nu}\right) d\xi \le C\widetilde{t}^{-\mu},$$

where $0 < \mu < \min(1, \delta, \nu)$. In the case $0 \le y \le \frac{|\xi|}{2}$ we have with $\xi^* = (\xi_1, ..., \xi_l + \lambda y, ..., \xi_n)$, $\lambda \in (0,1)$

$$\left|\psi\left(t,\breve{\xi}(y)\right) - \psi(t,\xi)\right| \le C|y|\left|\partial_{\xi_l}\psi(t,\xi^*)\right|$$
$$\le Cty\{\xi\}^{\delta-n-1}\langle\xi\rangle^{\nu-n-1} e^{-\frac{\alpha}{2}t\{\xi\}^\delta\langle\xi\rangle^\nu}$$

for all $\xi \in \mathbf{R}^n$, $0 \le y \le \frac{|\xi|}{2}$, since $|\xi| \le |\xi^*| \le |\xi| + y \le \frac{3}{2}|\xi|$. Hence we get

$$\|I_2\|_{\mathbf{L}^1_\xi} \le Ct\left\|\{\xi\}^{\delta-n-1}\langle\xi\rangle^{\nu-n-1} e^{-\frac{\alpha}{2}t\{\xi\}^\delta\langle\xi\rangle^\nu} \int_0^{\frac{|\xi|}{2}} y^{-\mu} dy\right\|_{\mathbf{L}^1_\xi}$$
$$\le Ct\left\|\{\xi\}^{\delta-\mu-n}\langle\xi\rangle^{\nu-\mu-n} e^{-\frac{\alpha}{2}t\{\xi\}^\delta\langle\xi\rangle^\nu}\right\|_{\mathbf{L}^1_\xi} \le C\widetilde{t}^{-\mu}.$$

Therefore estimate (1.45) is true. By virtue of (1.45) we find with $k \geq \max(b, [\beta])$, $\varrho = \beta - k$, and $\breve{x}(y) = (x_1, ..., x_j + y, ..., x_n)$

$$\left\| |\cdot|^b \partial_{x_j}^\beta G(t) \right\|_{\mathbf{L}^q} \leq C \left\| |x|^b \int_0^{\frac{1}{\widetilde{t}}} \left| \partial_{x_j}^k G(t, \breve{x}(y)) - \partial_{x_j}^k G(t, x) \right| y^{-1-\varrho} dy \right\|_{\mathbf{L}^q}$$
$$+ C \left\| |x|^b \int_{\frac{1}{\widetilde{t}}}^\infty \left| \partial_{x_j}^k G(t, \breve{x}(y)) - \partial_{x_j}^k G(t, x) \right| y^{-1-\varrho} dy \right\|_{\mathbf{L}^q}$$
$$\leq C\widetilde{t}^{k+1+n} \left\| |x|^b \langle x\widetilde{t} \rangle^{-\mu-n-k-1} \right\|_{\mathbf{L}^q} \int_0^{\frac{1}{\widetilde{t}}} y^{-\varrho} dy$$
$$+ C\widetilde{t}^{k+n} \left\| |x|^b \langle x\widetilde{t} \rangle^{-\mu-n-k} \right\|_{\mathbf{L}^q} \int_{\frac{1}{\widetilde{t}}}^\infty y^{-1-\varrho} dy \leq C\widetilde{t}^{n\left(1-\frac{1}{q}\right)+\beta-b};$$

hence by the Young inequality (1.2) with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1$, we obtain the first estimate of the lemma

$$\left\| |\cdot|^b \partial_{x_j}^\beta \mathcal{G}(t) \phi \right\|_{\mathbf{L}^p} \leq \left\| \int_{\mathbf{R}^n} |x-y|^b \partial_{x_j}^\beta G(t, x-y) \phi(y) dy \right\|_{\mathbf{L}^p}$$
$$+ \left\| \int_{\mathbf{R}^n} \partial_{x_j}^\beta G(t, x-y) |y|^b \phi(y) dy \right\|_{\mathbf{L}^p}$$
$$\leq C\widetilde{t}^{n\left(\frac{1}{r}-\frac{1}{p}\right)+\beta-b} \|\phi\|_{\mathbf{L}^r} + C\widetilde{t}^{n\left(\frac{1}{r}-\frac{1}{p}\right)+\beta} \|\phi\|_{\mathbf{L}^{r,b}}$$

for all $t > 0$.

We now prove the second estimate. In view of (1.45) we obtain for $0 \leq b \leq a$

$$|x|^b |G(t, x-y) - G(t, x)| \leq |x|^b |y| |\nabla_x G(t, x^*)|$$
$$\leq C\widetilde{t}^{n+1} |x|^b |y| \langle \widetilde{t}x^* \rangle^{-n-\mu-1} \leq C\widetilde{t}^{n+a-b} |y|^a \langle \widetilde{t}x \rangle^{-n-\mu+b-a}$$

for all $x, y \in \mathbf{R}^n$, $|y| \leq \frac{|x|}{2}$. As well for $|y| \geq \frac{|x|}{2}$ we get

$$|x|^b |G(t, x-y) - G(t, x)| \leq C |y|^b (|G(t, x-y)| + |G(t, x)|)$$
$$\leq C\widetilde{t}^{n+a-b} |y|^a \left(\left|\widetilde{t}(x-y)\right|^{b-a} \langle \widetilde{t}(x-y) \rangle^{-n-\mu} + \left|\widetilde{t}x\right|^{b-a} \langle \widetilde{t}x \rangle^{-n-\mu} \right).$$

Thus we obtain

$$|x|^b (G(t, x-y) - G(t, x))$$
$$\leq C\widetilde{t}^{n+a-b} |y|^a \left(\left|\widetilde{t}(x-y)\right|^{b-a} \langle \widetilde{t}(x-y) \rangle^{-n-\mu} + \left|\widetilde{t}x\right|^{b-a} \langle \widetilde{t}x \rangle^{-n-\mu} \right) \quad (1.46)$$

for all $x, y \in \mathbf{R}^n$. In view of (1.46) applying the Young inequality (1.2) we have

$$\left\| |\cdot|^b \left(\mathcal{G}(t)\phi - \vartheta G(t)\right)\right\|_{\mathbf{L}^p} \leq \left\| \int_{\mathbf{R}^n} |x|^b \left(G(t,x-y) - G(t,x)\right)\phi(y)dy \right\|_{\mathbf{L}^p}$$
$$\leq C\widetilde{t}^{n+a-b} \left\| \int_{\mathbf{R}^n} \left(\left|\widetilde{t}(x-y)\right|^{b-a} \left\langle \widetilde{t}(x-y)\right\rangle^{-n-\mu} \right.\right.$$
$$\left.\left. + \left|\widetilde{t}x\right|^{b-a} \left\langle \widetilde{t}x\right\rangle^{-n-\mu} \right) |y|^a \phi(y)dy \right\|_{\mathbf{L}^p}$$
$$\leq C\widetilde{t}^{n+a-b} \left\| \left|\widetilde{t}(\cdot)\right|^{b-a} \left\langle \widetilde{t}(\cdot)\right\rangle^{-n-\mu} \right\|_{\mathbf{L}^p} \left\| |\cdot|^a \phi \right\|_{\mathbf{L}^1} \leq C\widetilde{t}^{n\left(1-\frac{1}{p}\right)+a-b} \left\| |\cdot|^a \phi \right\|_{\mathbf{L}^1}$$

for all $t > 0$, where $1 \leq p \leq \infty$, $b \in [0,a]$, $a \in (0,1)$. Hence the second estimate of the lemma is true. Lemma 1.28 is proved.

To find the asymptotic formulas for the solution we assume that the symbol $L(\xi)$ has the following asymptotic representation in the origin

$$L(\xi) = L_0(\xi) + O\left(|\xi|^{\delta+\gamma}\right) \tag{1.47}$$

for $\xi \to 0$ with some $\gamma > 0$. We assume that the symbol $L_0(\xi)$ is homogeneous of order δ and satisfies (1.43) and (1.44) with $\nu = \delta > 0$. Also we define

$$\mathcal{G}_0(t)\phi = \overline{\mathcal{F}}_{\xi\to x} e^{-tL_0(\xi)}\hat{\phi}(\xi) = t^{-\frac{1}{\delta}} \int_{\mathbf{R}^n} G_0\left((x-y)t^{-\frac{1}{\delta}}\right)\phi(y)\,dy,$$

where

$$G_0(x) = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-L_0(\xi)}\right).$$

Lemma 1.30. *Let the symbol $L(\xi)$ satisfy conditions (1.43), (1.44) with $\nu > 0$ and (1.47). Then the estimate is true*

$$\left\| \mathcal{G}(t)\phi - \vartheta t^{-\frac{n}{\delta}} G_0\left((\cdot)t^{-\frac{1}{\delta}}\right)\right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{n+\mu}{\delta}} \|\phi\|_{\mathbf{L}^{1,a}}$$

for all $t \geq 1$, where $\vartheta = \int_{\mathbf{R}^n} \phi(x)\,dx$, $\mu = \min(a,\gamma)$.

Proof. Taking the second estimate of Lemma 1.28 with $p = \infty$ and $b = 0$, we get

$$\|\mathcal{G}(t)\phi - \vartheta G(t)\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{n+a}{\delta}} \|\phi\|_{\mathbf{L}^{1,a}}$$

for all $t \geq 1$. Then by condition (1.47) we obtain

$$\left\| G(t) - t^{-\frac{n}{\delta}} G_0\left((\cdot)t^{-\frac{1}{\delta}}\right)\right\|_{\mathbf{L}^\infty} \leq \left\| e^{-tL(\xi)} - e^{-tL_0(\xi)}\right\|_{\mathbf{L}^1}$$
$$\leq \int_{|\xi|\leq 1} \left| e^{-tL(\xi)} - e^{-tL_0(\xi)}\right| d\xi + \int_{|\xi|\geq 1} \left| e^{-tL(\xi)}\right| d\xi + \int_{|\xi|\geq 1} \left| e^{-tL_0(\xi)}\right| d\xi$$
$$\leq Ct \int_{|\xi|\leq 1} e^{-\alpha t|\xi|^\delta} |\xi|^{\delta+\gamma} d\xi + \int_{|\xi|\geq 1} e^{-\alpha t|\xi|^\nu} d\xi + \int_{|\xi|\geq 1} e^{-\alpha t|\xi|^\delta} d\xi$$
$$\leq Ct^{-\frac{n+\gamma}{\delta}} + Ce^{-\frac{\alpha}{2}t} \leq Ct^{-\frac{n+\gamma}{\delta}}$$

for all $t \geq 1$. Hence we find the estimate of the lemma. Lemma 1.30 is proved.

Now we will give some modifications of Lemma 1.28 in order to relax the restriction $\nu > 0$ in estimates (1.43) and (1.44). For example, when the symbol $L(\xi)$ does not grow at infinity, we may assume the asymptotic expansion

$$L(\xi) = \alpha + \sum_{k=1}^{n} a_k \langle \xi \rangle^{-2k} + O \langle \xi \rangle^{-2n-2} \tag{1.48}$$

for $\xi \to \infty$, where $\alpha > 0$, $a_k \in \mathbf{R}$.

Define the polynomials $b_k(t)$ of order $k \geq 0$ by the asymptotic expansion of $e^{-tL(\xi)}$ for $\xi \to \infty$ such that

$$e^{-tL(\xi)} = e^{-tL_0(\xi)} + e^{-\alpha t} \sum_{k=0}^{n} b_k(t) \langle \xi \rangle^{-2k} + R(t,\xi). \tag{1.49}$$

Thus instead of conditions (1.43), (1.44) with $\nu = 0$ and (1.47), (1.48) we can suppose the following condition. Let the remainder symbol $R(t,\xi) \in \mathbf{C}^N(\mathbf{R}^n \setminus \{0\})$, satisfy the estimate

$$\left| \partial_{\xi_j}^l R(t,\xi) \right| \leq C t^{\frac{l}{\delta}} \langle t \rangle^{-\frac{\gamma}{\delta}} e^{-\frac{\alpha}{2} t |\xi|^{\delta}} + C e^{-\frac{\alpha}{2} t} \langle \xi \rangle^{-2n-2} \tag{1.50}$$

for all $t > 0$, $\xi \in \mathbf{R}^n \setminus \{0\}$, $l = 0, 1, ..., N$, $j = 1, 2, ..., n$, with some $N \geq n + 2$, $\gamma > 0$.

Define $\mathcal{B}^0 = 1$ and

$$\mathcal{B}^k \phi = \int_{\mathbf{R}^n} B_k(x-y)\, \phi(y)\, dy$$

with kernels

$$B_k(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{i\xi x} \langle \xi \rangle^{-2k} d\xi = \frac{2^{1-k} |x|^{k-\frac{n}{2}}}{(k-1)!} K_{\frac{n}{2}-k}(|x|)$$

for $k \geq 1$ (see Titchmarsh [1986]), where

$$K_\nu(|x|) = K_{-\nu}(|x|) = 2^{-\nu-1} |x|^{\nu} \int_0^{\infty} \xi^{-\nu-1} e^{-\xi - \frac{|x|^2}{4\xi}} d\xi$$

is the Macdonald function (or modified Bessel function) of order $\nu \in \mathbf{R}$ (see Watson [1944]). By the estimates of the Macdonald function we see that for any $k \geq 1$

$$|B_k(x)| \leq \begin{cases} C |x|^{k - \frac{n+1}{2}} e^{-|x|}, \text{ for } |x| \geq 1, \\ C \int_{|x|}^{1} y^{2k-n-1} dy, \text{ for } |x| < 1 . \end{cases}$$

Hence for any $k \geq 0$

$$\left\| \mathcal{B}^k \phi \right\|_{\mathbf{L}^{p,b}} \leq C \|\phi\|_{\mathbf{L}^{p,b}} \tag{1.51}$$

for all $1 \leq p \leq \infty$ and $b \geq 0$.

Then we can obtain the following modifications of Lemmas 1.28 and 1.30.

Lemma 1.31. *Let the symbol $L(\xi)$ be such that condition (1.50) takes place. Then the estimates are true*

$$\left\| |\cdot|^b \mathcal{G}(t)\phi \right\|_{\mathbf{L}^p} \leq Ct^{-\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{p}\right)+\frac{b}{\delta}} \|\phi\|_{\mathbf{L}^r} + Ct^{-\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{p}\right)} \|\phi\|_{\mathbf{L}^{r,b}} + e^{-\frac{\alpha}{2}t} \|\phi\|_{\mathbf{L}^{p,b}}$$

and

$$\left\| |\cdot|^b \left(\mathcal{G}(t)\phi - \vartheta t^{-\frac{n}{\delta}} G_0\left((\cdot) t^{-\frac{1}{\delta}} \right) \right) \right\|_{\mathbf{L}^p} \leq Ct^{-\frac{n}{\delta}\left(1-\frac{1}{p}\right)+\frac{b}{\delta}} \left(t^{-\frac{a}{\delta}} + \langle t \rangle^{-\frac{\gamma}{\delta}} \right) \left\| |\cdot|^a \phi \right\|_{\mathbf{L}^1} + e^{-\frac{\alpha}{2}t} \left\| |\cdot|^b \phi \right\|_{\mathbf{L}^p}$$

for all $t > 0$, where $\vartheta = \int_{\mathbf{R}^n} \phi(x)\,dx$, $1 \leq r \leq p \leq \infty$, $0 \leq b \leq a$, provided that the right-hand sides are finite.

Remark 1.32. Conditions of Lemma 1.31 are fulfilled, for example, for the Sobolev type equation $\mathcal{L}u = -(1-\Delta)^{-1}\alpha\Delta u$, that is, when the symbol $L(\xi) = \frac{\alpha|\xi|^2}{1+|\xi|^2}$. In this case we have

$$R(t,\xi) = e^{-\alpha t \frac{|\xi|^2}{1+|\xi|^2}} - e^{-\alpha t|\xi|^2} - e^{-\alpha t} \sum_{k=0}^{n} \frac{\alpha^k t^k}{k!} \left(1+|\xi|^2\right)^{-k}.$$

So that representing

$$R(t,\xi) = e^{-\alpha t \frac{|\xi|^2}{1+|\xi|^2}} \left(1 - e^{-\alpha t \frac{|\xi|^4}{1+|\xi|^2}}\right) - e^{-\alpha t} \sum_{k=0}^{n} \frac{\alpha^k t^k}{k!} \left(1+|\xi|^2\right)^{-k}$$

for all $|\xi| \leq 1$, and

$$R(t,\xi) = -e^{-\alpha t|\xi|^2} + e^{-\alpha t} \left(e^{-\frac{\alpha t}{1+|\xi|^2}} - \sum_{k=0}^{n} \frac{\alpha^k t^k}{k!} \left(1+|\xi|^2\right)^{-k} \right)$$

for all $|\xi| \geq 1$, we can easily obtain condition (1.50).

Proof. We have due (1.49)

$$\mathcal{G}(t) = \mathcal{G}_0(t) + e^{-\alpha t} \sum_{k=0}^{n} b_k(t) \mathcal{B}^k + \mathcal{R}(t),$$

where the remainder operator $\mathcal{R}(t)$ is defined by its symbol $R(t,\xi)$

$$\mathcal{R}(t)\phi = \overline{\mathcal{F}}_{\xi \to x} R(t,\xi) \widehat{\phi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \widetilde{R}(t, x-y) \phi(y)\,dy;$$

here the kernel $\widetilde{R}(t,x) = \overline{\mathcal{F}}_{\xi\to x} R(t,\xi)$. Note that the Green operator $\mathcal{G}_0(t)$ satisfies conditions (1.43), (1.44) with $\nu = \delta > 0$. Hence by Lemma 1.28 we have

$$\left\| |\cdot|^b \mathcal{G}_0(t)\phi \right\|_{\mathbf{L}^p} \le Ct^{-\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{p}\right)+\frac{b}{\delta}} \|\phi\|_{\mathbf{L}^r} + Ct^{-\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{p}\right)} \|\phi\|_{\mathbf{L}^{r,b}} \tag{1.52}$$

and

$$\left\| |\cdot|^b \left(\mathcal{G}_0(t)\phi - \vartheta t^{-\frac{n}{\delta}} G_0\left((\cdot)\, t^{-\frac{1}{\delta}} \right) \right) \right\|_{\mathbf{L}^p} \le Ct^{-\frac{n}{\delta}\left(1-\frac{1}{p}\right)-\frac{a-b}{\delta}} \left\| |\cdot|^a \phi \right\|_{\mathbf{L}^1} \tag{1.53}$$

for all $t > 0$. By estimate (1.51) we have

$$\left\| |\cdot|^b e^{-\alpha t} \sum_{k=0}^{n} b_k(t)\, \mathcal{B}^k \phi \right\|_{\mathbf{L}^p} \le Ce^{-\frac{\alpha}{2}t} \left\| |\cdot|^b \phi \right\|_{\mathbf{L}^p} \tag{1.54}$$

for all $t > 0$, where $1 \le p \le \infty$, $b \ge 0$.

Now by virtue of condition (1.50) we can estimate the kernel $\widetilde{R}(t,x)$

$$\left| \widetilde{R}(t,x) \right| \le C \|R(t)\|_{\mathbf{L}^1} \le C \langle t \rangle^{-\frac{\gamma}{\delta}} \left\| e^{-\frac{\alpha}{2}t|\xi|^\delta} \right\|_{\mathbf{L}^1}$$

$$+ Ce^{-\frac{\alpha}{2}t} \left\| \langle \xi \rangle^{-2n-2} \right\|_{\mathbf{L}^1} \le C \langle t \rangle^{-\frac{\gamma}{\delta}} t^{-\frac{n}{\delta}}.$$

On the other hand, integrating $n+2$ times by parts with respect to ξ_l we obtain

$$\left| \widetilde{R}(t,x) \right| \le C |x|^{-n-2} \sum_{l=1}^{n} \left| \int_{\mathbf{R}^n} e^{i\xi x} \partial_{\xi_l}^{n+2} R(t,\xi)\, d\xi \right|$$

$$\le C |x|^{-n-2}\, t^{\frac{n+2}{\delta}} \langle t \rangle^{-\frac{\gamma}{\delta}} \left\| e^{-\frac{\alpha}{2}t|\xi|^\delta} \right\|_{\mathbf{L}^1} + C |x|^{-n-2} e^{-\frac{\alpha}{2}t} \left\| \langle \xi \rangle^{-2n-2} \right\|_{\mathbf{L}^1}$$

$$\le C \left(|x|\, t^{-\frac{1}{\delta}} \right)^{-n-2} \langle t \rangle^{-\frac{\gamma}{\delta}} t^{-\frac{n}{\delta}} + C |x|^{-n-2} e^{-\frac{\alpha}{2}t}.$$

Combining these two estimates we have

$$\left| \widetilde{R}(t,x) \right| \le C \left\langle |x|\, t^{-\frac{1}{\delta}} \right\rangle^{-n-2} \langle t \rangle^{-\frac{\gamma}{\delta}} t^{-\frac{n}{\delta}} + C |x|^{-n-2} e^{-\frac{\alpha}{2}t} \tag{1.55}$$

for all $x \in \mathbf{R}^n$, $t > 0$. In view of (1.55) applying the Young inequality with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1$ we get

$$\|\mathcal{R}(t)\phi\|_{\mathbf{L}^{p,b}} \leq C\left\|\int_{\mathbf{R}^n}\widetilde{R}(t,x-y)\langle y\rangle^b\phi(y)\,dy\right\|_{\mathbf{L}^p}$$
$$+C\left\|\int_{\mathbf{R}^n}|x-y|^b\,\widetilde{R}(t,x-y)\phi(y)\,dy\right\|_{\mathbf{L}^p}$$
$$\leq C\left\|\widetilde{R}(t)\right\|_{\mathbf{L}^q}\|\phi\|_{\mathbf{L}^{r,b}}+C\left\|\widetilde{R}(t)\right\|_{\mathbf{L}^{q,b}}\|\phi\|_{\mathbf{L}^r}$$
$$\leq C\langle t\rangle^{-\frac{\gamma}{\delta}}t^{-\frac{n}{\delta}}\left\|\left\langle|x|\,t^{-\frac{1}{\delta}}\right\rangle^{-n-2}\right\|_{\mathbf{L}^q}\|\phi\|_{\mathbf{L}^{r,b}}+Ce^{-\frac{\alpha}{2}t}\left\||x|^{-n-2}\right\|_{\mathbf{L}^{q,b}}\|\phi\|_{\mathbf{L}^{r,b}}$$
$$+C\langle t\rangle^{-\frac{\gamma}{\delta}}t^{-\frac{n}{\delta}}\left\|\left\langle|x|\,t^{-\frac{1}{\delta}}\right\rangle^{-n-2}\right\|_{\mathbf{L}^{q,b}}\|\phi\|_{\mathbf{L}^r}\leq C\langle t\rangle^{-\frac{\gamma}{\delta}}t^{-\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{p}\right)+\frac{b}{\delta}}\|\phi\|_{\mathbf{L}^{r,a}} \tag{1.56}$$

for all $t>0$, where $1\leq r\leq p\leq\infty$, $0\leq b\leq a$. By virtue of (1.52), (1.53), (1.54) and (1.56) the estimates of the lemma follow. Lemma 1.31 is proved.

Now we will give another modification of Lemma 1.28 to the case of oscillating symbols for $\xi\to\infty$. Consider the Green operator of the form

$$\mathcal{G}(t)\phi=\overline{\mathcal{F}}_{\xi\to x}\left(S(t,\xi)\,\widehat{\phi}(\xi)\right).$$

We restrict our attention only to the case of the space dimensions $n=1,2,3$. We assume the following asymptotic expansion for the symbols $S(t,\xi)$ for $n=1,2,3$

$$\partial_t^k S(t,\xi)=\langle t\rangle^{-k}e^{-tL_0(\xi)}+\partial_t^k\left(e^{-\alpha t}\frac{\sin(t\beta|\xi|)}{|\xi|}\right)+R_k(t,\xi)$$

for $\xi\to\infty$, where $\alpha>0$, $\beta\in\mathbf{R}$, $k=0,1$. We suppose that the remainder symbols $R_k(t,\xi)\in\mathbf{C}^N(\mathbf{R}^n\setminus\{0\})$ obey the estimate

$$\left|\partial_{\xi_j}^l R_k(t,\xi)\right|\leq Ct^{\frac{l}{\delta}}\langle t\rangle^{-\frac{\gamma}{\delta}-k}e^{-\frac{\alpha}{2}t|\xi|^{\delta}}+Ce^{-\frac{\alpha}{2}t}\langle\xi\rangle^{k-2} \tag{1.57}$$

for all $t>0$, $\xi\in\mathbf{R}^n\setminus\{0\}$, $l=0,1,...,N$, $j=1,2,...,n$, $k=0,1$, with some $N\geq n+2$, $\gamma>0$.

Note that $\frac{\sin(t\beta|\xi|)}{|\xi|}$ is the symbol for the wave equation. Hence using the well-known formulas for the Green operator of the wave equation

$$\mathcal{W}(t)\phi=\overline{\mathcal{F}}_{\xi\to x}\left(\frac{\sin(t\beta|\xi|)}{|\xi|}\widehat{\phi}(\xi)\right)$$

we obtain (see e.g. Mizohata [1973], Kanwal [2004]) for $n=1$

$$\mathcal{W}(t)\phi=\frac{1}{2}\int_{x-\beta t}^{x+\beta t}\phi(y)\,dy,$$

for $n=2$

$$\mathcal{W}(t)\,\phi = \frac{1}{2\pi}\int\int_{|x-y|\le\beta t}\frac{\phi(y)\,dy}{\sqrt{\beta^2t^2-|x-y|^2}},$$

and for $n=3$

$$\mathcal{W}(t)\,\phi = \frac{1}{4\pi t}\int\int_{|x-y|=\beta t}\phi(y)\,d\omega_y.$$

By these explicit formulas we easily see that

$$\left\|\partial_t^k\mathcal{W}(t)\,\phi\right\|_{\mathbf{L}^{p,b}} \le C\langle t\rangle^{b+2}\left\|\langle i\nabla\rangle^{\kappa k}\phi\right\|_{\mathbf{L}^{p,b}} \tag{1.58}$$

for all $1\le p\le\infty$ and $b\ge 0$, $k=0,1$, where $\kappa=0$ for the one dimensional case $n=1$ and $\kappa=1$ for the case of space dimensions $n=2,3$.

Then we can obtain the following modifications of Lemmas 1.28 and 1.30.

Lemma 1.33. *Let the symbol $S(t,\xi)$ of the operator $\mathcal{G}(t)$ be such that condition (1.57) takes place. Then the estimates are true provided that the right-hand sides are finite*

$$\begin{aligned}&\left\||\cdot|^b\,\partial_t^k\mathcal{G}(t)\,\phi\right\|_{\mathbf{L}^p} \le Ct^{-\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{p}\right)+\frac{b}{\delta}}\langle t\rangle^{-k}\left\|\langle i\nabla\rangle^{k\kappa}\phi\right\|_{\mathbf{L}^r}\\&+Ct^{-\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{p}\right)}\langle t\rangle^{-k}\left\|\langle i\nabla\rangle^{k\kappa}\phi\right\|_{\mathbf{L}^{r,b}}+e^{-\frac{\alpha}{2}t}\left\|\langle i\nabla\rangle^{k\kappa}\phi\right\|_{\mathbf{L}^{p,b}}\end{aligned}$$

and

$$\begin{aligned}&\left\||\cdot|^b\left(\mathcal{G}(t)\,\phi-\vartheta t^{-\frac{n}{\delta}}G_0\left((\cdot)\,t^{-\frac{1}{\delta}}\right)\right)\right\|_{\mathbf{L}^p}\\&\le Ct^{-\frac{n}{\delta}\left(1-\frac{1}{p}\right)+\frac{b-a}{\delta}}\|\phi\|_{\mathbf{L}^{1,a}}+t^{-\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{p}\right)+\frac{b}{\delta}}\langle t\rangle^{-\frac{\gamma}{\delta}}\|\phi\|_{\mathbf{L}^{r,b}}\\&+e^{-\frac{\alpha}{2}t}\|\phi\|_{\mathbf{L}^{p,b}}\end{aligned}$$

for all $t>0$, where $\vartheta=\int_{\mathbf{R}^n}\phi(x)\,dx$, $\max\left(1,\frac{2p}{2+p}\right)\le r\le p\le\infty$, $0\le b\le \min(\gamma,a)$, $k=0,1$, here $\kappa=0$ for the one dimensional case $n=1$ and $\kappa=1$ for the case of space dimensions $n=2,3$.

Remark 1.34. Condition (1.57) of Lemma 1.33 is fulfilled for the damped wave equation, when the symbol

$$S(t,\xi)=e^{-\frac{t}{2}}\frac{\sin\left(t\sqrt{|\xi|^2-\frac{1}{4}}\right)}{\sqrt{|\xi|^2-\frac{1}{4}}}.$$

In this case we have

$$\begin{aligned}R_k(t,\xi)&=\partial_t^kS(t,\xi)-\langle t\rangle^{-k}e^{-t|\xi|^2}-\partial_t^k\left(e^{-\frac{t}{2}}\frac{\sin(t|\xi|)}{|\xi|}\right)\\&=O\left(\langle t\rangle^{-1-k}e^{-t|\xi|^2}\right)+O\left(e^{-\frac{t}{4}}\langle\xi\rangle^{k-2}\right)\end{aligned}$$

for all $\xi\in\mathbf{R}^n$ and this asymptotic representation can be differentiated with respect to ξ. Therefore we can easily obtain condition (1.57).

Proof. We have

$$\partial_t^k \mathcal{G}(t) = \langle t\rangle^{-k} \mathcal{G}_0(t) + \partial_t^k \left(e^{-\alpha t}\mathcal{W}(t)\right) + \mathcal{R}_k(t),$$

where the remainder operators $\mathcal{R}_k(t)$ are defined by their symbols $R_k(t,\xi)$

$$\mathcal{R}_k(t)\phi = \overline{\mathcal{F}}_{\xi\to x} \langle\xi\rangle^{-\kappa k} R_k(t,\xi) \langle\xi\rangle^{\kappa k} \widehat{\phi}(\xi)$$
$$= (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \widetilde{R}_k(t, x-y) \langle i\nabla\rangle^{\kappa k} \phi(y)\, dy;$$

here the kernel $\widetilde{R}_k(t,x) = \overline{\mathcal{F}}_{\xi\to x}\left(\langle\xi\rangle^{-\kappa k} R_k(t,\xi)\right)$. Note that the Green operator $\mathcal{G}_0(t)$ satisfies conditions (1.43), (1.44) with $\nu = \delta > 0$. Hence by Lemma 1.28 we have

$$\left\| |\cdot|^b \mathcal{G}_0(t)\phi \right\|_{\mathbf{L}^p} \le Ct^{-\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{p}\right)+\frac{b}{\delta}} \|\phi\|_{\mathbf{L}^r} + Ct^{-\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{p}\right)} \|\phi\|_{\mathbf{L}^{r,b}} \tag{1.59}$$

and

$$\left\| |\cdot|^b \left(\mathcal{G}_0(t)\phi - \vartheta t^{-\frac{n}{\delta}} G_0\left((\cdot)\, t^{-\frac{1}{\delta}}\right)\right)\right\|_{\mathbf{L}^p} \le Ct^{-\frac{n}{\delta}\left(1-\frac{1}{p}\right)-\frac{a-b}{\delta}} \|\phi\|_{\mathbf{L}^{1,a}} \tag{1.60}$$

for all $t > 0$. By estimate (1.58) we have

$$\left\|\partial_t^k \left(e^{-\alpha t}\mathcal{W}(t)\phi\right)\right\|_{\mathbf{L}^{p,b}} \le Ce^{-\frac{\alpha}{2}t} \left\|\langle i\nabla\rangle^{\kappa k}\phi\right\|_{\mathbf{L}^{p,b}} \tag{1.61}$$

for all $1 \le p \le \infty$ and $b \ge 0$, $k = 0,1$, where $\kappa = 0$ for the one dimensional case $n = 1$ and $\kappa = 1$ for the case of space dimensions $n = 2, 3$.

Now by virtue of condition (1.57) we can estimate the kernel $\widetilde{R}_k(t,x)$

$$\left\|\widetilde{R}_k(t)\right\|_{\mathbf{L}^2} = C\left\|\langle\cdot\rangle^{-\kappa k} R_k(t)\right\|_{\mathbf{L}^2} \le C\langle t\rangle^{-\frac{\gamma}{\delta}-k} \left\|e^{-\frac{\alpha}{2}t|\xi|^\delta}\right\|_{\mathbf{L}^2}$$
$$+ Ce^{-\frac{\alpha}{2}t}\left\|\langle\cdot\rangle^{-\kappa k+k-2}\right\|_{\mathbf{L}^2} \le Ct^{-\frac{n}{2\delta}} \langle t\rangle^{-\frac{\gamma}{\delta}-k}. \tag{1.62}$$

On the other hand

$$\left\||x|^{n+2}\widetilde{R}_k(t)\right\|_{\mathbf{L}^2} \le C\sum_{l=1}^n \left\|\partial_{\xi_l}^{n+2}\langle\cdot\rangle^{-\kappa k} R_k(t)\right\|_{\mathbf{L}^2}$$
$$\le Ct^{\frac{n+2}{\delta}}\langle t\rangle^{-\frac{\gamma}{\delta}-k}\left\|e^{-\frac{\alpha}{2}t|\xi|^\delta}\right\|_{\mathbf{L}^2} + Ce^{-\frac{\alpha}{2}t}\left\|\langle\cdot\rangle^{-\kappa k+k-2}\right\|_{\mathbf{L}^2}$$
$$\le Ct^{\frac{n}{2\delta}+\frac{2}{\delta}}\langle t\rangle^{-\frac{\gamma}{\delta}-k} + Ce^{-\frac{\alpha}{2}t}. \tag{1.63}$$

By virtue of the Hölder inequality, taking $a = \||\cdot|^\omega\varphi\|_{\mathbf{L}^2}\|\varphi\|_{\mathbf{L}^2}^{-1} > 0$, we obtain for $p \in [1,2]$

$$\|\phi\|_{\mathbf{L}^p} \le \|(a+|\cdot|^\omega)\phi\|_{\mathbf{L}^2}\left\|(a+|\cdot|^\omega)^{-1}\right\|_{\mathbf{L}^{\frac{2p}{2-p}}}$$
$$\le Ca^{\frac{n}{\omega}\left(\frac{1}{p}-\frac{1}{2}\right)}\|\varphi\|_{\mathbf{L}^2} + Ca^{\frac{n}{\omega}\left(\frac{1}{p}-\frac{1}{2}\right)-1}\||\cdot|^\omega\varphi\|_{\mathbf{L}^2}$$
$$\le \||\cdot|^\omega\phi\|_{\mathbf{L}^2}^{\frac{n}{\omega}\left(\frac{1}{p}-\frac{1}{2}\right)}\|\phi\|_{\mathbf{L}^2}^{1-\frac{n}{\omega}\left(\frac{1}{p}-\frac{1}{2}\right)}. \tag{1.64}$$

Hence the substitution of (1.62) and (1.63) into interpolation inequality (1.64) yields the estimate

$$\left\| |\cdot|^b \widetilde{R}_k(t) \right\|_{\mathbf{L}^q} \le Ct^{-\frac{n}{\delta}\left(1-\frac{1}{q}\right)+\frac{b}{\delta}} \langle t \rangle^{-\frac{\gamma}{\delta}-k} + Ce^{-\frac{\alpha}{2}t} \tag{1.65}$$

for $1 \le q \le 2$, $b \in [0,a]$. In view of (1.65) applying the Young inequality with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1$ we get

$$\begin{aligned}
\|\mathcal{R}_k(t)\phi\|_{\mathbf{L}^{p,b}} &\le C \left\| \int_{\mathbf{R}^n} \widetilde{R}_k(t, x-y) \langle y \rangle^b \langle i\nabla \rangle^{k\kappa} \phi(y)\, dy \right\|_{\mathbf{L}^p} \\
&+ C \left\| \int_{\mathbf{R}^n} |x-y|^b \widetilde{R}(t, x-y) \langle i\nabla \rangle^{k\kappa} \phi(y)\, dy \right\|_{\mathbf{L}^p} \\
&\le C \left\| \widetilde{R}(t) \right\|_{\mathbf{L}^q} \left\| \langle i\nabla \rangle^{k\kappa} \phi \right\|_{\mathbf{L}^{r,b}} + C \left\| \widetilde{R}(t) \right\|_{\mathbf{L}^{q,b}} \left\| \langle i\nabla \rangle^{k\kappa} \phi \right\|_{\mathbf{L}^r} \\
&\le Ct^{-\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{p}\right)} \langle t \rangle^{-\frac{\gamma}{\delta}-k} \left\| \langle i\nabla \rangle^{k\kappa} \phi \right\|_{\mathbf{L}^{r,b}} \\
&+ Ct^{-\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{p}\right)+\frac{b}{\delta}} \langle t \rangle^{-\frac{\gamma}{\delta}-k} \left\| \langle i\nabla \rangle^{k\kappa} \phi \right\|_{\mathbf{L}^r}
\end{aligned} \tag{1.66}$$

for all $t > 0$, where $\max\left(1, \frac{2p}{2+p}\right) \le r \le p \le \infty$, $0 \le b \le a$. By virtue of (1.59), (1.60), (1.61) and (1.66) the estimates of the lemma follow. Lemma 1.33 is proved.

1.5.2 Estimates in the $\mathbf{L}^2$ - theory

By the Sobolev Imbedding Theorem the $\mathbf{L}^p$ norms can be estimated by derivatives in $\mathbf{L}^2$ norms and weighted $\mathbf{L}^{1,a}$ norms can be estimates by weighted $\mathbf{L}^{2,b}$ norms with $b > a + \frac{n}{2}$. And the $\mathbf{L}^2$ - theory is convenient for passing to Fourier transform representations.

Consider the Green operator of the form

$$\mathcal{G}(t)\phi = \overline{\mathcal{F}}_{\xi\to x}\left(S(t,\xi)\widehat{\phi}(\xi)\right),$$

where the symbol $S(t,\xi)$ satisfies estimates

$$|S(t,\xi)| \le \langle \xi \rangle^{-\mu} e^{-\alpha t \{\xi\}^\delta} \tag{1.67}$$

for all $\xi \in \mathbf{R}^n$, where $\alpha > 0$, $\delta > 0$, $\mu \ge 0$. Also we suppose that the symbol is sufficiently smooth except the origin: $S(t,\xi) \in \mathbf{C}^N(\mathbf{R}^+ \times \mathbf{R}^n \setminus \{0\})$ and obeys the estimate

$$\left| \partial_{\xi_j}^l \left(\langle \xi \rangle^\mu S(t,\xi) \right) \right| \le C \langle t \rangle^{\frac{l}{\delta}} e^{-\frac{\alpha}{2} t\{\xi\}^\delta} \tag{1.68}$$

for all $t > 0$, $\xi \in \mathbf{R}^n \setminus \{0\}$, $l = 0, 1, ..., N$, $j = 1, 2, ..., n$, with some $N \ge n + 2$. We now prove $\mathbf{L}^2$ estimates for the Green operator $\mathcal{G}(t)$.

Lemma 1.35. *Let the symbol $S(t,\xi)$ satisfy conditions (1.67) and (1.68). Then the estimates are true*

$$\left\| |\nabla|^{\rho} \mathcal{G}(t)\phi \right\|_{\mathbf{L}^2} \le C \langle t\rangle^{-\frac{\rho-\beta}{\delta}-\frac{n}{\delta}\left(\frac{1}{q}-\frac{1}{2}\right)} \left\| |\nabla|^{\beta} \langle i\nabla\rangle^{-\mu} \phi \right\|_{\mathbf{L}^q} + Ce^{-\alpha t} \left\| |\nabla|^{\rho} \langle i\nabla\rangle^{-\mu} \phi \right\|_{\mathbf{L}^2}$$

for all $t>0$, where $1 \le q \le 2$, $0 \le \beta \le \rho$, and

$$\left\| |\cdot|^{\omega} \mathcal{G}(t)\phi \right\|_{\mathbf{L}^2} \le C \langle t\rangle^{\frac{\omega}{\delta}-\frac{n}{2\delta}} \left\| \langle i\nabla\rangle^{-\mu} \phi \right\|_{\mathbf{L}^1} + C \langle t\rangle^{-\frac{n}{\delta}\left(\frac{1}{q}-\frac{1}{2}\right)} \left\| |\cdot|^{\omega} \langle i\nabla\rangle^{-\mu} \phi \right\|_{\mathbf{L}^q} + Ce^{-\frac{\alpha}{2}t} \left\| \langle\cdot\rangle^{\omega} \langle i\nabla\rangle^{-\mu} \phi \right\|_{\mathbf{L}^2}$$

for all $t>0$, where $1 \le q \le 2$, $\frac{n}{2} < \omega \le N$, provided that the right-hand sides are finite.

Remark 1.36. The conditions of Lemma 1.35 are fulfilled for the damped wave equation for any space dimensions

$$S(t,\xi) = e^{-\frac{t}{2}} \frac{\sin\left(t\sqrt{|\xi|^2-\frac{1}{4}}\right)}{\sqrt{|\xi|^2-\frac{1}{4}}}$$

if we take $\delta = 2$, $\mu = 1$ and $\alpha = \frac{1}{2}$.

Proof. By condition (1.67) we get

$$\left\| |\xi|^{\rho} \langle\xi\rangle^{\mu} S(t,\xi) \right\|_{\mathbf{L}^r(|\xi|\le 1)} \le C \langle t\rangle^{-\frac{\rho}{\delta}-\frac{n}{\delta r}}$$

and

$$\left\| \langle\xi\rangle^{\mu} S(t,\xi) \right\|_{\mathbf{L}^\infty(|\xi|\ge 1)} \le Ce^{-\alpha t}$$

for all $t>0$; hence by the Hölder inequality and estimate for the Fourier transform (see Titchmarsh [1986])

$$\left\| \mathcal{F}\phi \right\|_{\mathbf{L}^{\frac{q}{q-1}}} \le C \left\| \phi \right\|_{\mathbf{L}^q}$$

for $1 \le q \le 2$, the first estimate of the lemma follows

$$\begin{aligned} &\left\| |\nabla|^{\rho} \mathcal{G}(t)\phi \right\|_{\mathbf{L}^2} \\ &\le C \left\| |\xi|^{\rho-\beta} \langle\xi\rangle^{\mu} S(t,\xi) \right\|_{\mathbf{L}^{\frac{2q}{2-q}}(|\xi|\le 1)} \left\| |\xi|^{\beta} \langle\xi\rangle^{-\mu} \widehat{\phi}(\xi) \right\|_{\mathbf{L}^{\frac{q}{q-1}}(|\xi|\le 1)} \\ &+ C \left\| \langle\xi\rangle^{\mu} S(t,\xi) \right\|_{\mathbf{L}^\infty(|\xi|\ge 1)} \left\| |\xi|^{\rho} \langle\xi\rangle^{-\mu} \widehat{\phi}(\xi) \right\|_{\mathbf{L}^2(|\xi|\ge 1)} \\ &\le C \langle t\rangle^{-\frac{\rho-\beta}{\delta}-\frac{n}{\delta}\left(\frac{1}{q}-\frac{1}{2}\right)} \left\| |\nabla|^{\beta} \langle i\nabla\rangle^{-\mu} \phi \right\|_{\mathbf{L}^q} + Ce^{-\alpha t} \left\| |\nabla|^{\rho} \langle i\nabla\rangle^{-\mu} \phi \right\|_{\mathbf{L}^2}. \end{aligned}$$

To prove the second estimate of the lemma we introduce a cut off function $\chi_1 \in \mathbf{C}^\infty(\mathbf{R}^n)$ such that $\chi_1(\xi) = 1$ for $|\xi| \le 1$ and $\chi_1(\xi) = 0$ for $|\xi| \ge 2$, also let $\chi_2(\xi) = 1 - \chi_1(\xi)$. Note that there exists a smooth and rapidly decaying kernel

$$K(t,x) = \overline{\mathcal{F}}_{\xi \to x}\left(\langle \xi \rangle^\mu S(t,\xi)\chi_1(\xi)\right).$$

By condition (1.68) we have

$$\left|\partial_{\xi_j}^l \left(\langle \xi \rangle^\mu S(t,\xi)\chi_1(\xi)\right)\right| \le C\langle t\rangle^{\frac{l}{\delta}} e^{-\frac{\alpha}{2}t\{\xi\}^\delta}$$

for all $t > 0$, $|\xi| \le 2$, $l = 0,1,...,N$, $j = 1,2,...,n$. Hence

$$\left\|x_j^l K(t,x)\right\|_{\mathbf{L}^2} \le C\left\|\partial_{\xi_j}^l \left(\langle \xi \rangle^\mu S(t,\xi)\chi_1(\xi)\right)\right\|_{\mathbf{L}^2(|\xi|\le 2)} \le C\langle t\rangle^{\frac{l}{\delta}-\frac{n}{2\delta}},$$

so that by the Hölder inequality

$$\left\||\cdot|^\sigma K(t)\right\|_{\mathbf{L}^2} \le C\langle t\rangle^{\frac{\sigma}{\delta}-\frac{n}{2\delta}} \tag{1.69}$$

with $0 \le \sigma \le \omega$. By interpolation inequality (1.64) with $p = \frac{2q}{3q-2}$ we get

$$\begin{aligned}\|K(t)\|_{\mathbf{L}^{\frac{2q}{3q-2}}} &\le \left\||\cdot|^\omega K(t)\right\|_{\mathbf{L}^2}^{\frac{n}{\omega}\left(1-\frac{1}{q}\right)} \|K(t)\|_{\mathbf{L}^2}^{1-\frac{n}{\omega}\left(1-\frac{1}{q}\right)} \\ &\le C\langle t\rangle^{-\frac{n}{\delta}\left(\frac{1}{q}-\frac{1}{2}\right)}.\end{aligned} \tag{1.70}$$

Then in view of (1.69) and (1.70), and by applying the Young inequality and estimate $|x|^\omega \le C|x-y|^\omega + C|y|^\omega$, we obtain

$$\begin{aligned}&\left\||\cdot|^\omega \overline{\mathcal{F}}_{\xi\to x}\left(\langle \xi\rangle^\mu S(t,\xi)\chi_1(\xi)\right)\left(\langle\xi\rangle^{-\mu}\widehat{\phi}(\xi)\right)\right\|_{\mathbf{L}^2} \\ &= \left\||\cdot|^\omega \int_{\mathbf{R}^n} K(t,\cdot - y)\langle i\nabla\rangle^{-\mu}\phi(y)\,dy\right\|_{\mathbf{L}_x^2} \\ &\le C\left\||\cdot|^\omega K(t)\right\|_{\mathbf{L}^2}\left\|\langle i\nabla\rangle^{-\mu}\phi\right\|_{\mathbf{L}^1} + C\|K(t)\|_{\mathbf{L}^{\frac{2q}{3q-2}}}\left\||\cdot|^\omega\langle i\nabla\rangle^{-\mu}\phi\right\|_{\mathbf{L}^q} \\ &\le C\langle t\rangle^{\frac{\omega}{\delta}-\frac{n}{2\delta}}\left\|\langle i\nabla\rangle^{-\mu}\phi\right\|_{\mathbf{L}^1} + C\langle t\rangle^{-\frac{n}{\delta}\left(\frac{1}{q}-\frac{1}{2}\right)}\left\||\cdot|^\omega\langle i\nabla\rangle^{-\mu}\phi\right\|_{\mathbf{L}^q}.\end{aligned} \tag{1.71}$$

Employing condition (1.68) we have

$$\left|\partial_{\xi_j}^l\left(\langle\xi\rangle^\mu S(t,\xi)\chi_2(\xi)\right)\right| \le Ce^{-\frac{\alpha}{2}t}$$

for all $t > 0$, $|\xi| \le 2$, $l = 0,1,...,N$, $j = 1,2,...,n$. Denote $m = [\omega]$, $\varrho = \omega - m$, then for the fractional derivative $\partial_{\xi_j}^\omega$ we can write

$$\begin{aligned}\partial_{\xi_j}^\omega(\psi\phi) &= \sum_{k=1}^m\left(\partial_{\xi_j}^k\psi\right)\partial_{\xi_j}^{\omega-k}\phi \\ &+ C\sum_{k=1}^m\int_0^\infty\left(\partial_{\xi_j}^k\psi(\xi) - \partial_{\xi_j}^k\psi\left(\breve{\xi}(\eta)\right)\right)\partial_{\xi_j}^{m-k}\phi\left(\breve{\xi}(\eta)\right)\eta^{-1-\varrho}d\eta,\end{aligned}$$

where $\breve{\xi}(\eta) \equiv (\xi_1, ..., \xi_j + \eta, ..., \xi_n)$. Then we have

$$\left\|\partial_{\xi_j}^{\omega}(\psi\phi)\right\|_{\mathbf{L}^2} \leq C\sum_{k=1}^{m}\left\|\left(\partial_{\xi_j}^{k}\psi\right)\partial_{\xi_j}^{\omega-k}\phi\right\|_{\mathbf{L}^2}$$
$$+ C\sum_{k=1}^{m}\left\|\sum_{k=1}^{m}\int_0^{\infty}\left(\partial_{\xi_j}^{k}\psi(\xi) - \partial_{\xi_j}^{k}\psi\left(\breve{\xi}(\eta)\right)\right)\partial_{\xi_j}^{m-k}\phi\left(\breve{\xi}(\eta)\right)\eta^{-1-\varrho}d\eta y\right\|_{\mathbf{L}^2}$$
$$\leq C\sum_{k=1}^{m}\left\|\partial_{\xi_j}^{k}\psi\right\|_{\mathbf{L}^\infty}\left\|\partial_{\xi_j}^{\omega-k}\phi\right\|_{\mathbf{L}^2} + C\sum_{k=1}^{m}\left\|\partial_{\xi_j}^{k+1}\psi\right\|_{\mathbf{L}^\infty}\left\|\partial_{\xi_j}^{m-k}\phi\right\|_{\mathbf{L}^2}$$
$$\leq C\left\|\langle i\nabla\rangle^{[\omega]+1}\psi\right\|_{\mathbf{L}^\infty}\left\|\langle i\nabla\rangle^{\omega}\phi\right\|_{\mathbf{L}^2}.$$

Therefore we get

$$\left\|\partial_{\xi_j}^{\omega}\left(\langle\xi\rangle^{\mu}S(t,\xi)\chi_2(\xi)\right)\left(\langle\xi\rangle^{-\mu}\widehat{\phi}(\xi)\right)\right\|_{\mathbf{L}^2}$$
$$\leq Ce^{-\frac{\alpha}{2}t}\left\|\langle i\nabla\rangle^{\omega}\langle\xi\rangle^{-\mu}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2} \leq Ce^{-\frac{\alpha}{2}t}\left\|\langle\cdot\rangle^{\omega}\langle i\nabla\rangle^{-\mu}\phi\right\|_{\mathbf{L}^2}. \tag{1.72}$$

Now estimates (1.71) and (1.72) yield the second estimate of the lemma. Lemma 1.35 is proved.

To find the large time asymptotic behavior of the Green operator we assume that the symbol $S(t,\xi)$ has the following asymptotic representation in the origin

$$S(t,\xi) = e^{-tL_0(\xi)} + O\left(|\xi|^{\gamma}\langle\xi\rangle^{-\mu}e^{-\alpha t\{\xi\}^{\delta}}\right) \tag{1.73}$$

for $\xi \to 0$ with some $\gamma > 0$. We assume that the symbol $L_0(\xi)$ is homogeneous of order δ and satisfies estimate (1.43) with $\nu = \delta > 0$. Also we define

$$\mathcal{G}_0(t)\phi = \overline{\mathcal{F}}_{\xi\to x}e^{-tL_0(\xi)}\hat{\phi}(\xi) = t^{-\frac{1}{\delta}}\int_{\mathbf{R}^n}G_0\left((x-y)t^{-\frac{1}{\delta}}\right)\phi(y)\,dy,$$

where the kernel

$$G_0(x) = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-L_0(\xi)}\right).$$

Denote the mean value

$$\theta = (2\pi)^{-\frac{n}{2}}\int_{\mathbf{R}^n}\phi(x)\,dx.$$

Lemma 1.37. *Let the symbol $S(t,\xi)$ satisfy conditions (1.67) and (1.73). Then the estimate is true*

$$\left\||\nabla|^{\rho}\left(\mathcal{G}(t)\phi - \theta G_0(t)\right)\right\|_{\mathbf{L}^2} \leq Ct^{-\frac{\rho+\gamma}{\delta}-\frac{n}{\delta}\left(\frac{1}{q}-\frac{1}{2}\right)}\left\|\langle i\nabla\rangle^{-\mu}\phi\right\|_{\mathbf{L}^q}$$
$$+ Ct^{-\frac{\rho+1}{\delta}-\frac{n}{\delta}\left(\frac{1}{q}-\frac{1}{2}\right)}\left\|\langle\cdot\rangle\langle i\nabla\rangle^{-\mu}\phi\right\|_{\mathbf{L}^q} + Ce^{-\alpha t}\left\||\nabla|^{\rho}\langle i\nabla\rangle^{-\mu}\phi\right\|_{\mathbf{L}^2}$$

for all $t \geq 1$, where $1 \leq q \leq 2$, $\rho \geq 0$, provided that the right-hand side is finite.

Proof. By (1.73) we obtain

$$\left\| |\xi|^{\rho} \langle\xi\rangle^{\mu} \left(S(t,\xi) - e^{-tL_0(\xi)} \right) \right\|_{\mathbf{L}^r(|\xi|\le 1)} \le C \langle t\rangle^{-\frac{\rho+\gamma}{\delta} - \frac{n}{\delta r}}$$

and in view of (1.67) we have

$$\left\| \langle\xi\rangle^{\mu} \left(S(t,\xi) + e^{-tL_0(\xi)} \right) \right\|_{\mathbf{L}^{\infty}(|\xi|\ge 1)} \le Ce^{-\alpha t}$$

for all $t \ge 1$. Hence we get for $1 \le q \le 2$

$$\begin{aligned}
& \| |\nabla|^{\rho} (\mathcal{G}(t)\phi - \theta G_0(t)) \|_{\mathbf{L}^2} \\
& \le C \left\| |\xi|^{\rho} \langle\xi\rangle^{\mu} \left(S(t,\xi) - e^{-tL_0(\xi)} \right) \langle\xi\rangle^{-\mu} \widehat{\phi}(\xi) \right\|_{\mathbf{L}^2(|\xi|\le 1)} \\
& + C \left\| |\xi|^{\rho} e^{-\alpha t|\xi|^{\delta}} \left(\widehat{\phi}(\xi) - \widehat{\phi}(0) \right) \right\|_{\mathbf{L}^2(|\xi|\le 1)} \\
& + \left\| \langle\xi\rangle^{\mu} \left(S(t,\xi) + e^{-tL_0(\xi)} \right) |\xi|^{\rho} \langle\xi\rangle^{-\mu} \widehat{\phi}(\xi) \right\|_{\mathbf{L}^2(|\xi|\ge 1)} \\
& \le C \left\| |\xi|^{\rho} \langle\xi\rangle^{\mu} \left(S(t,\xi) - e^{-tL_0(\xi)} \right) \right\|_{\mathbf{L}^{\frac{2q}{2-q}}(|\xi|\le 1)} \left\| \langle\xi\rangle^{-\mu} \widehat{\phi} \right\|_{\mathbf{L}^{\frac{q}{q-1}}(|\xi|\le 1)} \\
& + C \left\| |\xi|^{\rho+1} e^{-\alpha t|\xi|^{\delta}} \right\|_{\mathbf{L}^{\frac{2q}{2-q}}(|\xi|\le 1)} \left\| \partial_{\xi_j} \langle\xi\rangle^{-\mu} \widehat{\phi} \right\|_{\mathbf{L}^{\frac{q}{q-1}}(|\xi|\le 1)} \\
& + C \| \langle\xi\rangle^{\mu} S(t,\xi) \|_{\mathbf{L}^{\infty}(|\xi|\ge 1)} \left\| |\xi|^{\rho} \langle\xi\rangle^{-\mu} \widehat{\phi} \right\|_{\mathbf{L}^2(|\xi|\ge 1)} \\
& \le Ct^{-\frac{\rho+\gamma}{\delta} - \frac{n}{\delta}\left(\frac{1}{q} - \frac{1}{2}\right)} \left\| \langle i\nabla\rangle^{-\mu} \phi \right\|_{\mathbf{L}^q} \\
& + Ct^{-\frac{\rho+1}{\delta} - \frac{n}{\delta}\left(\frac{1}{q} - \frac{1}{2}\right)} \left\| \langle\cdot\rangle \langle i\nabla\rangle^{-\mu} \phi \right\|_{\mathbf{L}^q} + Ce^{-\alpha t} \left\| |\nabla|^{\rho} \langle i\nabla\rangle^{-\mu} \phi \right\|_{\mathbf{L}^2}.
\end{aligned}$$

Therefore the estimate of the lemma is true. Lemma 1.37 is proved.

1.5.3 Estimates in Fourier spaces

We now obtain estimates for the Green operator $\mathcal{G}(t)$ in the norms

$$\begin{aligned}
\|\varphi\|_{\mathbf{A}^{\rho,p}} &= \| |\xi|^{\rho} \widehat{\varphi}(\xi) \|_{\mathbf{L}^p_{\xi}(|\xi|\le 1)}, \\
\|\varphi\|_{\mathbf{B}^{s,p}} &= \| |\xi|^{s} \widehat{\varphi}(\xi) \|_{\mathbf{L}^p_{\xi}(|\xi|\ge 1)}, \\
\|\varphi\|_{\mathbf{\Gamma}^{\rho,s}_{\omega}} &= \left\| \left\| \mathcal{D}^{\omega}_{\eta} \{\xi\}^{\rho} \langle\xi\rangle^{s} \widehat{\varphi}(\xi) \right\|_{\mathbf{L}^{\infty}_{\eta}(|\eta|\le 1)} \right\|_{\mathbf{L}^{\infty}_{\xi}},
\end{aligned}$$

where $\rho, s \in \mathbf{R}$, $\omega \in (0,1)$, $1 \le p \le \infty$, and

$$\mathcal{D}^{\omega}_{\eta} \phi(\xi) \equiv |\eta|^{-\omega} |\phi(\xi+\eta) - \phi(\xi)|.$$

The norm $\mathbf{A}^{\rho,p}$ is responsible for the large time asymptotic properties of solutions and the norm $\mathbf{B}^{s,p}$ describes the regularity of solutions. These norms allow us to relax condition $\nu > 0$.

Lemma 1.38. *Let the symbol $L(\xi)$ satisfy conditions (1.43) and (1.44) with $\nu \geq 0$. Then the following estimates are valid for all $t > 0$ provided that the right-hand sides are finite*

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{A}^{\rho,p}} \leq C\langle t\rangle^{-\frac{\rho}{\delta}-\frac{1}{\delta}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\varphi\|_{\mathbf{A}^{0,q}}$$

for $\rho \geq 0$ if $p = q$ and for $\rho + \frac{1}{p} - \frac{1}{q} > 0$ if $1 \leq p < q \leq \infty$,

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{B}^{s\nu,p}} \leq Ce^{-\frac{\alpha}{2}t}\{t\}^{-s}\|\varphi\|_{\mathbf{B}^{0,p}}$$

for $1 \leq p \leq \infty$, $s \geq 0$ ($s = 0$ if $\nu = 0$),

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{\Gamma}_{\omega}^{\rho,s\nu}} \leq C\langle t\rangle^{-\frac{\rho}{\delta}}\{t\}^{-s}\|\varphi\|_{\mathbf{\Gamma}_{\omega}^{0,0}}$$
$$+ C\langle t\rangle^{\frac{\omega-\rho}{\delta}}\{t\}^{-s}\left(\|\varphi\|_{\mathbf{A}^{0,\infty}} + \|\varphi\|_{\mathbf{B}^{0,\infty}}\right)$$

for $s \geq 0$ ($s = 0$ if $\nu = 0$), $\rho \geq 0$, $\omega < \delta$ if $\rho = 0$ and $\omega < \rho$ if $\rho > 0$. Moreover if $\widehat{\varphi}(0) = 0$, then

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{A}^{\rho,p}} \leq C\langle t\rangle^{-\frac{1}{\delta}\left(\rho+\omega+\frac{1}{p}\right)}\|\varphi\|_{\mathbf{\Gamma}_{\omega}^{0,0}},$$

for $\rho + \omega \geq 0$ if $p = \infty$ and for $\rho + \omega + \frac{1}{p} > 0$ if $1 \leq p < \infty$ and

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{\Gamma}_{\omega}^{\rho,s\nu}} \leq C\langle t\rangle^{-\frac{\rho}{\delta}}\{t\}^{-s}\|\varphi\|_{\mathbf{\Gamma}_{\omega}^{0,0}}$$

where $s \geq 0$ ($s = 0$ if $\nu = 0$), $\rho \geq 0$, $\omega < \delta$ if $\rho = 0$ and $\omega < \rho$ if $\rho > 0$.

Proof. By virtue of dissipation condition (1.43) we have

$$|\xi|^{\rho}\left|e^{-tL(\xi)}\right| \leq C|\xi|^{\rho}e^{-t\alpha|\xi|^{\delta}} \leq C\langle t\rangle^{-\frac{\rho}{\delta}}e^{-t\frac{\alpha}{2}|\xi|^{\delta}} \tag{1.74}$$

for all $t > 0$, $|\xi| \leq 1$, where $\rho \geq 0$. By (1.74) changing the variable $\eta = \xi\langle t\rangle^{\frac{1}{\delta}}$ we get in the case $1 \leq r < \infty$, $\rho + \frac{1}{r} > 0$

$$\left\||\xi|^{\rho}e^{-tL(\xi)}\right\|_{\mathbf{L}_{\xi}^{r}(|\xi|\leq 1)} = \left(\int_{|\xi|\leq 1}|\xi|^{\rho r}e^{-rt\alpha|\xi|^{\delta}}d\xi\right)^{\frac{1}{r}}$$
$$\leq C\langle t\rangle^{-\frac{\rho}{\delta}-\frac{1}{\delta r}}\left(\int_{|\eta|\leq\langle t\rangle^{\frac{1}{\delta}}}|\eta|^{\rho r}e^{-r\alpha|\eta|^{\delta}}d\eta\right)^{\frac{1}{r}} \leq C\langle t\rangle^{-\frac{\rho}{\delta}-\frac{1}{\delta r}}, \tag{1.75}$$

and in the case $r = \infty$, $\rho \geq 0$

$$\left\||\xi|^{\rho}e^{-tL(\xi)}\right\|_{\mathbf{L}_{\xi}^{\infty}(|\xi|\leq 1)} \leq C\langle t\rangle^{-\frac{\rho}{\delta}}\left\|e^{-t\frac{\alpha}{2}|\xi|^{\delta}}\right\|_{\mathbf{L}_{\xi}^{\infty}(|\xi|\leq 1)} \leq C\langle t\rangle^{-\frac{\rho}{\delta}}; \tag{1.76}$$

hence by the Hölder inequality we get

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{A}^{\rho,p}} = \left\||\xi|^{\rho} e^{-tL(\xi)}\widehat{\varphi}(\xi)\right\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)}$$

$$\leq C\left\||\xi|^{\rho} e^{-tL(\xi)}\right\|_{\mathbf{L}_\xi^{\frac{pq}{q-p}}(|\xi|\leq 1)} \|\widehat{\varphi}\|_{\mathbf{L}^q_\xi(|\xi|\leq 1)} \leq C\langle t\rangle^{-\frac{\rho}{\delta}-\frac{1}{\delta}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\varphi\|_{\mathbf{A}^{0,q}}$$

for $\rho + \frac{1}{p} - \frac{1}{q} > 0$ if $1 \leq p < q \leq \infty$ and for $\rho \geq 0$ if $p = q$.

To prove the second estimate of the lemma we write by condition (1.43)

$$|\xi|^{s\nu}\left|e^{-tL(\xi)}\right| \leq C|\xi|^{s\nu} e^{-t\alpha|\xi|^{\nu}} \leq Ce^{-\frac{\alpha}{2}t}\{t\}^{-s}$$

for all $t > 0$, $|\xi| \geq 1$, where $s \geq 0$ ($s = 0$ if $\nu = 0$). Therefore

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{B}^{s\nu,p}} \leq C\left\||\xi|^{s\nu} e^{-tL(\xi)}\right\|_{\mathbf{L}^\infty_\xi(|\xi|\geq 1)} \|\widehat{\varphi}\|_{\mathbf{L}^p_\xi(|\xi|\geq 1)}$$

$$\leq Ce^{-\frac{\alpha}{2}t}\{t\}^{-s}\|\varphi\|_{\mathbf{B}^{0,p}}.$$

Thus the second estimate of the lemma is true.

We now prove the third estimate of the lemma. We have

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{\Gamma}^{\rho,s\nu}_\omega} = \left\|\left\|\mathcal{D}^\omega_\eta\{\xi\}^{\rho}\langle\xi\rangle^{s\nu} e^{-tL(\xi)}\widehat{\varphi}(\xi)\right\|_{\mathbf{L}^\infty_\eta(|\eta|\leq 1)}\right\|_{\mathbf{L}^\infty_\xi}$$

$$\leq \left\|\{\xi\}^{\rho}\langle\xi\rangle^{s\nu} e^{-L(\xi)t}\left\|\mathcal{D}^\omega_\eta\widehat{\varphi}(\xi)\right\|_{\mathbf{L}^\infty_\eta(|\eta|\leq 1)}\right\|_{\mathbf{L}^\infty_\xi}$$

$$+\left\|\left\|\widehat{\varphi}(\xi-\eta)\mathcal{D}^\omega_\eta\{\xi\}^{\rho}\langle\xi\rangle^{s\nu} e^{-L(\xi)t}\right\|_{\mathbf{L}^\infty_\eta(|\eta|\leq 1)}\right\|_{\mathbf{L}^\infty_\xi}. \tag{1.77}$$

Via the first and second estimates of the lemma we have for the first summand in the right-hand side of (1.77)

$$\left\|\{\xi\}^{\rho}\langle\xi\rangle^{s\nu} e^{-L(\xi)t}\left\|\mathcal{D}^\omega_\eta\widehat{\varphi}(\xi)\right\|_{\mathbf{L}^\infty_\eta(|\eta|\leq 1)}\right\|_{\mathbf{L}^\infty_\xi}$$

$$\leq C\langle t\rangle^{-\frac{\rho}{\delta}}\left\|\left\|\mathcal{D}^\omega_\eta\widehat{\varphi}(\xi)\right\|_{\mathbf{L}^\infty_\eta(|\eta|\leq 1)}\right\|_{\mathbf{L}^\infty_\xi(|\xi|\leq 1)}$$

$$+Ce^{-\frac{\alpha}{2}t}\{t\}^{-s}\left\|\left\|\mathcal{D}^\omega_\eta\widehat{\varphi}(\xi)\right\|_{\mathbf{L}^\infty_\eta(|\eta|\leq 1)}\right\|_{\mathbf{L}^\infty_\xi(|\xi|\geq 1)}$$

$$\leq C\langle t\rangle^{-\frac{\rho}{\delta}}\{t\}^{-s}\|\varphi\|_{\mathbf{\Gamma}^{0,0}_\omega}. \tag{1.78}$$

Consequently for the second summand in the right-hand side of (1.77) we find

$$\left\|\left\|\widehat{\varphi}(\xi-\eta)\mathcal{D}^\omega_\eta\{\xi\}^{\rho}\langle\xi\rangle^{s\nu} e^{-L(\xi)t}\right\|_{\mathbf{L}^\infty_\eta(|\eta|\leq 1)}\right\|_{\mathbf{L}^\infty_\xi}$$

$$\leq C\|\widehat{\varphi}\|_{\mathbf{L}^\infty}\left\|\left\|\mathcal{D}^\omega_\eta\{\xi\}^{\rho}\langle\xi\rangle^{s\nu} e^{-L(\xi)t}\right\|_{\mathbf{L}^\infty_\eta(|\eta|\leq 1)}\right\|_{\mathbf{L}^\infty_\xi}. \tag{1.79}$$

In the case $|\xi| \le 2$, $|\eta| \le 1$, by condition (1.44) we have for all $0 < t \le 1$

$$\mathcal{D}_\eta^\omega \{\xi\}^\rho \langle\xi\rangle^{s\nu} e^{-L(\xi)t} = |\eta|^{-\omega} \left| \int_\xi^{\xi-\eta} \frac{\partial}{\partial y} \left(\{y\}^\rho \langle y\rangle^{s\nu} e^{-L(y)t} \right) dy \right|$$
$$\le C |\eta|^{-\omega} \left| \int_\xi^{\xi-\eta} \left(|y|^{\delta+\rho-1} + \rho |y|^{\rho-1} \right) dy \right| \le C$$

and then for all $t \ge 1$ changing the variables $\xi = t^{-\frac{1}{\delta}} \widetilde{\xi}$, $\eta = t^{-\frac{1}{\delta}} \widetilde{\eta}$, and $y = t^{-\frac{1}{\delta}} \widetilde{y}$ we get

$$\mathcal{D}_\eta^\omega \{\xi\}^\rho \langle\xi\rangle^{s\nu} e^{-L(\xi)t} \le C |\eta|^{-\omega} \left| \int_\xi^{\xi-\eta} e^{-Ct|y|^\delta} \left(t |y|^{\delta+\rho-1} + \rho |y|^{\rho-1} \right) dy \right|$$
$$\le C t^{\frac{\omega-\rho}{\delta}} |\widetilde{\eta}|^{-\omega} \left| \int_{\widetilde{\xi}}^{\widetilde{\xi}-\widetilde{\eta}} e^{-C|\widetilde{y}|^\delta} \left(|\widetilde{y}|^{\delta+\rho-1} + \rho |\widetilde{y}|^{\rho-1} \right) d\widetilde{y} \right| \le C t^{\frac{\omega-\rho}{\delta}}$$

where $\omega < \delta$ if $\rho = 0$ and $\omega < \rho$ if $\rho > 0$. And in the case $|\xi| \ge 2$, $|\eta| \le 1$, by condition (1.44) we find

$$\mathcal{D}_\eta^\omega \{\xi\}^\rho \langle\xi\rangle^{s\nu} e^{-L(\xi)t} \le C e^{-\frac{\alpha}{2}t} |\eta|^{-\omega} \left| \int_\xi^{\xi-\eta} e^{-Ct|y|^\nu} |y|^{s\nu-1} dy \right|$$
$$\le C e^{-\frac{\alpha}{2}t} \{t\}^{-s} |\eta|^{-\omega} \left| \int_\xi^{\xi-\eta} \frac{dy}{|y|} \right| \le C e^{-\frac{\alpha}{2}t} \{t\}^{-s} \tag{1.80}$$

for all $t > 0$.Thus we get the estimate

$$\left\| \left\| \mathcal{D}_\eta^\omega \{\xi\}^\rho \langle\xi\rangle^{s\nu} e^{-L(\xi)t} \right\|_{\mathbf{L}_\eta^\infty(|\eta|\le 1)} \right\|_{\mathbf{L}_\xi^\infty} \le C \{t\}^{-s} \langle t\rangle^{\frac{\omega-\rho}{\delta}} \tag{1.81}$$

for all $t > 0$. The substitution of (1.78), (1.79) and (1.81) into (1.77) yields the third estimate of the lemma.

Now we suppose that $\widehat{\varphi}(0) = 0$, then via (1.75) and (1.76) we obtain the fourth estimate of the lemma

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{A}^{\rho,p}} = \left\| |\xi|^\rho e^{-tL(\xi)} \left(\widehat{\varphi}(\xi) - \widehat{\varphi}(0) \right) \right\|_{\mathbf{L}_\xi^p(|\xi|\le 1)}$$
$$\le \left\| |\xi|^{\rho+\omega} e^{-tL(\xi)} \mathcal{D}_\xi^\omega \widehat{\varphi}(0) \right\|_{\mathbf{L}_\xi^p(|\xi|\le 1)}$$
$$\le C \left\| |\xi|^{\rho+\omega} e^{-tL(\xi)} \right\|_{\mathbf{L}_\xi^p(|\xi|\le 1)} \left\| \mathcal{D}_\eta^\omega \widehat{\varphi}(0) \right\|_{\mathbf{L}_\xi^\infty(|\eta|\le 1)}$$
$$\le C \langle t\rangle^{-\frac{1}{\delta}\left(\rho+\omega+\frac{1}{p}\right)} \|\varphi\|_{\mathbf{\Gamma}_\omega^{0,0}}$$

for $\rho + \omega \ge 0$ if $p = \infty$ and for $\rho + \omega + \frac{1}{p} > 0$ if $1 \le p < \infty$.

To prove the last estimate in view of the condition $\widehat{\varphi}(0) = 0$, instead of (1.79) we write

$$\left\| \left\| \widehat{\varphi}(\xi - \eta) \mathcal{D}_\eta^\omega \{\xi\}^\rho \langle\xi\rangle^{s\nu} e^{-L(\xi)t} \right\|_{\mathbf{L}_\eta^\infty(|\eta|\leq 1)} \right\|_{\mathbf{L}_\xi^\infty}$$
$$\leq C \|\varphi\|_{\mathbf{\Gamma}_\omega^{0,0}} \left\| \left\| |\xi - \eta|^\omega \mathcal{D}_\eta^\omega \{\xi\}^\rho \langle\xi\rangle^{s\nu} e^{-L(\xi)t} \right\|_{\mathbf{L}_\eta^\infty(|\eta|\leq 1)} \right\|_{\mathbf{L}_\xi^\infty(|\xi|\leq 2)}$$
$$+ \|\widehat{\varphi}\|_{\mathbf{L}^\infty} \left\| \left\| \mathcal{D}_\eta^\omega \{\xi\}^\rho \langle\xi\rangle^{s\nu} e^{-L(\xi)t} \right\|_{\mathbf{L}_\eta^\infty(|\eta|\leq 1)} \right\|_{\mathbf{L}_\xi^\infty(|\xi|\geq 2)}. \quad (1.82)$$

Using (1.81) with ρ replaced by $\rho + \omega$ we find

$$\left\| \left\| |\xi - \eta|^\omega \mathcal{D}_\eta^\omega \{\xi\}^\rho \langle\xi\rangle^{s\nu} e^{-L(\xi)t} \right\|_{\mathbf{L}_\eta^\infty(|\eta|\leq 1)} \right\|_{\mathbf{L}_\xi^\infty(|\xi|\leq 2)}$$
$$\leq \left\| \left\| \mathcal{D}_\eta^\omega \{\xi\}^{\rho+\omega} \langle\xi\rangle^{s\nu} e^{-L(\xi)t} \right\|_{\mathbf{L}_\eta^\infty(|\eta|\leq 1)} \right\|_{\mathbf{L}_\xi^\infty(|\xi|\leq 2)}$$
$$+ \left\| \{\xi\}^\rho \langle\xi\rangle^{s\nu} e^{-L(\xi)t} \left\| \left| |\xi - \eta|^\omega - |\xi|^\omega \right| |\eta|^{-\omega} \right\|_{\mathbf{L}_\eta^\infty(|\eta|\leq 1)} \right\|_{\mathbf{L}_\xi^\infty(|\xi|\leq 2)}$$
$$\leq \left\| \left\| \mathcal{D}_\eta^\omega \{\xi\}^{\rho+\omega} \langle\xi\rangle^{s\nu} e^{-L(\xi)t} \right\|_{\mathbf{L}_\eta^\infty(|\eta|\leq 1)} \right\|_{\mathbf{L}_\xi^\infty}$$
$$+ C \left\| \{\xi\}^\rho \langle\xi\rangle^{s\nu} e^{-L(\xi)t} \right\|_{\mathbf{L}_\xi^\infty} \leq C \{t\}^{-s} \langle t\rangle^{-\frac{\rho}{\delta}}. \quad (1.83)$$

The substitution of (1.78), (1.80), (1.82) and (1.83) into (1.77) yields the last estimate of the lemma which is then proved.

In the next lemma we give the large time asymptotics for the Green operator $\mathcal{G}(t)$.

Lemma 1.39. *Let the symbol $L(\xi)$ satisfy conditions (1.43), (1.44) and (1.47) with $\nu \geq 0$. Then the estimate is true*

$$\left\| \mathcal{G}(t)\phi - t^{-\frac{1}{\delta}} \widehat{\phi}(0) G_0\left(t^{-\frac{1}{\delta}}(\cdot)\right) \right\|_{\mathbf{A}^{\rho,p}}$$
$$\leq C \langle t\rangle^{-\frac{\rho+\mu}{\delta} - \frac{1}{\delta p}} \left(\|\phi\|_{\mathbf{\Gamma}_\omega^{0,0}} + \|\phi\|_{\mathbf{A}^{0,\infty}} \right)$$

for all $t \geq 1$, where $\vartheta = \widehat{\phi}(0)$, $\mu = \min(\gamma, \omega)$, $1 \leq p \leq \infty$, $\rho + \omega \geq 0$ if $p = \infty$ and $\rho + \omega + \frac{1}{p} > 0$ if $1 \leq p < \infty$.

Proof. By virtue of dissipation condition (1.43) and asymptotics (1.47) we have

$$\left| e^{-tL(\xi)} - e^{-tL_0(\xi)} \right| \leq Ct \left| L(\xi) - L_0(\xi) \right| e^{-Ct|\xi|^\delta} \leq Ct |\xi|^{\delta+\gamma} e^{-Ct|\xi|^\delta}$$

for all $t > 0$ and $|\xi| \leq 1$. Thus

$$\|(\mathcal{G}(t) - \mathcal{G}_0(t))\phi\|_{\mathbf{A}^{\rho,p}} = \left\| |\xi|^{\rho} \left(e^{-tL(\xi)} - e^{-tL_0(\xi)}\right) \widehat{\phi}(\xi) \right\|_{\mathbf{L}^p_{\xi}(|\xi|\leq 1)}$$
$$\leq Ct \left\| |\xi|^{\delta+\rho+\gamma} e^{-Ct|\xi|^{\delta}} \right\|_{\mathbf{L}^p_{\xi}(|\xi|\leq 1)} \left\| \widehat{\phi}(\xi) \right\|_{\mathbf{L}^{\infty}_{\xi}(|\xi|\leq 1)} \leq C \langle t \rangle^{-\frac{\rho+\gamma}{\delta} - \frac{1}{\delta p}} \|\phi\|_{\mathbf{A}^{0,\infty}}.$$

By (1.53) and (1.54), we get

$$\begin{aligned}
&\left\| \mathcal{G}_0(t)\phi - t^{-\frac{1}{\delta}} \widehat{\phi}(0) G_0\left(t^{-\frac{1}{\delta}}(\cdot)\right) \right\|_{\mathbf{A}^{\rho,p}} \\
&= \left\| |\xi|^{\rho+\omega} e^{-tL_0(\xi)} |\xi|^{-\omega} \left(\widehat{\phi}(\xi) - \widehat{\phi}(0)\right) \right\|_{\mathbf{L}^p_{\xi}(|\xi|\leq 1)} \\
&\leq C \left\| |\xi|^{\rho+\omega} e^{-tL_0(\xi)} \right\|_{\mathbf{L}^p_{\xi}(|\xi|\leq 1)} \left\| \mathcal{D}^{\omega}_{\xi} \widehat{\phi}(0) \right\|_{\mathbf{L}^{\infty}_{\xi}(|\xi|\leq 1)} \\
&\leq C \langle t \rangle^{-\frac{\rho+\omega}{\delta} - \frac{1}{\delta p}} \|\phi\|_{\mathbf{\Gamma}^{0,0}_{\omega}}.
\end{aligned}$$

Therefore the estimate of the lemma is valid and Lemma 1.39 is proved.

1.5.4 Estimates for large x and t

Consider the Green operator $\mathcal{G}(t)$ of the form

$$\mathcal{G}(t)\phi = \overline{\mathcal{F}}_{\eta\to x} e^{-|\eta|^{\alpha} t} \hat{\phi}(\eta) = t^{-\frac{1}{\alpha}} \int_{\mathbf{R}} G\left(t^{-\frac{1}{\alpha}}(x-y)\right) \phi(y) dy \tag{1.84}$$

with a kernel $G(x) = \overline{\mathcal{F}}_{\eta\to x}\left(e^{-|\eta|^{\alpha}}\right)$.

In the next lemma we collect some estimates of the Green operator $\mathcal{G}(t)$ in the weighted Lebesgue norms $\|\phi\|_{\mathbf{L}^{p,\beta}}$, where $\beta \geq 0$, $1 \leq p \leq \infty$.

Lemma 1.40. *Suppose that the function $\phi \in \mathbf{L}^{\infty,\beta}(\mathbf{R})$, where $\beta \in (1, 1+\alpha]$. Then the estimates*

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^{\infty}} \leq \min\left(t^{-\frac{1}{\alpha}} \|\phi\|_{\mathbf{L}^1}, \|\phi\|_{\mathbf{L}^{\infty}}\right)$$

and

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^{\infty,\beta}} \leq C \langle t \rangle^{\frac{\beta-1}{\alpha}} \|\phi\|_{\mathbf{L}^1} + C \|\phi\|_{\mathbf{L}^{\infty,\beta}}$$

are valid for all $t > 0$, and $1 < \beta \leq 1+\alpha$. Moreover if $\phi \in \mathbf{L}^{\infty,\beta}(\mathbf{R})$, where $\beta \in (2, 2+\alpha]$, then the estimate

$$\begin{aligned}
&\left\| \left\langle t^{-\frac{1}{\alpha}}(\cdot) \right\rangle^{\beta} \left(\mathcal{G}(t)\phi - \vartheta t^{-\frac{1}{\alpha}} G\left(t^{-\frac{1}{\alpha}}(\cdot)\right)\right) \right\|_{\mathbf{L}^{\infty}} \\
&\leq Ct^{-\frac{2}{\alpha}} \|\phi\|_{\mathbf{L}^{1,1}} + Ct^{-\frac{\beta}{\alpha}} \|\phi\|_{\mathbf{L}^{\infty,\beta}}
\end{aligned}$$

is true for all $t \geq 1$, $2 < \beta \leq 2+\alpha$, where $\vartheta = \int_{\mathbf{R}} \phi(y) dy$. Also the following two estimates are valid

$$\left\| \left| t^{-\frac{1}{\alpha}} (\cdot) \right|^{\beta} \left(\mathcal{G}(t-\tau)\phi(\tau) - \vartheta(\tau)(t-\tau)^{-\frac{1}{\alpha}} G\left((t-\tau)^{-\frac{1}{\alpha}} (\cdot) \right) \right) \right\|_{\mathbf{L}^{\infty}}$$
$$\leq Ct^{-\frac{2}{\alpha}} \|\phi(\tau)\|_{\mathbf{L}^{1,1}} + Ct^{-\frac{\beta}{\alpha}} \|\phi(\tau)\|_{\mathbf{L}^{\infty,\beta}}$$

for all $0 < \tau < \frac{t}{2}$ *and*

$$\left\| \left| t^{-\frac{1}{\alpha}} (\cdot) \right|^{\beta} \mathcal{G}(t-\tau)\phi(\tau) \right\|_{\mathbf{L}^{\infty}} \leq C \|\phi(\tau)\|_{\mathbf{L}^{\infty}} + C\tau^{-\frac{\beta}{\alpha}} \|\phi(\tau)\|_{\mathbf{L}^{\infty,\beta}}$$

for all $\frac{t}{2} \leq \tau < t$, *where* $2 < \beta \leq 2 + \alpha$, *and* $\vartheta(\tau) = \int_{\mathbf{R}} \phi(\tau, y) dy$.

Proof. We have the estimate

$$\|G\|_{\mathbf{L}^{\infty,1+\alpha}} \leq C. \tag{1.85}$$

In the same manner we can obtain the estimate for the derivative

$$\|G'\|_{\mathbf{L}^{\infty,2+\alpha}} \leq C. \tag{1.86}$$

By virtue of (1.85) we find

$$t^{-\frac{1}{\alpha}} \left\| G\left(t^{-\frac{1}{\alpha}} (\cdot) \right) \right\|_{\mathbf{L}^1} = \|G\|_{\mathbf{L}^1} \leq C \left\| \langle \cdot \rangle^{-1-\alpha} \right\|_{\mathbf{L}^1} \leq C,$$

and

$$t^{-\frac{\beta}{\alpha}} \left\| |\cdot|^{\beta} G\left(t^{-\frac{1}{\alpha}} (\cdot) \right) \right\|_{\mathbf{L}^{\infty}} = \left\| |\cdot|^{\beta} G \right\|_{\mathbf{L}^{\infty}} \leq C \left\| \langle \cdot \rangle^{\beta-1-\alpha} \right\|_{\mathbf{L}^{\infty}} \leq C,$$

for $\beta \in [0, 1+\alpha]$. Hence by the Young inequality for convolutions we obtain the first two estimates of the lemma

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^{\infty}} \leq \left\| t^{-\frac{1}{\alpha}} G\left(t^{-\frac{1}{\alpha}} (\cdot) \right) \right\|_{\mathbf{L}^1} \|\phi\|_{\mathbf{L}^{\infty}} \leq C \|\phi\|_{\mathbf{L}^{\infty}},$$

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^{\infty}} \leq \left\| t^{-\frac{1}{\alpha}} G\left(t^{-\frac{1}{\alpha}} (\cdot) \right) \right\|_{\mathbf{L}^{\infty}} \|\phi\|_{\mathbf{L}^1} \leq Ct^{-\frac{1}{\alpha}} \|\phi\|_{\mathbf{L}^1}$$

and

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^{\infty,\beta}} \leq C \left\| |\cdot|^{\beta} t^{-\frac{1}{\alpha}} G\left(t^{-\frac{1}{\alpha}} (\cdot) \right) \right\|_{\mathbf{L}^{\infty}} \|\phi\|_{\mathbf{L}^1}$$
$$+ C \left\| t^{-\frac{1}{\alpha}} G\left(t^{-\frac{1}{\alpha}} (\cdot) \right) \right\|_{\mathbf{L}^1} \|\phi\|_{\mathbf{L}^{\infty,\beta}} \leq C \langle t \rangle^{\frac{\beta-1}{\alpha}} \|\phi\|_{\mathbf{L}^1} + C \|\phi\|_{\mathbf{L}^{\infty,\beta}}$$

for all $t > 0$, where $0 \leq \beta \leq 1 + \alpha$.

To prove the third estimate we write

$$\left| t^{-\frac{1}{\alpha}} x \right|^{\beta} \left(\mathcal{G}(t)\phi - \vartheta t^{-\frac{1}{\alpha}} G\left(t^{-\frac{1}{\alpha}} x \right) \right)$$
$$= t^{-\frac{1}{\alpha}} \int_{\mathbf{R}} \left| t^{-\frac{1}{\alpha}} x \right|^{\beta} \left(G\left(t^{-\frac{1}{\alpha}} (x-y) \right) - G\left(t^{-\frac{1}{\alpha}} x \right) \right) \phi(y) dy$$

for any $\beta \geq 0$. By applying the Lagrange Theorem, in view of (1.86), we obtain

$$\begin{aligned} |\xi|^{\beta} |G(\xi-\eta) - G(\xi)| &\leq |\xi|^{\beta} |\eta| |G'(\xi^*)| \\ &\leq C |\xi|^{\beta} |\eta| \langle \xi^* \rangle^{-2-\alpha} \leq C |\eta| \langle \xi \rangle^{\beta-2-\alpha} \end{aligned}$$

for all $\xi, \eta \in \mathbf{R}$, $|\eta| \leq \frac{|\xi|}{2}$. For all $|\eta| \geq \frac{|\xi|}{2}$ we get

$$|\xi|^{\beta} |G(\xi-\eta) - G(\xi)| \leq C |\eta|^{\beta} \left(\langle \xi - \eta \rangle^{-1-\alpha} + \langle \eta \rangle^{-1-\alpha} \right).$$

Thus we see that

$$\begin{aligned} &\left| t^{-\frac{1}{\alpha}} x \right|^{\beta} \left(G\left(t^{-\frac{1}{\alpha}} (x-y) \right) - G\left(t^{-\frac{1}{\alpha}} x \right) \right) \\ &\leq C \left| t^{-\frac{1}{\alpha}} y \right| \left\langle t^{-\frac{1}{\alpha}} x \right\rangle^{\beta-2-\alpha} \\ &+ C \left| t^{-\frac{1}{\alpha}} y \right|^{\beta} \left(\left\langle t^{-\frac{1}{\alpha}} (x-y) \right\rangle^{-1-\alpha} + \left\langle t^{-\frac{1}{\alpha}} \eta \right\rangle^{-1-\alpha} \right) \end{aligned}$$

for all $x, y \in \mathbf{R}$, $t > 0$. Hence by the Young inequality for convolutions

$$\begin{aligned} &\left\| \left| t^{-\frac{1}{\alpha}} (\cdot) \right|^{\beta} \left(\mathcal{G}(t) \phi - \vartheta t^{-\frac{1}{\alpha}} G\left(t^{-\frac{1}{\alpha}} (\cdot) \right) \right) \right\|_{\mathbf{L}^{\infty}} \\ &\leq C t^{-\frac{1}{\alpha}} \left\| \int_{\mathbf{R}} \left\langle t^{-\frac{1}{\alpha}} \cdot \right\rangle^{\beta-2-\alpha} \left| t^{-\frac{1}{\alpha}} y \right| \phi(y) dy \right\|_{\mathbf{L}^{\infty}} \\ &+ C t^{-\frac{1}{\alpha}} \left\| \int_{\mathbf{R}} \left(\left\langle t^{-\frac{1}{\alpha}} (\cdot - y) \right\rangle^{-1-\alpha} + \left\langle t^{-\frac{1}{\alpha}} y \right\rangle^{-1-\alpha} \right) \left| t^{-\frac{1}{\alpha}} y \right|^{\beta} \phi(y) dy \right\|_{\mathbf{L}^{\infty}} \\ &\leq C t^{-\frac{1}{\alpha}} \left\| \left\langle t^{-\frac{1}{\alpha}} (\cdot) \right\rangle^{\beta-2-\alpha} \right\|_{\mathbf{L}^{\infty}} \left\| \left| t^{-\frac{1}{\alpha}} (\cdot) \right| \phi \right\|_{\mathbf{L}^{1}} \\ &+ C t^{-\frac{1}{\alpha}} \left\| \left\langle t^{-\frac{1}{\alpha}} (\cdot) \right\rangle^{-1-\alpha} \right\|_{\mathbf{L}^{1}} \left\| \left| t^{-\frac{1}{\alpha}} (\cdot) \right|^{\beta} \phi \right\|_{\mathbf{L}^{\infty}} \\ &\leq C t^{-\frac{2}{\alpha}} \|\phi\|_{\mathbf{L}^{1,1}} + C t^{-\frac{\beta}{\alpha}} \|\phi\|_{\mathbf{L}^{\infty,\beta}} \leq C t^{-\frac{2}{\alpha}} \|\phi\|_{\mathbf{L}^{\infty,\beta}} \end{aligned}$$

for all $t \geq 1$, where $2 < \beta \leq 2 + \alpha$. On the other hand using estimate

$$|G(\xi-\eta) - G(\xi)| \leq |\eta| |G'(\xi^*)| \leq C |\eta|$$

we have

$$\begin{aligned} &\left\| \mathcal{G}(t) \phi - \vartheta t^{-\frac{1}{\alpha}} G\left(t^{-\frac{1}{\alpha}} (\cdot) \right) \right\|_{\mathbf{L}^{\infty}} \\ &\leq C t^{-\frac{1}{\alpha}} \left\| \int_{\mathbf{R}} \left(G\left(t^{-\frac{1}{\alpha}} (\cdot - y) \right) - G\left(t^{-\frac{1}{\alpha}} x \right) \right) \phi(y) dy \right\|_{\mathbf{L}^{\infty}} \\ &\leq C t^{-\frac{1}{\alpha}} \int_{\mathbf{R}} \left| t^{-\frac{1}{\alpha}} y \right| \phi(y) dy = C t^{-\frac{2}{\alpha}} \| |\cdot| \phi \|_{\mathbf{L}^{1}} \leq C t^{-\frac{2}{\alpha}} \|\phi\|_{\mathbf{L}^{\infty,\beta}}. \end{aligned}$$

Thus the third estimate of the lemma is true.

To prove the fourth estimate we write

$$\begin{aligned}
&\left|t^{-\frac{1}{\alpha}}x\right|^{\beta}\left(G\left((t-\tau)^{-\frac{1}{\alpha}}(x-y)\right)-G\left((t-\tau)^{-\frac{1}{\alpha}}x\right)\right)\\
&\leq\left|(t-\tau)^{-\frac{1}{\alpha}}x\right|^{\beta}\left(G\left((t-\tau)^{-\frac{1}{\alpha}}(x-y)\right)-G\left((t-\tau)^{-\frac{1}{\alpha}}x\right)\right)\\
&\leq C\left|(t-\tau)^{-\frac{1}{\alpha}}y\right|\left\langle(t-\tau)^{-\frac{1}{\alpha}}x\right\rangle^{\beta-2-\alpha}\\
&+C\left|(t-\tau)^{-\frac{1}{\alpha}}y\right|^{\beta}\left(\left\langle(t-\tau)^{-\frac{1}{\alpha}}(x-y)\right\rangle^{-1-\alpha}+\left\langle(t-\tau)^{-\frac{1}{\alpha}}y\right\rangle^{-1-\alpha}\right)
\end{aligned}$$

for all $x,y\in\mathbf{R}$, $t>0$. Then for all $0<\tau<\frac{t}{2}$ we have

$$\begin{aligned}
&\left\|\left|t^{-\frac{1}{\alpha}}(\cdot)\right|^{\beta}\left(\mathcal{G}(t-\tau)\phi(\tau)-\vartheta(\tau)(t-\tau)^{-\frac{1}{\alpha}}G\left((t-\tau)^{-\frac{1}{\alpha}}(\cdot)\right)\right)\right\|_{\mathbf{L}^{\infty}}\\
&\leq C(t-\tau)^{-\frac{1}{\alpha}}\left\|\int_{\mathbf{R}}\left\langle(t-\tau)^{-\frac{1}{\alpha}}\cdot\right\rangle^{\beta-2-\alpha}\left|(t-\tau)^{-\frac{1}{\alpha}}y\right|\phi(\tau,y)dy\right\|_{\mathbf{L}^{\infty}}\\
&+C(t-\tau)^{-\frac{1}{\alpha}}\left\|\int_{\mathbf{R}}\left(\left\langle(t-\tau)^{-\frac{1}{\alpha}}(\cdot-y)\right\rangle^{-1-\alpha}\right.\right.\\
&\left.\left.+\left\langle(t-\tau)^{-\frac{1}{\alpha}}y\right\rangle^{-1-\alpha}\right)\left|(t-\tau)^{-\frac{1}{\alpha}}y\right|^{\beta}\phi(\tau,y)dy\right\|_{\mathbf{L}^{\infty}}\\
&\leq C(t-\tau)^{-\frac{1}{\alpha}}\left\|\left\langle(t-\tau)^{-\frac{1}{\alpha}}(\cdot)\right\rangle^{\beta-2-\alpha}\right\|_{\mathbf{L}^{\infty}}\left\|\left|(t-\tau)^{-\frac{1}{\alpha}}(\cdot)\right|\phi\right\|_{\mathbf{L}^{1}}\\
&+C(t-\tau)^{-\frac{1}{\alpha}}\left\|\left\langle(t-\tau)^{-\frac{1}{\alpha}}(\cdot)\right\rangle^{-1-\alpha}\right\|_{\mathbf{L}^{1}}\left\|\left|(t-\tau)^{-\frac{1}{\alpha}}(\cdot)\right|^{\beta}\phi\right\|_{\mathbf{L}^{\infty}}\\
&\leq Ct^{-\frac{2}{\alpha}}\|\phi(\tau)\|_{\mathbf{L}^{1,1}}+Ct^{-\frac{\beta}{\alpha}}\|\phi(\tau)\|_{\mathbf{L}^{\infty,\beta}}.
\end{aligned}$$

In addition for all $\frac{t}{2}\leq\tau<t$ we obtain

$$\begin{aligned}
&\left\|\left|t^{-\frac{1}{\alpha}}(\cdot)\right|^{\beta}\mathcal{G}(t-\tau)\phi(\tau)\right\|_{\mathbf{L}^{\infty}}\leq t^{-\frac{\beta}{\alpha}}\left\||\cdot|^{\beta}\mathcal{G}(t-\tau)\phi(\tau)\right\|_{\mathbf{L}^{\infty}}\\
&\leq t^{-\frac{\beta}{\alpha}}(t-\tau)^{\frac{\beta}{\alpha}}\|\phi(\tau)\|_{\mathbf{L}^{\infty}}+t^{-\frac{\beta}{\alpha}}\left\||\cdot|^{\beta}\phi(\tau)\right\|_{\mathbf{L}^{\infty}}\\
&\leq C\|\phi(\tau)\|_{\mathbf{L}^{\infty}}+C\tau^{-\frac{\beta}{\alpha}}\|\phi(\tau)\|_{\mathbf{L}^{\infty,\beta}}.
\end{aligned}$$

where $2<\beta\leq2+\alpha$, with $\vartheta(\tau)=\int_{\mathbf{R}}\phi(\tau,y)dy$. Lemma 1.40 is then proved.

Let us now compute the asymptotics of the Green function

$$G(\xi)=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}e^{i\xi\eta}e^{-|\eta|^{\alpha}}d\eta$$

for large values of ξ.

Lemma 1.41. *Let $\alpha > 0$. Then the asymptotics*

$$G(\xi) = \frac{\sqrt{2}}{\sqrt{\pi}} \Gamma(\alpha+1) |\xi|^{-1-\alpha} \sin \frac{\pi\alpha}{2} + O\left(|\xi|^{-1-2\alpha}\right) \tag{1.87}$$

is true for $|\xi| \to \infty$.

Proof. Denote $m = [a] + 1$. We integrate by parts m times with respect to η to get

$$\begin{aligned}
G(\xi) &= \frac{1}{\sqrt{2\pi}} (i\xi)^{-m} \int_{\mathbf{R}} e^{i\xi\eta} \partial_\eta^m e^{-|\eta|^\alpha} d\eta \\
&= \frac{\alpha(\alpha-1)\cdots(\alpha+1-m)}{\xi^m \sqrt{2\pi}} \int_{\mathbf{R}} e^{i\xi\eta} |\eta|^{\alpha-m} d\eta \\
&+ C\xi^{-m} \int_{\mathbf{R}} e^{i\xi\eta} \left(|\eta|^{\alpha-m} - \partial_\eta^m e^{-|\eta|^\alpha} \right) d\eta \\
&= I_1 + I_2.
\end{aligned}$$

The first summand I_1 gives the main term of the asymptotics (1.87); it can be computed explicitly (see Erdélyi et al. [1954])

$$\begin{aligned}
I_1 &= \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{\xi^m \sqrt{2\pi}} \int_{\mathbf{R}} e^{i\xi\eta} |\eta|^{\alpha-m} d\eta \\
&= \frac{\sqrt{2}}{\sqrt{\pi}} \Gamma(\alpha+1) |\xi|^{-1-\alpha} \sin \frac{\pi\alpha}{2}.
\end{aligned} \tag{1.88}$$

For the second term I_2 we obtain

$$\begin{aligned}
I_2 &= O\left(\xi^{-m}\right) \int_{\mathbf{R}} e^{i\xi\eta} \left(|\eta|^{\alpha-m} - \partial_\eta^m e^{-|\eta|^\alpha} \right) d\eta \\
&= O\left(\xi^{-m}\right) \int_{\mathbf{R}} e^{i\xi\eta} |\eta|^{\alpha-m} \left(e^{-|\eta|^\alpha} - 1 \right) d\eta \\
&+ \sum_{k=2}^{m} O\left(\xi^{-m}\right) \int_{\mathbf{R}} e^{i\xi\eta} |\eta|^{k\alpha-m} e^{-|\eta|^\alpha} d\eta.
\end{aligned}$$

In each of the integrals we can again repeat integration by parts with respect to η and use the explicit formula (1.88) to get estimate $O\left(|\xi|^{-1-2\alpha}\right)$. Thus we have asymptotics (1.87), and Lemma 1.41 is proved.

2

Asymptotically weak nonlinearity

The aim of this chapter is to find the large time asymptotic representations of solutions in the supercritical case, that is when the nonlinear term decays in time faster than the linear part of the equation. This type of nonlinearity we call the asymptotically weak one. We intend to find the main term of the asymptotics and to give an estimate of the remainder term in the uniform norm. We will see that the large time asymptotic behavior has a quasi linear character, that is the nonlinearity alters only the coefficient of the main term of the asymptotic formula. Taking into account some additional symmetry of the nonlinearity in equation we will be able to consider the case of large initial data.

2.1 General approach

Now we give a general approach for obtaining the large time asymptotic representation of solutions to the Cauchy problem (1.7)

$$\begin{cases} u_t + \mathcal{N}(u) + \mathcal{L}u = 0, \ x \in \mathbf{R}^n, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^n, \end{cases} \tag{2.1}$$

in the case of asymptotically weak nonlinearity (we often call this case supercritical). By the Duhamel principle we rewrite the Cauchy problem (2.1) as the following integral equation

$$u(t) = \mathcal{G}(t) u_0 - \int_0^t \mathcal{G}(t-\tau) \mathcal{N}(u(\tau)) \, d\tau, \tag{2.2}$$

where $\mathcal{G}$ is the Green operator of the corresponding linear problem. We fix a metric space $\mathbf{Z}$ of functions defined on $\mathbf{R}^n$ and a complete metric space $\mathbf{X}$ of functions defined on $[0,\infty) \times \mathbf{R}^n$.

Definition 2.1. *We call function $G_0 \in \mathbf{X}$ an asymptotic kernel for the Green operator $\mathcal{G}$ in spaces $\mathbf{X}$, $\mathbf{Z}$ if there exists a continuous linear functional $f : \mathbf{Z} \to \mathbf{R}$ such that the estimate is true*

$$\left\| \langle t \rangle^{\gamma} \left(\mathcal{G}(t)\, \phi - G_0(t)\, f(\phi) \right) \right\|_{\mathbf{X}} \leq C \left\| \phi \right\|_{\mathbf{Z}} \tag{2.3}$$

for any $\phi \in \mathbf{Z}$, where $\gamma > 0$.

Remark 2.2. Below the functional $f(\phi)$ often will be chosen in the form $f(\phi) = \int_{\mathbf{R}^n} \phi(x)\, dx = (2\pi)^{\frac{n}{2}} \widehat{\phi}(0)$. Sometimes the functional f also might have the form of the moments.

Definition 2.3. *We call the nonlinearity $\mathcal{N}(u)$ in equation (2.2) as asymptotically weak in space $\mathbf{X}$ if the integral $\int_0^{\infty} f(\mathcal{N}(u(\tau)))\, d\tau$ converges for any $u \in \mathbf{X}$.*

Now we state some sufficient conditions for obtaining the large time asymptotics of solutions to the Cauchy problem (2.1) in the supercritical case.

Theorem 2.4. *Let the initial data $u_0 \in \mathbf{Z}$. Assume that there exists an asymptotic kernel G_0 for the Green operator $\mathcal{G}$ in spaces $\mathbf{X}$, $\mathbf{Z}$. Let the nonlinearity $\mathcal{N}(u)$ of equation (2.1) be asymptotically weak in space $\mathbf{X}$. Suppose that there exists a unique global solution $u \in \mathbf{X}$ to the Cauchy problem (2.1) and that the following estimates are valid*

$$\begin{aligned} &\left\| \langle t \rangle^{\gamma} \int_0^{\frac{t}{2}} \left(\mathcal{G}(t-\tau)\, \mathcal{N}(u(\tau)) - G_0(t-\tau)\, f(\mathcal{N}(u(\tau))) \right) d\tau \right\|_{\mathbf{X}} \\ &+ \left\| \langle t \rangle^{\gamma} \int_{\frac{t}{2}}^{t} \mathcal{G}(t-\tau)\, \mathcal{N}(u(\tau))\, d\tau \right\|_{\mathbf{X}} \leq C \left\| u \right\|_{\mathbf{X}}^{\sigma} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &\left\| \langle t \rangle^{\gamma} G_0(t) \int_{\frac{t}{2}}^{\infty} f(\mathcal{N}(u(\tau)))\, d\tau \right\|_{\mathbf{X}} \\ &+ \left\| \langle t \rangle^{\gamma} \int_0^{\frac{t}{2}} \left(G_0(t-\tau) - G_0(t) \right) f(\mathcal{N}(u(\tau)))\, d\tau \right\|_{\mathbf{X}} \leq C \left\| u \right\|_{\mathbf{X}}^{\sigma} \end{aligned} \tag{2.5}$$

for any $u \in \mathbf{X}$, where $\sigma > 0$, $\gamma > 0$. Then this solution has the following large time asymptotics

$$\left\| \langle t \rangle^{\gamma} \left(u(t) - A G_0(t) \right) \right\|_{\mathbf{X}} \leq C \left\| u_0 \right\|_{\mathbf{Z}} + C \left\| u \right\|_{\mathbf{X}}^{\sigma}, \tag{2.6}$$

where $\sigma > 0$, $\gamma > 0$ are taken from condition (2.4), and the constant

$$A = f(u_0) - \int_0^{\infty} f(\mathcal{N}(u(\tau)))\, d\tau.$$

Remark 2.5. We can guarantee that the coefficient $A \neq 0$ in the asymptotic representation (2.6) if $f(u_0) \neq 0$ and $\mathcal{N}(u)$ is small. It happens, for example, in the case of small solutions or for convective type equations, when $f(\mathcal{N}(u)) \equiv 0$ (see Example 2.15 below.) It can occur that $A = 0$, for instance, for convective equations $f(\mathcal{N}(u)) \equiv 0$ if the initial data have zero mean value $f(u_0) = 0$ (see Example 2.18 and Example 2.10). In the last case formula (2.6) gives us only some time decay estimate for the solutions.

Proof. By virtue of the integral equation (2.2) we get

$$\begin{aligned}
&\left\|\langle t\rangle^{\gamma}(u(t) - AG_0(t))\right\|_{\mathbf{X}} \leq \left\|\langle t\rangle^{\gamma}(\mathcal{G}(t)u_0 - G_0(t)f(u_0)\right\|_{\mathbf{X}} \\
&+ \left\|\langle t\rangle^{\gamma}\int_0^{\frac{t}{2}}(\mathcal{G}(t-\tau)\mathcal{N}(u(\tau)) - G_0(t-\tau)\vartheta(\tau))\,d\tau\right\|_{\mathbf{X}} \\
&+ \left\|\langle t\rangle^{\gamma}\int_{\frac{t}{2}}^{t}\mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\,d\tau\right\|_{\mathbf{X}} + \left\|\langle t\rangle^{\gamma}G_0(t)\int_{\frac{t}{2}}^{\infty}\vartheta(\tau)\,d\tau\right\|_{\mathbf{X}} \\
&+ \left\|\langle t\rangle^{\gamma}\int_0^{\frac{t}{2}}(G_0(t-\tau) - G_0(t))\vartheta(\tau)\,d\tau\right\|_{\mathbf{X}}, \qquad (2.7)
\end{aligned}$$

where

$$\vartheta(\tau) = f(\mathcal{N}(u(\tau))).$$

All summands in the right-hand side of (2.7) are estimated by $C\|u_0\|_{\mathbf{Z}} + C\|u\|_{\mathbf{X}}^{\sigma}$ via estimates (2.3) - (2.5). Thus by (2.7) the asymptotics (2.6) is valid. Theorem 2.4 is proved.

Example 2.6. **Large time asymptotics for global solutions of the nonlinear heat equation**

We now apply Theorem 2.4 for obtaining the large time asymptotics of global solutions to the Cauchy problem (1.13)

$$\begin{cases} u_t - \Delta u = \lambda |u|^{\sigma} u, \ x \in \mathbf{R}^n, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^n \end{cases} \qquad (2.8)$$

in the supercritical case $\sigma > \frac{2}{n}$, where $\lambda \in \mathbf{R}$. Define the space $\mathbf{Z}$

$$\mathbf{Z} = \left\{\phi \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^p(\mathbf{R}^n)\right\},$$

where now $a \in (0,1]$, and $p > \max\left(1, \frac{n}{2}\sigma\right)$ and the space

$$\mathbf{X} = \{\phi \in \mathbf{C}([0,\infty);\mathbf{Z}) \cap \mathbf{C}((0,\infty);\mathbf{L}^{\infty}(\mathbf{R}^n)) : \ \|\phi\|_{\mathbf{X}} < \infty\},$$

where the norm

$$\begin{aligned}
\|\phi\|_{\mathbf{X}} &= \sup_{t\geq 0}\left(\langle t\rangle^{-\frac{a}{2}}\|\phi(t)\|_{\mathbf{L}^{1,a}} + \langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{p}\right)}\|\phi(t)\|_{\mathbf{L}^p}\right) \\
&+ \sup_{t>0}\{t\}^{\frac{n}{2p}}\langle t\rangle^{\frac{n}{2}}\|\phi(t)\|_{\mathbf{L}^{\infty}}.
\end{aligned}$$

Also we consider the norm

$$\|\phi\|_{\mathbf{Y}} = \sup_{t>0} \{t\}^{\frac{n\sigma}{2p}} \langle t\rangle^{\frac{n\sigma}{2}} \Big(\langle t\rangle^{-\frac{a}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} + \langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{p}\right)} \|\phi(t)\|_{\mathbf{L}^p} + \{t\}^{\frac{n}{2p}} \langle t\rangle^{\frac{n}{2}} \|\phi(t)\|_{\mathbf{L}^\infty} \Big).$$

Theorem 2.7. *Let $\sigma > \frac{2}{n}$. Assume that the initial data $u_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^p(\mathbf{R}^n)$, $a \in (0,1]$, and $p > \max\left(1, \frac{n}{2}\sigma\right)$. Suppose that there exists a unique global solution $u \in \mathbf{X}$ to the Cauchy problem (2.8). Then this solution has the following large time asymptotics*

$$u(t,x) = At^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} + O\left(t^{-\frac{n}{2}-\gamma}\right) \tag{2.9}$$

as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$, where $0 < \gamma < \min\left(\frac{a}{2}, \frac{n}{2}\sigma - 1\right)$, and the constant

$$A = \int_{\mathbf{R}^n} u_0(x)\,dx - \lambda \int_0^\infty d\tau \int_{\mathbf{R}^n} |u(\tau,x)|^\sigma u(\tau,x)\,dx.$$

Remark 2.8. The existence of a unique global solution $u \in \mathbf{X}$ to the Cauchy problem (2.8) was obtained in Theorem 1.19 for the case of small initial data and any sign of $\lambda \in \mathbf{R}$ and in Theorems 1.22 and 1.25 for the case of large initial data and $\lambda < 0$.

Before proving Theorem 2.7 we prepare the following lemma.

Lemma 2.9. *The Green operator*

$$\mathcal{G}(t)\phi = \int_{\mathbf{R}^n} G(t, x-y)\phi(y)\,dy,$$

where $G(t,x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$, has the asymptotic kernel (see Definition 2.1)

$$G_0(t,x) = (4\pi(t+1))^{-\frac{n}{2}} e^{-\frac{|x|^2}{4(t+1)}}$$

in spaces $\mathbf{X}$, $\mathbf{Z}$. *Moreover, estimate (2.5) is valid.*

Proof. By a direct calculation we have

$$\|G_0(t)\|_{\mathbf{L}^q} = (4\pi(t+1))^{-\frac{n}{2}} \left(\int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4(t+1)}q} dx \right)^{\frac{1}{q}} = C(t+1)^{\frac{n}{2q}-\frac{n}{2}} \left(\int_{\mathbf{R}^n} e^{-|y|^2} dy \right)^{\frac{1}{q}} \le C(t+1)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}$$

for all $t \ge 0$, where $1 \le q \le \infty$, and in the same manner

$$\|G_0(t)\|_{\mathbf{L}^{1,a}} = (4\pi(t+1))^{-\frac{n}{2}} \int_{\mathbf{R}^n} \langle x\rangle^a e^{-\frac{|x|^2}{4(t+1)}} dx \le C(t+1)^{\frac{a}{2}}$$

for all $t \geq 0$. Hence we see that $G_0 \in \mathbf{X}$. Moreover, let us define the functional $f : \mathbf{Z} \to \mathbf{R}$, by

$$f(\phi) = \int_{\mathbf{R}^n} \phi(x)\,dx,$$

and prove estimate (2.3) with $\gamma = \frac{a}{2} > 0$. We now use the estimates of Lemma 1.28 with $\delta = \nu = 2$ to obtain

$$\left\| |\cdot|^b \partial_{x_j}^{\beta} \mathcal{G}(t)\phi \right\|_{\mathbf{L}^q} \leq Ct^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{\beta-b}{2}} \|\phi\|_{\mathbf{L}^r} + Ct^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{\beta}{2}} \|\phi\|_{\mathbf{L}^{r,b}} \quad (2.10)$$

and

$$\left\| |\cdot|^b \partial_{x_j}^{\beta} \left(\mathcal{G}(t)\phi - \vartheta G_0(t)\right) \right\|_{\mathbf{L}^q} \leq Ct^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-\frac{a+\beta-b}{2}} \|\phi\|_{\mathbf{L}^{1,a}} \quad (2.11)$$

for all $t > 0$, where $\vartheta = \int_{\mathbf{R}^n} \phi(x)\,dx$, $1 \leq r \leq q \leq \infty$, $\beta \geq 0$, $0 \leq b \leq a$. Since

$$\|\mathcal{G}(t)\phi\|_{\mathbf{X}} \leq C\|\phi\|_{\mathbf{Z}}$$

and

$$\|G_0(t) f(\phi)\|_{\mathbf{X}} \leq \|G_0\|_{\mathbf{X}} |f(\phi)| \leq C\|\phi\|_{\mathbf{Z}},$$

we get the estimates

$$\begin{aligned} &\|\mathcal{G}(t)\phi - G_0(t) f(\phi)\|_{\mathbf{L}^{1,a}} + \|\mathcal{G}(t)\phi - G_0(t) f(\phi)\|_{\mathbf{L}^p} \\ &+ t^{\frac{n}{2p}} \|\mathcal{G}(t)\phi - G_0(t) f(\phi)\|_{\mathbf{L}^\infty} \leq C\|\phi\|_{\mathbf{Z}} \end{aligned} \quad (2.12)$$

for all $t \in (0, 1]$. Now by (2.11) we write the estimate

$$\begin{aligned} &t^{-\frac{a}{2}} \|\mathcal{G}(t)\phi - G_0(t) f(\phi)\|_{\mathbf{L}^{1,a}} + t^{\frac{n}{2}\left(1-\frac{1}{p}\right)} \|\mathcal{G}(t)\phi - G_0(t) f(\phi)\|_{\mathbf{L}^p} \\ &+ t^{\frac{n}{2}} \|\mathcal{G}(t)\phi - G_0(t) f(\phi)\|_{\mathbf{L}^\infty} \leq C\|\phi\|_{\mathbf{Z}} \end{aligned} \quad (2.13)$$

for all $t > 1$. Combining (2.12) and (2.13) we obtain estimate (2.3) with $\gamma = \frac{a}{2}$. Now let us prove estimate (2.5). Also in view of the definition of the norm $\mathbf{Y}$ we have

$$\begin{aligned} |f(\mathcal{N}(u(\tau)))| &\leq \|\mathcal{N}(u(\tau))\|_{\mathbf{L}^1} \leq C\{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{-\frac{n\sigma}{2}} \|\mathcal{N}(u)\|_{\mathbf{Y}} \\ &\leq C\{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{-\frac{n\sigma}{2}} \|u\|_{\mathbf{X}}^{\sigma}. \end{aligned}$$

By a direct calculation we have

$$\begin{aligned} &\left\| |\cdot|^b \int_0^{\frac{t}{2}} |G_0(t-\tau) - G_0(t)| f(\mathcal{N}(u(\tau)))\,d\tau \right\|_{\mathbf{L}^q} \\ &\leq \langle t\rangle^{-1} C\|u\|_{\mathbf{X}}^{\sigma} \int_0^{\frac{t}{2}} \left\| |\cdot|^b (G_0(t-\tau) + G_0(t)) \right\|_{\mathbf{L}^q} \{\tau\}^{-\alpha} \langle\tau\rangle^{-\gamma}\,d\tau \\ &\leq C\langle t\rangle^{-1+\frac{b}{2}-\frac{n}{2}\left(1-\frac{1}{q}\right)} \int_0^{\frac{t}{2}} \{\tau\}^{-\alpha} \langle\tau\rangle^{-\gamma}\,d\tau \leq C\langle t\rangle^{-\gamma+\frac{b}{2}-\frac{n}{2}\left(1-\frac{1}{q}\right)}, \end{aligned}$$

and in the same way

$$\left\| \langle t \rangle^{\gamma} G_0(t) \int_{\frac{t}{2}}^{\infty} f(\mathcal{N}(u(\tau)))\, d\tau \right\|_{\mathbf{X}} \leq C \|u\|_{\mathbf{X}}^{\sigma}.$$

Hence estimate (2.5) is true. Lemma 2.9 is proved.

We now turn to the proof of Theorem 2.4. Since $G_0 \in \mathbf{X}$ we have

$$\left\| \langle t \rangle^{\gamma} G_0(t) \int_{\frac{t}{2}}^{\infty} \vartheta(\tau)\, d\tau \right\|_{\mathbf{X}} \leq C \|u\|_{\mathbf{X}}^{\sigma} \left\| \langle t \rangle^{\gamma} G_0(t) \int_{\frac{t}{2}}^{\infty} \{\tau\}^{-\alpha} \langle \tau \rangle^{-1-\gamma}\, d\tau \right\|_{\mathbf{X}}$$
$$\leq C \|u\|_{\mathbf{X}}^{\sigma} \|G_0\|_{\mathbf{X}} \leq C \|u\|_{\mathbf{X}}^{\sigma}.$$

From (2.5) we find

$$\left\| \langle t \rangle^{\gamma} \int_0^{\frac{t}{2}} (G_0(t-\tau) - G_0(t))\, \vartheta(\tau)\, d\tau \right\|_{\mathbf{X}}$$
$$\leq C \|u\|_{\mathbf{X}}^{\sigma} \left\| \langle t \rangle^{\gamma} \int_0^{\frac{t}{2}} |G_0(t-\tau) - G_0(t)| \{\tau\}^{-\alpha} \langle \tau \rangle^{-1-\gamma}\, d\tau \right\|_{\mathbf{X}} \leq C \|u\|_{\mathbf{X}}^{\sigma}.$$

Let us prove the estimates

$$\left\| \langle t \rangle^{\gamma} \int_0^{\frac{t}{2}} (\mathcal{G}(t-\tau)\phi(\tau) - G_0(t-\tau)\vartheta(\tau))\, d\tau \right\|_{\mathbf{X}}$$
$$+ \left\| \langle t \rangle^{\gamma} \int_{\frac{t}{2}}^{t} \mathcal{G}(t-\tau)\phi(\tau)\, d\tau \right\|_{\mathbf{X}} \leq C \|\phi\|_{\mathbf{Y}}, \tag{2.14}$$

where $0 < \gamma < \min\left(\frac{a}{2}, \frac{n}{2}\sigma - 1\right)$. By estimate (2.12) we have

$$\left\| \int_0^{\frac{t}{2}} (\mathcal{G}(t-\tau)\phi(\tau) - G_0(t-\tau)\vartheta(\tau))\, d\tau \right\|_{\mathbf{L}^{1,a}}$$
$$+ \left\| \int_{\frac{t}{2}}^{t} \mathcal{G}(t-\tau)\phi(\tau)\, d\tau \right\|_{\mathbf{L}^{1,a}} \leq C \int_0^t \left(\|\phi(\tau)\|_{\mathbf{L}^{1,a}} + \|\phi(\tau)\|_{\mathbf{L}^p} \right) d\tau$$
$$\leq C \|\phi\|_{\mathbf{Y}} \int_0^t \{\tau\}^{-\frac{n\sigma}{2p}}\, d\tau \leq C \|\phi\|_{\mathbf{Y}},$$

$$\left\| \int_0^{\frac{t}{2}} (\mathcal{G}(t-\tau)\phi(\tau) - G_0(t-\tau)\vartheta(\tau))\, d\tau \right\|_{\mathbf{L}^p}$$
$$+ \left\| \int_{\frac{t}{2}}^{t} \mathcal{G}(t-\tau)\phi(\tau)\, d\tau \right\|_{\mathbf{L}^p}$$
$$\leq C \int_0^t \left(\|\phi(\tau)\|_{\mathbf{L}^{1,a}} + \|\phi(\tau)\|_{\mathbf{L}^p} \right) d\tau \leq C \|\phi\|_{\mathbf{Y}},$$

and

$$
\begin{aligned}
& t^{-\frac{n}{2p}} \left\| \int_0^{\frac{t}{2}} \left(\mathcal{G}(t-\tau)\phi(\tau) - G_0(t-\tau)\vartheta(\tau) \right) d\tau \right\|_{\mathbf{L}^\infty} \\
& + t^{-\frac{n}{2p}} \left\| \int_{\frac{t}{2}}^t \mathcal{G}(t-\tau)\phi(\tau)\, d\tau \right\|_{\mathbf{L}^\infty} \\
& \leq C \int_0^t \left(\|\phi(\tau)\|_{\mathbf{L}^{1,a}} + \|\phi(\tau)\|_{\mathbf{L}^p} \right) d\tau \leq C \|\phi\|_{\mathbf{Y}}
\end{aligned}
$$

for all $t \in (0,1]$. In addition by (2.11) we get

$$
\begin{aligned}
& \left\| |\cdot|^b \int_0^{\frac{t}{2}} \left(\mathcal{G}(t-\tau)\phi(\tau) - G_0(t-\tau)\vartheta(\tau) \right) d\tau \right\|_{\mathbf{L}^q} \\
& + \left\| |\cdot|^b \int_{\frac{t}{2}}^t \mathcal{G}(t-\tau)\phi(\tau)\, d\tau \right\|_{\mathbf{L}^q} \\
& \leq C \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-\frac{a-b}{2}} \|\phi(\tau)\|_{\mathbf{L}^{1,a}}\, d\tau \\
& + C \int_{\frac{t}{2}}^t \left((t-\tau)^{\frac{b}{2}} \|\phi(\tau)\|_{\mathbf{L}^q} + \|\phi(\tau)\|_{\mathbf{L}^{q,b}} \right) d\tau \\
& \leq C t^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-\frac{a-b}{2}} \|\phi\|_{\mathbf{Y}} \int_0^{\frac{t}{2}} \{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{\frac{a}{2}-\frac{n\sigma}{2}}\, d\tau \\
& + C t^{1-\frac{n}{2}\left(1-\frac{1}{q}\right)+\frac{b}{2}-\frac{n\sigma}{2}} \|\phi\|_{\mathbf{Y}} \leq C t^{-\frac{n}{2}\left(1-\frac{1}{q}\right)+\frac{b}{2}-\gamma} \|\phi\|_{\mathbf{Y}}
\end{aligned}
$$

for all $t > 1$, where $\vartheta(\tau) = \int_{\mathbf{R}^n} \phi(\tau, x)\, dx$, $1 \leq q \leq \infty$, $0 \leq b \leq a$, since $\frac{n\sigma}{2} > 1 + \gamma$. Hence estimate (2.14) is valid.
Via (1.24) and (2.14) we obtain condition (2.4). Now the result of the theorem follows by application of Theorem 2.4. Theorem 2.7 is proved.

Example 2.10. **The case of odd solutions to the nonlinear heat equation**

Now let us consider problem (2.8) with some special initial data $u_0(x)$, which are odd functions in $\mathbf{R}^n$, that is

$$
u_0(x_1, ..., -x_j, ..., x_n) = -u_0(x_1, ..., x_j, ..., x_n),
$$

for every $j = 1, 2, ..., n$. In this case the solutions $u(t,x)$ will also be odd functions with respect to $x \in \mathbf{R}^n$. Then we will show that the critical value is shifted $\sigma > \frac{1}{n}$.

Define the space $\mathbf{Z}$

$$\mathbf{Z} = \left\{ \phi \in \mathbf{L}^{1,a}\left(\mathbf{R}^n\right) \cap \mathbf{L}^p\left(\mathbf{R}^n\right) : \ \phi \text{ is odd function in } \mathbf{R}^n \right\},$$

where now $a \in (n, n+1]$, and $p > \max\left(1, \frac{n}{2}\sigma\right)$ and the space

$$\mathbf{X} = \{ \phi \in \mathbf{C}\left([0,\infty); \mathbf{Z}\right) \cap \mathbf{C}\left((0,\infty); \mathbf{L}^\infty\left(\mathbf{R}^n\right)\right) :$$
$$\phi \text{ is odd function in } \mathbf{R}^n \text{ and } \|\phi\|_{\mathbf{X}} < \infty \},$$

where the norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t \geq 0} \left(\langle t \rangle^{-\frac{a-n}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} + \langle t \rangle^{n - \frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^p} \right)$$
$$+ \sup_{t>0} \{t\}^{\frac{n}{2p}} \langle t \rangle^n \|\phi(t)\|_{\mathbf{L}^\infty}.$$

Also we define the norm to estimate the nonlinearity

$$\|\phi\|_{\mathbf{Y}} = \sup_{t>0} \{t\}^{\frac{n\sigma}{2p}} \langle t \rangle^{n\sigma} \left(\langle t \rangle^{-\frac{a-n}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} \right.$$
$$\left. + \langle t \rangle^{n - \frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^p} + \{t\}^{\frac{n}{2p}} \langle t \rangle^n \|\phi(t)\|_{\mathbf{L}^\infty} \right).$$

Theorem 2.11. *Let $\sigma > \frac{1}{n}$. Assume that the initial data u_0 are odd functions in $\mathbf{R}^n$, and $u_0 \in \mathbf{L}^{1,a}\left(\mathbf{R}^n\right) \cap \mathbf{L}^p\left(\mathbf{R}^n\right)$, with $a \in (n, n+1]$, and $p > \max\left(1, \frac{n}{2}\sigma\right)$. Suppose that there exists a unique global solution $u \in \mathbf{X}$ to the Cauchy problem (2.8). Then this solution has the following large time asymptotics*

$$u(t,x) = \frac{A}{4^n \pi^{\frac{n}{2}} t^{\frac{3n}{2}}} e^{-\frac{|x|^2}{4t}} \prod_{j=1}^n x_j + O\left(t^{-n-\gamma}\right) \tag{2.15}$$

as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$, where $0 < \gamma < \min\left(\frac{a-n}{2}, n\sigma - 1\right)$, and the constant

$$A = \int_{\mathbf{R}^n} u_0(x) \prod_{j=1}^n x_j dx$$
$$- \lambda \int_0^\infty d\tau \int_{\mathbf{R}^n} |u(\tau,x)|^\sigma u(\tau,x) \prod_{j=1}^n x_j dx.$$

Remark 2.12. The existence of a unique global solution $u \in \mathbf{X}$ to the Cauchy problem (2.8) can be obtained in the same way as in the proof of Theorem 1.19 for the case of small initial data and any sign of $\lambda \in \mathbf{R}$ and as in Theorem 1.25 for the case of large initial data and $\lambda < 0$. Therefore we are interested here in the large time asymptotic representation of the solutions.

Remark 2.13. Note that in the domain $x = O\left(\sqrt{t}\right)$ the main term of the asymptotics (2.15) behaves like $O\left(t^{-n}\right)$, so the remainder term decays faster. In the domains $x = o\left(\sqrt{t}\right)$ and $\frac{|x|}{\sqrt{t}} \to \infty$ the main term of the asymptotic representation (2.15) decays faster than the remainder term. So formula (2.15) gives only a decay estimate for these cases. Below in Section 2.2 we will obtain uniform asymptotic representations for solutions.

Before proving Theorem 2.11 we prepare the following lemma.

Lemma 2.14. *The Green operator*

$$\mathcal{G}(t)\phi = \int_{\mathbf{R}^n} G(t, x-y)\phi(y)\,dy,$$

where $G(t,x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$*, has the asymptotic kernel (see Definition 2.1)*

$$G_0(t,x) = \frac{1}{4^n \pi^{\frac{n}{2}} (t+1)^{\frac{3n}{2}}} e^{-\frac{|x|^2}{4(t+1)}} \prod_{j=1}^{n} x_j$$

in spaces **X**, **Z**. *Moreover, estimate (2.5) is valid.*

Proof. By a direct calculation we have

$$\begin{aligned} \|G_0(t)\|_{\mathbf{L}^q} &\le C(t+1)^{-\frac{3n}{2}} \left(\int_{\mathbf{R}^n} |x|^{nq} e^{-\frac{|x|^2}{4(t+1)}q} dx \right)^{\frac{1}{q}} \\ &\le C(t+1)^{\frac{n}{2q}-n} \left(\int_{\mathbf{R}^n} |y|^{nq} e^{-|y|^2} dy \right)^{\frac{1}{q}} \le C(t+1)^{-n+\frac{n}{2q}} \end{aligned}$$

for all $t \ge 0$, where $1 \le q \le \infty$, and similarly

$$\|G_0(t)\|_{\mathbf{L}^{1,a}} \le C(t+1)^{-\frac{3n}{2}} \int_{\mathbf{R}^n} \langle x \rangle^{n+a} e^{-\frac{|x|^2}{4(t+1)}} dx \le C(t+1)^{\frac{a-n}{2}}$$

for all $t \ge 0$. Hence we see that $G_0 \in \mathbf{X}$. We define the functional $f : \mathbf{Z} \to \mathbf{R}$, by

$$f(\phi) = \int_{\mathbf{R}^n} \phi(x) \prod_{j=1}^{n} x_j dx,$$

and prove estimate (2.3) with $\gamma = \frac{a-n}{2} > 0$. Using Lemma 1.30 with $\delta = \nu = 2$ we get for any odd function ϕ

$$\left\| |\cdot|^b \partial_{x_j}^{\beta} \left(\mathcal{G}(t)\phi - \vartheta G_0(t) \right) \right\|_{\mathbf{L}^q} \le C t^{-n+\frac{n}{2q}-\frac{a+\beta-b}{2}} \|\phi\|_{\mathbf{L}^{1,a}} \tag{2.16}$$

for all $t > 0$, where

$$\vartheta = \int_{\mathbf{R}^n} \phi(x) \prod_{j=1}^{n} x_j dx,$$

$1 \le r \le q \le \infty$, $\beta \ge 0$, $0 \le b \le a$. Since

$$\|\mathcal{G}(t)\phi\|_{\mathbf{X}} \le C\|\phi\|_{\mathbf{Z}}$$

and

$$\|G_0(t) f(\phi)\|_{\mathbf{X}} \le \|G_0\|_{\mathbf{X}} |f(\phi)| \le C\|\phi\|_{\mathbf{Z}},$$

we get the estimates

$$\begin{aligned} &\|\mathcal{G}(t)\phi - G_0(t) f(\phi)\|_{\mathbf{L}^{1,a}} + \|\mathcal{G}(t)\phi - G_0(t) f(\phi)\|_{\mathbf{L}^p} \\ &+ t^{\frac{n}{2p}} \|\mathcal{G}(t)\phi - G_0(t) f(\phi)\|_{\mathbf{L}^\infty} \le C\|\phi\|_{\mathbf{Z}} \end{aligned} \tag{2.17}$$

for all $t \in (0,1]$. Now by (2.16) we write the estimate

$$\begin{aligned} &t^{-\frac{a-n}{2}} \|\mathcal{G}(t)\phi - G_0(t) f(\phi)\|_{\mathbf{L}^{1,a}} + t^{n-\frac{n}{2p}} \|\mathcal{G}(t)\phi - G_0(t) f(\phi)\|_{\mathbf{L}^p} \\ &+ t^n \|\mathcal{G}(t)\phi - G_0(t) f(\phi)\|_{\mathbf{L}^\infty} \le C\|\phi\|_{\mathbf{Z}} \end{aligned} \tag{2.18}$$

for all $t > 1$. Combining (2.17) and (2.18) we obtain estimate (2.3) with $\gamma = \frac{a-n}{2}$. Now let us prove estimate (2.5). In view of the definition of the norm $\mathbf{Y}$ we have

$$\begin{aligned} |f(\mathcal{N}(u(\tau)))| &\le \|\mathcal{N}(u(\tau))\|_{\mathbf{L}^{1,n}} \le C\{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{-n\sigma} \|\mathcal{N}(u)\|_{\mathbf{Y}} \\ &\le C\{\tau\}^{-\frac{n\sigma}{2p}} \langle\tau\rangle^{-n\sigma} \|u\|_{\mathbf{X}}^{\sigma}. \end{aligned}$$

By a direct calculation we have

$$\begin{aligned} &\left\| |\cdot|^b \int_0^{\frac{t}{2}} |G_0(t-\tau) - G_0(t)| f(\mathcal{N}(u(\tau))) d\tau \right\|_{\mathbf{L}^q} \\ &\le \langle t\rangle^{-1} C\|u\|_{\mathbf{X}}^{\sigma} \int_0^{\frac{t}{2}} \left\| |\cdot|^b (G_0(t-\tau) + G_0(t)) \right\|_{\mathbf{L}^q} \{\tau\}^{-\alpha} \langle\tau\rangle^{-\gamma} d\tau \\ &\le C\langle t\rangle^{-1+\frac{b}{2}-n+\frac{n}{2q}} \int_0^{\frac{t}{2}} \{\tau\}^{-\alpha} \langle\tau\rangle^{-\gamma} d\tau \le C\langle t\rangle^{-\gamma+\frac{b}{2}-n+\frac{n}{2q}}. \end{aligned}$$

In the same way

$$\left\| \langle t\rangle^{\gamma} G_0(t) \int_{\frac{t}{2}}^{\infty} f(\mathcal{N}(u(\tau))) d\tau \right\|_{\mathbf{X}} \le C\|u\|_{\mathbf{X}}^{\sigma}.$$

Hence estimate (2.5) is true, and Lemma 2.14 is therefore proved.

We now turn to the proof of Theorem 2.11. Since $G_0 \in \mathbf{X}$ we have

$$\left\| \langle t\rangle^{\gamma} G_0(t) \int_{\frac{t}{2}}^{\infty} \vartheta(\tau) d\tau \right\|_{\mathbf{X}} \le C\|u\|_{\mathbf{X}}^{\sigma} \|G_0\|_{\mathbf{X}} \le C\|u\|_{\mathbf{X}}^{\sigma}.$$

From (2.5) we find

$$\left\| \langle t\rangle^{\gamma} \int_0^{\frac{t}{2}} \left(G_0\left(t-\tau\right)-G_0\left(t\right)\right)\vartheta\left(\tau\right)d\tau \right\|_{\mathbf{X}}$$
$$\leq C\left\|u\right\|_{\mathbf{X}}^{\sigma} \left\| \langle t\rangle^{\gamma} \int_0^{\frac{t}{2}} \left|G_0\left(t-\tau\right)-G_0\left(t\right)\right| \{\tau\}^{-\alpha} \langle\tau\rangle^{-1-\gamma} d\tau \right\|_{\mathbf{X}} \leq C\left\|u\right\|_{\mathbf{X}}^{\sigma}.$$

Let us prove the estimate (2.14) with $0<\gamma<\min\left(\frac{a-n}{2}, n\sigma-1\right)$. Estimates for the case of $t\in(0,1]$ are the same as in the proof of Theorem 2.7. By (2.16) we get

$$\left\| |\cdot|^{b} \int_0^{\frac{t}{2}} \left(\mathcal{G}\left(t-\tau\right)\phi\left(\tau\right)-G_0\left(t-\tau\right)\vartheta\left(\tau\right)\right)d\tau \right\|_{\mathbf{L}^q}$$
$$+\left\| |\cdot|^{b} \int_{\frac{t}{2}}^{t} \mathcal{G}\left(t-\tau\right)\phi\left(\tau\right)d\tau \right\|_{\mathbf{L}^q}$$
$$\leq C\int_0^{\frac{t}{2}} \left(t-\tau\right)^{-n+\frac{n}{2q}-\frac{a-b}{2}} \left\|\phi\left(\tau\right)\right\|_{\mathbf{L}^{1,a}} d\tau$$
$$+C\int_{\frac{t}{2}}^{t}\left(\left(t-\tau\right)^{\frac{b-n}{2}}\left\|\phi\left(\tau\right)\right\|_{\mathbf{L}^{q,n}}+\left\|\phi\left(\tau\right)\right\|_{\mathbf{L}^{q,b}}\right)d\tau$$
$$\leq Ct^{-n+\frac{n}{2q}-\frac{a-b}{2}}\left\|\phi\right\|_{\mathbf{Y}}\int_0^{\frac{t}{2}}\{\tau\}^{-\frac{n\sigma}{2p}}\langle\tau\rangle^{\frac{a-n}{2}-n\sigma}d\tau$$
$$+Ct^{1-n+\frac{n}{2q}+\frac{b-n}{2}-n\sigma}\left\|\phi\right\|_{\mathbf{Y}}\leq Ct^{-n+\frac{n}{2q}+\frac{b-n}{2}-\gamma}\left\|\phi\right\|_{\mathbf{Y}}$$

for all $t>1$, $1\leq q\leq\infty$, $n\leq b\leq a$, since $n\sigma>1+\gamma$. Hence estimate (2.14) is valid. Via (1.24) and (2.14) we obtain condition (2.4).
Hence the condition of the Definition 2.1 is fulfilled. Now the result of the theorem follows by application of Theorem 2.4. Theorem 2.11 is proved.

Example 2.15. **Large time asymptotics for global solutions of the Burgers-type equations**

In the next theorem we obtain large time asymptotics for the global solutions to the Cauchy problem for the Burgers type equation (1.18)

$$\begin{cases} u_t-\Delta u=\left(\lambda\cdot\nabla\right)\left|u\right|^{\sigma}u,\ x\in\mathbf{R}^n,\ t>0, \\ u\left(0,x\right)=u_0\left(x\right),\ x\in\mathbf{R}^n, \end{cases} \tag{2.19}$$

in the supercritical case $\sigma > \frac{1}{n}$, where $\lambda \in \mathbf{R}^n$. Define the space $\mathbf{Z}$

$$\mathbf{Z} = \left\{ \phi \in \mathbf{L}^{1,a}\left(\mathbf{R}^n\right) \cap \mathbf{L}^p\left(\mathbf{R}^n\right) \right\},$$

where now $a \in (0,1]$, and $p > n\sigma$ and the space

$$\mathbf{X} = \left\{ \phi \in \mathbf{C}\left([0,\infty);\mathbf{Z}\right) \cap \mathbf{C}\left((0,\infty);\mathbf{W}^1_\infty\left(\mathbf{R}^n\right)\right) : \ \|\phi\|_{\mathbf{X}} < \infty \right\},$$

where the norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t\geq 0}\left(\langle t\rangle^{-\frac{a}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} + \langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{p}\right)} \|\phi(t)\|_{\mathbf{L}^p} \right)$$
$$+ \sup_{t>0}\left(\{t\}^{\frac{n}{2p}} \langle t\rangle^{\frac{n}{2}} \|\phi(t)\|_{\mathbf{L}^\infty} + \{t\}^{\frac{n}{2p}+\frac{1}{2}} \langle t\rangle^{\frac{n+1}{2}} \|\nabla\phi(t)\|_{\mathbf{L}^\infty} \right).$$

Also we consider the norm

$$\|\phi\|_{\mathbf{Y}} = \sup_{t>0} \{t\}^{\frac{n\sigma}{2p}+\frac{1}{2}} \langle t\rangle^{\frac{n\sigma}{2}+\frac{1}{2}} \left(\langle t\rangle^{-\frac{a}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} \right.$$
$$\left. + \langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{p}\right)} \|\phi(t)\|_{\mathbf{L}^p} + \{t\}^{\frac{n}{2p}} \langle t\rangle^{\frac{n}{2}} \|\phi(t)\|_{\mathbf{L}^\infty} \right).$$

Theorem 2.16. *Let $\sigma > \frac{1}{n}$. Assume that the initial data $u_0 \in \mathbf{L}^p\left(\mathbf{R}^n\right) \cap \mathbf{L}^{1,a}\left(\mathbf{R}^n\right)$, with $a \in (0,1]$ and $p > n\sigma$. Suppose that there exists a unique global solution $u \in \mathbf{X}$ to the Cauchy problem for the Burgers type equation (2.19). Then this solution has asymptotics (2.9) with $0 < \gamma < \min\left(\frac{a}{2}, \frac{n}{2}(\sigma - 1)\right)$ and a constant $A = \int_{\mathbf{R}} u_0(x)\,dx$.*

Remark 2.17. The existence of a unique global solution $u \in \mathbf{X}$ to the Cauchy problem (2.19) was obtained in Theorem 1.19 for the case of small initial data and in Theorem 1.27 for the case of large initial data.

Proof. Revising the proof of Lemma 2.9 we add the estimates of the derivative

$$\|\nabla G_0(t)\|_{\mathbf{L}^q} = C(t+1)^{\frac{n}{2q}-\frac{n+1}{2}} \left(\int_{\mathbf{R}^n} e^{-|y|^2} dy \right)^{\frac{1}{q}}$$
$$\leq C(t+1)^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{q}\right)}$$

for all $t \geq 0$, to see that $G_0 \in \mathbf{X}$. Using estimate (2.10) we add (2.12) and (2.13) to the estimates

$$t^{\frac{n}{2p}+\frac{1}{2}} \|\nabla\left(\mathcal{G}(t)\phi - G_0(t) f(\phi)\right)\|_{\mathbf{L}^\infty} \leq C\|\phi\|_{\mathbf{Z}} \tag{2.20}$$

for all $t \in (0,1]$ and

$$t^{\frac{n+1+a}{2}} \|\nabla\left(\mathcal{G}(t)\phi - G_0(t) f(\phi)\right)\|_{\mathbf{L}^\infty} \leq C\|\phi\|_{\mathbf{Z}} \tag{2.21}$$

for all $t > 1$. Then by (2.12), (2.13), (2.20) and (2.21) we can see that condition (2.3) is fulfilled in spaces $\mathbf{X}$, $\mathbf{Z}$. Likewise we have

$$\left\| \nabla \int_0^{\frac{t}{2}} \left| G_0 (t-\tau) - G_0 (t) \right| \{\tau\}^{-\alpha} \langle \tau \rangle^{-1-\gamma} d\tau \right\|_{\mathbf{L}^q}$$
$$\leq C \langle t \rangle^{-1} \int_0^{\frac{t}{2}} \left\| \nabla G_0 (t-\tau) + \nabla G_0 (t) \right\|_{\mathbf{L}^q} \{\tau\}^{-\alpha} \langle \tau \rangle^{-\gamma} d\tau$$
$$\leq C \langle t \rangle^{-\frac{3}{2}-\frac{n}{2}\left(1-\frac{1}{q}\right)} \int_0^{\frac{t}{2}} \{\tau\}^{-\alpha} \langle \tau \rangle^{-\gamma} d\tau \leq C \langle t \rangle^{-\gamma-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{q}\right)}.$$

Hence estimate (2.5) is true.

To prove (2.14) we add the estimates of the derivatives in view of (2.20)

$$t^{-\frac{n}{2p}-\frac{1}{2}} \left\| \nabla \int_0^{\frac{t}{2}} \left(\mathcal{G} (t-\tau) \phi (\tau) - G_0 (t-\tau) \vartheta (\tau) \right) d\tau \right\|_{\mathbf{L}^\infty}$$
$$+ t^{-\frac{n}{2p}} \left\| \nabla \int_{\frac{t}{2}}^{t} \mathcal{G} (t-\tau) \phi (\tau) d\tau \right\|_{\mathbf{L}^\infty}$$
$$\leq C \int_0^t \left(\|\phi (\tau)\|_{\mathbf{L}^{1,a}} + \|\phi (\tau)\|_{\mathbf{L}^p} \right) d\tau \leq C \|\phi\|_{\mathbf{Y}} \int_0^t \{\tau\}^{-\frac{n\sigma}{2p}-\frac{1}{2}} d\tau \leq C \|\phi\|_{\mathbf{Y}}$$

for all $t \in (0,1]$ and by (2.21) we get

$$\left\| \nabla \int_0^{\frac{t}{2}} \left(\mathcal{G} (t-\tau) \phi (\tau) - G_0 (t-\tau) \vartheta (\tau) \right) d\tau \right\|_{\mathbf{L}^q}$$
$$+ \left\| \nabla \int_{\frac{t}{2}}^{t} \mathcal{G} (t-\tau) \phi (\tau) d\tau \right\|_{\mathbf{L}^q}$$
$$\leq C \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-\frac{a+1}{2}} \|\phi (\tau)\|_{\mathbf{L}^{1,a}} d\tau$$
$$+ C \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} \|\phi (\tau)\|_{\mathbf{L}^q} d\tau$$
$$\leq C t^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-\frac{a+1}{2}} \|\phi\|_{\mathbf{Y}} \int_0^{\frac{t}{2}} \{\tau\}^{-\frac{n\sigma}{2p}-\frac{1}{2}} \langle \tau \rangle^{\frac{a}{2}-\frac{n\sigma}{2}-\frac{1}{2}} d\tau$$
$$+ C t^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-\frac{n\sigma}{2}} \|\phi\|_{\mathbf{Y}} \leq C t^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-\frac{1}{2}-\gamma} \|\phi\|_{\mathbf{Y}}$$

for all $t > 1$, where $\vartheta (\tau) = \int_{\mathbf{R}^n} \phi (\tau, x) dx$, $1 \leq q \leq \infty$, $0 \leq b \leq a$, since $\frac{n\sigma}{2} > \frac{1}{2} + \gamma$. Hence estimate (2.14) is valid. Via (1.24) and (2.14) we obtain condition (2.4). Also in view of (1.24) and by the definition of the norm $\mathbf{Y}$ we have

$$\left| f \left(\mathcal{N} (u (\tau)) \right) \right| \leq \| \mathcal{N} (u (\tau)) \|_{\mathbf{L}^1} \leq C \{\tau\}^{-\frac{n\sigma}{2p}-\frac{1}{2}} \langle \tau \rangle^{-\frac{n\sigma}{2}-\frac{1}{2}} \| \mathcal{N} (u) \|_{\mathbf{Y}}$$
$$\leq C \{\tau\}^{-\frac{n\sigma}{2p}-\frac{1}{2}} \langle \tau \rangle^{-\frac{n\sigma}{2}-\frac{1}{2}} \|u\|_{\mathbf{X}}^{\sigma}.$$

Since $\alpha = \frac{n\sigma}{2p} + \frac{1}{2} < 1$ and $\frac{n\sigma}{2} > \frac{1}{2} + \gamma$, the condition of Definition 2.1 is fulfilled. Then by Theorem 2.4 we see that asymptotics (2.9) is valid with a constant $A = \int_{\mathbf{R}} u_0(x)\,dx$, because the nonlinearity has the form of the full derivative so that

$$\int_{\mathbf{R}^n} (\lambda \cdot \nabla) |u(t,x)|^{\sigma} u(t,x)\,dx = 0.$$

Theorem 2.16 is proved.

Example 2.18. **Burgers type equations with initial data having zero mean value**

Now we consider the Cauchy problem for the Burgers type equation (2.19) with initial data having zero mean value $\int_{\mathbf{R}^n} u_0(x)\,dx = 0$. Since the nonlinearity of equation (2.19) has the form of the full derivative, then the solutions $u(t,x)$ also have a zero mean value for all $t > 0$. Define the space $\mathbf{Z}$

$$\mathbf{Z} = \left\{ \phi \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^p(\mathbf{R}^n) \right\},$$

where now $a \in (1,2]$, $p > \max(1, n\sigma)$ and the space

$$\mathbf{X} = \left\{ \phi \in \mathbf{C}([0,\infty); \mathbf{Z}) \cap \mathbf{C}\left((0,\infty); \mathbf{W}^1_\infty(\mathbf{R}^n)\right) : \ \|\phi\|_{\mathbf{X}} < \infty \right\},$$

where the norm

$$\begin{aligned} \|\phi\|_{\mathbf{X}} &= \sup_{t\geq 0} \left(\langle t \rangle^{-\frac{a-1}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} + \langle t \rangle^{\frac{n+1}{2} - \frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^p} \right) \\ &+ \sup_{t>0} \left(\{t\}^{\frac{n}{2p}} \langle t \rangle^{\frac{n+1}{2}} \|\phi(t)\|_{\mathbf{L}^\infty} + \{t\}^{\frac{n}{2p}+\frac{1}{2}} \langle t \rangle^{\frac{n}{2}+1} \|\nabla \phi(t)\|_{\mathbf{L}^\infty} \right). \end{aligned}$$

Also we consider the norm

$$\begin{aligned} \|\phi\|_{\mathbf{Y}} &= \sup_{t>0} \{t\}^{\frac{n\sigma}{2p}+\frac{1}{2}} \langle t \rangle^{\frac{(n+1)\sigma}{2}+\frac{1}{2}} \left(\langle t \rangle^{-\frac{a-1}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} \right. \\ &\left. + \langle t \rangle^{\frac{n+1}{2}-\frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^p} + \{t\}^{\frac{n}{2p}} \langle t \rangle^{\frac{n+1}{2}} \|\phi(t)\|_{\mathbf{L}^\infty} \right). \end{aligned}$$

We will need a modification of the definition of the asymptotic kernel.

Definition 2.19. *We call the functions $G_j \in \mathbf{X}$, $j = 1, ..., n$, by the asymptotic kernels for the Green operator $\mathcal{G}$ in spaces $\mathbf{X}$, $\mathbf{Z}$ if there exist continuous linear functionals $f_j : \mathbf{Z} \to \mathbf{R}$ such that the estimate is true*

$$\left\| \langle t \rangle^{\gamma} \left(\mathcal{G}(t)\phi - \sum_{j=1}^{n} G_j(t) f_j(\phi) \right) \right\|_{\mathbf{X}} \leq C \|\phi\|_{\mathbf{Z}} \tag{2.22}$$

for any $\phi \in \mathbf{Z}$, where $\gamma > 0$.

In the case under consideration we choose the asymptotic kernels

$$G_j(t,x) = \frac{x_j}{(4\pi)^{\frac{n}{2}}(t+1)^{1+\frac{n}{2}}} e^{-\frac{|x|^2}{4(t+1)}}$$

and the functionals $f_j(\phi) \equiv \int_{\mathbf{R}^n} \phi(x) x_j dx$.

Theorem 2.20. *Let $\sigma > \frac{1}{n+1}$. Assume that the initial data have zero mean value $\int_{\mathbf{R}^n} u_0(x)\,dx = 0$ and $u_0 \in \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, with $a \in (1,2]$ and $p > \max(1, n\sigma)$. Suppose that there exists a unique global solution $u \in \mathbf{X}$ to the Cauchy problem for the Burgers type equation (2.19). Then this solution has asymptotics*

$$u(t,x) = \sum_{j=1}^{n} A_j x_j \frac{1}{(4\pi)^{\frac{n}{2}} t^{1+\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} + O\left(t^{-n-\frac{1}{2}-\gamma}\right) \qquad (2.23)$$

as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$, where

$$0 < \gamma < \min\left(\frac{a-1}{2}, \frac{(n+1)\sigma - 1}{2}\right)$$

and the constants

$$A_j = \int_{\mathbf{R}^n} u_0(x) x_j dx - \int_0^\infty d\tau \int_{\mathbf{R}^n} dx x_j (\lambda \cdot \nabla) |u(\tau,x)|^\sigma u(\tau,x).$$

Remark 2.21. The existence of a unique global solution $u \in \mathbf{X}$ to the Cauchy problem (2.19) can be obtained in the same manner as in Theorem 1.19 for the case of small initial data and as in Theorem 1.27 for the case of large initial data.

Proof. As in the proof of Lemma 2.9 we obtain the estimates

$$\|\nabla G_j(t)\|_{\mathbf{L}^q} \le C(t+1)^{-1-\frac{n}{2}\left(1-\frac{1}{q}\right)}$$

for all $t \ge 0$, to see that $G_0 \in \mathbf{X}$. We also have the estimates

$$t^{\frac{n}{2p}+1} \left\| \nabla \left(\mathcal{G}(t)\phi - \sum_{j=1}^{n} G_j(t) f_j(\phi) \right) \right\|_{\mathbf{L}^\infty} \le C \|\phi\|_{\mathbf{Z}} \qquad (2.24)$$

for all $t \in (0,1]$ and

$$t^{\frac{n+1+a}{2}} \left\| \nabla \left(\mathcal{G}(t)\phi - \sum_{j=1}^{n} G_j(t) f_j(\phi) \right) \right\|_{\mathbf{L}^\infty} \le C \|\phi\|_{\mathbf{Z}} \qquad (2.25)$$

for all $t > 1$. Then we can see that condition (2.3) is fulfilled in spaces $\mathbf{X}$, $\mathbf{Z}$. In the same manner we obtain

$$\begin{aligned}
&\left\| \nabla \int_0^{\frac{t}{2}} |G_j(t-\tau) - G_j(t)| \{\tau\}^{-\alpha} \langle\tau\rangle^{-1-\gamma} d\tau \right\|_{\mathbf{L}^q} \\
&\leq C \langle t\rangle^{-1} \int_0^{\frac{t}{2}} \|\nabla G_j(t-\tau) + \nabla G_j(t)\|_{\mathbf{L}^q} \{\tau\}^{-\alpha} \langle\tau\rangle^{-\gamma} d\tau \\
&\leq C \langle t\rangle^{-2-\frac{n}{2}\left(1-\frac{1}{q}\right)} \int_0^{\frac{t}{2}} \{\tau\}^{-\alpha} \langle\tau\rangle^{-\gamma} d\tau \leq C \langle t\rangle^{-\gamma-1-\frac{n}{2}\left(1-\frac{1}{q}\right)}.
\end{aligned}$$

Hence estimate (2.5) is true.

To prove (2.14) we add the estimates

$$\begin{aligned}
&\left\| \nabla \int_0^{\frac{t}{2}} \left(\mathcal{G}(t-\tau) \phi(\tau) - \sum_{j=1}^n G_j(t-\tau) \vartheta_j(\tau) \right) d\tau \right\|_{\mathbf{L}^q} \\
&+ \left\| \nabla \int_{\frac{t}{2}}^t \mathcal{G}(t-\tau) \phi(\tau) d\tau \right\|_{\mathbf{L}^q} \\
&\leq C \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n+1}{2}+\frac{n}{2q}-\frac{a+1}{2}} \|\phi(\tau)\|_{\mathbf{L}^{1,a}} d\tau \\
&+ C \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}} \|\phi(\tau)\|_{\mathbf{L}^q} d\tau \\
&\leq C t^{-\frac{n+1}{2}+\frac{n}{2q}-\frac{a+1}{2}} \|\phi\|_{\mathbf{Y}} \int_0^{\frac{t}{2}} \{\tau\}^{-\frac{n\sigma}{2p}-\frac{1}{2}} \langle\tau\rangle^{\frac{a-1}{2}-\frac{(n+1)\sigma}{2}-\frac{1}{2}} d\tau \\
&+ C t^{-\frac{n+1}{2}+\frac{n}{2q}-\frac{(n+1)\sigma}{2}} \|\phi\|_{\mathbf{Y}} \leq C t^{-\frac{n+1}{2}+\frac{n}{2q}-\frac{1}{2}-\gamma} \|\phi\|_{\mathbf{Y}}
\end{aligned}$$

for all $t > 1$ where $\vartheta_j(\tau) = \int_{\mathbf{R}^n} \phi(\tau, x) x_j dx$, $1 \leq q \leq \infty$, $0 \leq b \leq a$, since $\frac{(n+1)\sigma}{2} > \frac{1}{2} + \gamma$. Hence estimate (2.14) is valid. Via (1.24) and (2.14) we obtain condition (2.4). Also in view of (1.24) and by the definition of the norm $\mathbf{Y}$ we have

$$\begin{aligned}
|f_j(\mathcal{N}(u(\tau)))| &\leq \|\mathcal{N}(u(\tau))\|_{\mathbf{L}^{1,1}} \leq C \{\tau\}^{-\frac{n\sigma}{2p}-\frac{1}{2}} \langle\tau\rangle^{-\frac{(n+1)\sigma}{2}-\frac{1}{2}} \|\mathcal{N}(u)\|_{\mathbf{Y}} \\
&\leq C \{\tau\}^{-\frac{n\sigma}{2p}-\frac{1}{2}} \langle\tau\rangle^{-\frac{(n+1)\sigma}{2}-\frac{1}{2}} \|u\|_{\mathbf{X}}^{\sigma}.
\end{aligned}$$

Since $\alpha = \frac{n\sigma}{2p} + \frac{1}{2} < 1$ and $(n+1)\sigma > 1 + \gamma$, the condition of Definition 2.1 is fulfilled. Then by Theorem 2.4 we see that asymptotics (2.23) is valid. Theorem 2.20 is proved.

2.2 Asymptotics for large x and t

In this section we study the fractional nonlinear equations

$$\begin{cases} u_t + \mathcal{L}u + \mathcal{N}(u) = 0, \ x \in \mathbf{R}, t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}, \end{cases} \tag{2.26}$$

here the linear part $\mathcal{L}$ is a derivative of a fractional order $\alpha > 1$

$$\mathcal{L}u = |\partial_x|^{\alpha} u = \mathcal{F}^{-1}_{\xi\to x} |\xi|^{\alpha} \mathcal{F}_{x\to\xi} u.$$

For simplicity we consider here the one dimensional case $x \in \mathbf{R}$. The case of higher dimensions also can be treated by this method. The nonlinear operator $\mathcal{N}$ is

$$\mathcal{N}(u) = a|u|^{\sigma} u + b|u|^{\rho} u_x,$$

where $a,\ b \in \mathbf{R}$. We suppose that the nonlinearity is asymptotically weak, that is $\sigma > \alpha$ and $\rho > \alpha - 1$. Equation (2.26) with $\alpha = 2$, $b = 0$ is a nonlinear heat equation $u_t - u_{xx} + a|u|^{\sigma} u = 0$, it was studied in Section 2.1.

If the initial data $u_0 \in \mathbf{L}^{1,\beta}(\mathbf{R})$, $\beta > 0$, by the method of Section 2.1 we calculate the following asymptotics for solutions of the Cauchy problem (2.26)

$$u(t,x) = At^{-\frac{1}{\alpha}} G(\xi) + O\left(t^{-\frac{1+\gamma}{\alpha}}\right) \tag{2.27}$$

for large time $t \to \infty$ uniformly with respect to $x \in R$, where $\xi = t^{-\frac{1}{\alpha}} x$, $\gamma > 0$, and

$$G(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{i\xi\eta} e^{-|\eta|^{\alpha}} d\eta.$$

Observe that formula (2.27) does not give us an answer to the question of the behavior of the solution $u(t,x)$, when t and x tend to infinity simultaneously, because the main term of asymptotics (2.27) vanishes as $|\xi| \to \infty$, but the dependence on ξ of the remainder in the right-hand side of (2.27) is not evaluated explicitly.

In this section we fill this gap and calculate the asymptotics of solutions $u(t,x)$ to the Cauchy problem for Equation (2.26) as $t \to \infty$ and $\xi = xt^{-\frac{1}{\alpha}} \to \infty$.

2.2.1 Small initial data

In this section we prove the following theorem.

Theorem 2.22. *Let $\sigma > \alpha > 1$ and $\rho > \alpha - 1$. Suppose that the initial data $u_0 \in \mathbf{L}^{\infty,\alpha+1}(\mathbf{R})$ have a sufficiently small norm $\|u_0\|_{\mathbf{L}^{\infty,\alpha+1}}$. Then there exists a unique solution $u \in \mathbf{C}\left([0,\infty); \mathbf{L}^{\infty,\alpha+1}(\mathbf{R})\right) \cap \mathbf{C}\left((0,\infty); \mathbf{W}^1_{\infty}(\mathbf{R})\right)$ to the Cauchy problem (2.26). Moreover there exists a constant A such that the asymptotics*

$$u(t,x) = At^{-\frac{1}{\alpha}} G(\xi) + O\left(t^{-\frac{1}{\alpha}-\gamma} |\xi|^{-\alpha-1}\right) \tag{2.28}$$

is true for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $G(\xi) = \overline{\mathcal{F}}_{\eta\to\xi}\left(e^{-|\eta|^{\alpha}}\right)$, $\xi = t^{-\frac{1}{\alpha}} x$, $\gamma \in \left(0, \frac{1}{2}\right)$.

Remark 2.23. Since the asymptotics of the function $G(\xi)$ is (see Lemma 1.41)

$$G(\xi) = \frac{\sqrt{2}}{\sqrt{\pi}} \Gamma(\alpha+1) |\xi|^{-1-\alpha} \sin \frac{\pi\alpha}{2} + O\left(|\xi|^{-1-2\alpha}\right) \tag{2.29}$$

for $\xi \to \infty$, then we see that the second term in the right-hand side of asymptotic formula (2.28) remains to be the remainder, when simultaneously $t \to \infty$ and $|\xi| \to \infty$. When the parameter $\alpha = 2k$, $k \in \mathbf{N}$, the main term in the asymptotics (2.29) vanishes. Indeed for this case the function $G(\xi)$ decays exponentially, so that formula (2.28) gives only the estimate of the solution as $t \to \infty$, and $\xi = xt^{-\frac{1}{\alpha}} \to \infty$. Some additional decay conditions for the initial data $u_0(x)$ must be fulfilled to get the asymptotic representation for large t and x (see Theorem 2.27 below).

Remark 2.24. In the case of the nonlinear heat equation $b = 0$, by the same approach we can consider all powers $\alpha > 0$.

We write the Cauchy problem (2.26) as an integral equation

$$u(t) = \mathcal{G}(t) u_0 - \int_0^t \mathcal{G}(t-\tau) \mathcal{N}(u)(\tau)\, d\tau, \tag{2.30}$$

with

$$\mathcal{N}(u) = a|u|^\sigma u + b|u|^\rho v_x,$$

and the Green operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t)\phi = t^{-\frac{1}{\alpha}} \int_{\mathbf{R}} G\left(t^{-\frac{1}{\alpha}}(x-y)\right) \phi(y) dy$$

with a kernel $G(x) = \overline{\mathcal{F}}_{\eta \to x}\left(e^{-|\eta|^\alpha}\right)$. We apply Theorem 1.17 to prove the existence of global solutions. Denote

$$\mathbf{X} = \left\{\phi \in \mathbf{C}\left([0,\infty); \mathbf{L}^{\infty,\alpha+1}(\mathbf{R})\right) : \ \|\phi\|_{\mathbf{X}} < \infty\right\},$$

where

$$\begin{aligned} \|\phi\|_{\mathbf{X}} = \sup_{t>0} \Big(& \langle t\rangle^{-1} \|\phi(t)\|_{\mathbf{L}^{\infty,\alpha+1}} + \langle t\rangle^{\frac{1}{\alpha}} \|\phi(t)\|_{\mathbf{L}^\infty} \\ & + t^{\frac{1}{\alpha}} \langle t\rangle^{\frac{1}{\alpha}} \|\phi_x(t)\|_{\mathbf{L}^\infty} + t^{\frac{1}{\alpha}} \langle t\rangle^{-1} \|\phi_x\|_{\mathbf{L}^{\infty,\alpha+1}} \Big). \end{aligned}$$

Changing $y = \langle t\rangle^{-\frac{1}{\alpha}} x$, the $\mathbf{L}^1(\mathbf{R})$ norm can be estimated as follows

$$\begin{aligned} \|u(t)\|_{\mathbf{L}^1} &= \int_{|x| \le \langle t\rangle^{\frac{1}{\alpha}}} |u(t,x)|\, dx \\ &+ \int_{|x| > \langle t\rangle^{\frac{1}{\alpha}}} \left|\langle t\rangle^{-\frac{1}{\alpha}} x\right|^{\alpha+1} |u(t,x)| \left|\langle t\rangle^{-\frac{1}{\alpha}} x\right|^{-\alpha-1} dx \\ &\le C \langle t\rangle^{\frac{1}{\alpha}} \|u(t)\|_{\mathbf{L}^\infty} + C\langle t\rangle^{-1} \|u(t)\|_{\mathbf{L}^{\infty,\alpha+1}} \int_{|y|>1} |y|^{-\alpha-1} dy \\ &\le C \|u\|_{\mathbf{X}}. \end{aligned}$$

Applying Lemma 1.40 we obtain the estimates

$$\|\mathcal{G}\phi\|_{\mathbf{X}} \leq C \|\phi\|_{\mathbf{L}^{\infty,\alpha+1}}$$

and

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(w)(\tau) - \mathcal{N}(v)(\tau))\, d\tau \right\|_{\mathbf{L}^\infty} \\
&\leq C \int_0^{\frac{t}{2}} \|\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau))\|_{\mathbf{L}^1} (t-\tau)^{-\frac{1}{\alpha}}\, d\tau \\
&+ C \int_{\frac{t}{2}}^t \|\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau))\|_{\mathbf{L}^\infty}\, d\tau.
\end{aligned}$$

Then using the estimates

$$\begin{aligned}
&\|\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau))\|_{\mathbf{L}^\infty} \\
&\leq C \|w-v\|_{\mathbf{L}^\infty} \left(\|w\|_{\mathbf{L}^\infty}^\sigma + \|v\|_{\mathbf{L}^\infty}^\sigma\right) \\
&+ \|w_x - v_x\|_{\mathbf{L}^\infty} \left(\|w\|_{\mathbf{L}^\infty}^\rho + \|v\|_{\mathbf{L}^\infty}^\rho\right) \\
&\leq C \langle\tau\rangle^{-\frac{\sigma+1}{\alpha}} \|w-v\|_{\mathbf{X}} \left(\|w\|_{\mathbf{X}}^\sigma + \|v\|_{\mathbf{X}}^\sigma\right) \\
&+ C\tau^{-\frac{1}{\alpha}} \langle\tau\rangle^{-\frac{\rho+1}{\alpha}} \|w-v\|_{\mathbf{X}} \left(\|w\|_{\mathbf{X}}^\rho + \|v\|_{\mathbf{X}}^\rho\right)
\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}
&\|\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau))\|_{\mathbf{L}^1} \\
&\leq C \langle\tau\rangle^{-\frac{\sigma}{\alpha}} \|w-v\|_{\mathbf{X}} \left(\|w\|_{\mathbf{X}}^\sigma + \|v\|_{\mathbf{X}}^\sigma\right) \\
&+ C\tau^{-\frac{1}{\alpha}} \langle\tau\rangle^{-\frac{\rho}{\alpha}} \|w-v\|_{\mathbf{X}} \left(\|w\|_{\mathbf{X}}^\rho + \|v\|_{\mathbf{X}}^\rho\right)
\end{aligned} \tag{2.32}$$

we get

$$\begin{aligned}
&t^{\frac{1}{\alpha}} \left\| \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(u_1)(\tau) - \mathcal{N}(u_2)(\tau))\, d\tau \right\|_{\mathbf{L}^\infty} \\
&\leq C \|w-v\|_{\mathbf{X}} \left(\|w\|_{\mathbf{X}}^\rho + \|v\|_{\mathbf{X}}^\rho + \|w\|_{\mathbf{X}}^\sigma + \|v\|_{\mathbf{X}}^\sigma\right) \\
&\times \left(\int_0^{\frac{t}{2}} \left(\langle\tau\rangle^{-\frac{\sigma}{\alpha}} + \tau^{-\frac{1}{\alpha}} \langle\tau\rangle^{-\frac{\rho}{\alpha}} \right) (t-\tau)^{-\frac{1}{\alpha}}\, d\tau \right. \\
&\left. + \int_{\frac{t}{2}}^t \left(\langle\tau\rangle^{-\frac{\sigma+1}{\alpha}} + \tau^{-\frac{1}{\alpha}} \langle\tau\rangle^{-\frac{\rho+1}{\alpha}} \right) d\tau \right) \\
&\leq C \|w-v\|_{\mathbf{X}} \left(\|w\|_{\mathbf{X}}^\rho + \|v\|_{\mathbf{X}}^\rho + \|w\|_{\mathbf{X}}^\sigma + \|v\|_{\mathbf{X}}^\sigma\right)
\end{aligned}$$

for $t > 1$, since $\sigma > \alpha$ and $\rho > \alpha - 1$. For all $0 < t < 1$ we have

$$\left\| \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)))\, d\tau \right\|_{\mathbf{L}^\infty}$$
$$\leq C \|w-v\|_{\mathbf{X}} \left(\|w\|_{\mathbf{X}}^\rho + \|v\|_{\mathbf{X}}^\rho + \|w\|_{\mathbf{X}}^\sigma + \|v\|_{\mathbf{X}}^\sigma \right) \int_0^t \left(1 + \tau^{-\frac{1}{\alpha}}\right) d\tau$$
$$< C \|w-v\|_{\mathbf{X}} \left(\|w\|_{\mathbf{X}}^\rho + \|v\|_{\mathbf{X}}^\rho + \|w\|_{\mathbf{X}}^\sigma + \|v\|_{\mathbf{X}}^\sigma \right).$$

Combining these two estimates we get

$$\langle t \rangle^{\frac{1}{\alpha}} \left\| \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)))\, d\tau \right\|_{\mathbf{L}^\infty}$$
$$< C \|w-v\|_{\mathbf{X}} \left(\|w\|_{\mathbf{X}}^\rho + \|v\|_{\mathbf{X}}^\rho + \|w\|_{\mathbf{X}}^\sigma + \|v\|_{\mathbf{X}}^\sigma \right)$$

for all $t > 0$. Similarly we obtain the estimate (for simplicity we consider only one function $\mathcal{N}(v)$)

$$\left\| \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(v)(\tau))\, d\tau \right\|_{\mathbf{L}^{\infty,1+a}}$$
$$\leq \int_0^t \|\mathcal{N}(v(\tau))\|_{\mathbf{L}^1} \langle t-\tau \rangle\, d\tau + \int_0^t \|\mathcal{N}(v(\tau))\|_{\mathbf{L}^{\infty,\alpha+1}}\, d\tau; \tag{2.33}$$

hence via

$$\|\mathcal{N}(v(\tau))\|_{\mathbf{L}^{\infty,\alpha+1}} \leq C \|v\|_{\mathbf{L}^\infty}^\sigma \|v\|_{\mathbf{L}^{\infty,\alpha+1}} + C \|v\|_{\mathbf{L}^\infty}^\rho \|v_x\|_{\mathbf{L}^{\infty,\alpha+1}}$$
$$\leq C \|v\|_{\mathbf{X}}^{\sigma+1} \langle \tau \rangle^{1-\frac{\sigma}{\alpha}} + C \|v\|_{\mathbf{X}}^{\rho+1} \tau^{-\frac{1}{\alpha}} \langle \tau \rangle^{1-\frac{\rho}{\alpha}}, \tag{2.34}$$

we have for $t > 0$

$$\left\| \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(v)(\tau))\, d\tau \right\|_{\mathbf{L}^{\infty,1+a}}$$
$$\leq C \int_0^t \left(\|v\|_{\mathbf{X}}^{\sigma+1} \langle \tau \rangle^{-\frac{\sigma}{\alpha}} + \|v\|_{\mathbf{X}}^{\rho+1} \tau^{-\frac{1}{\alpha}} \langle \tau \rangle^{-\frac{\rho}{\alpha}} \right) \langle t-\tau \rangle\, d\tau$$
$$+ \int_0^t \left(\|v\|_{\mathbf{X}}^{\sigma+1} \langle \tau \rangle^{1-\frac{\sigma}{\alpha}} + \|v\|_{\mathbf{X}}^{\rho+1} \tau^{-\frac{1}{\alpha}} \langle \tau \rangle^{1-\frac{\rho}{\alpha}} \right) d\tau$$
$$\leq C \langle t \rangle \left(\|v\|_{\mathbf{X}}^{\sigma+1} + \|v\|_{\mathbf{X}}^{\rho+1} \right).$$

In the identical manner we obtain

$$\begin{aligned}
&\left\|\partial_x \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(v)(\tau))\,d\tau\right\|_{\mathbf{L}^\infty} \\
&\le C\int_0^{\frac{t}{2}} \|\mathcal{N}(v(\tau))\|_{\mathbf{L}^1}(t-\tau)^{-\frac{2}{\alpha}}\,d\tau \\
&+ C\int_{\frac{t}{2}}^t \|\mathcal{N}(v(\tau))\|_{\mathbf{L}^\infty}(t-\tau)^{-\frac{1}{\alpha}}\,d\tau \\
&\le C(\|v\|_{\mathbf{X}}^{\sigma+1}+\|v\|_{\mathbf{X}}^{\rho+1})\left(\int_0^{\frac{t}{2}}\left(\langle\tau\rangle^{-\frac{\sigma}{\alpha}}+\tau^{-\frac{1}{\alpha}}\langle\tau\rangle^{-\frac{\rho}{\alpha}}\right)(t-\tau)^{-\frac{2}{\alpha}}\,d\tau\right. \\
&\left.+\int_{\frac{t}{2}}^t\left(\langle\tau\rangle^{-\frac{\sigma+1}{\alpha}}+\tau^{-\frac{1}{\alpha}}\langle\tau\rangle^{-\frac{\rho+1}{\alpha}}\right)(t-\tau)^{-\frac{1}{\alpha}}\,d\tau\right) \\
&\le C(\|v\|_{\mathbf{X}}^{\sigma+1}+\|v\|_{\mathbf{X}}^{\rho+1})t^{-\frac{2}{\alpha}},
\end{aligned}$$

for all $t>1$ and for all $0<t<1$ we have

$$\begin{aligned}
&\left\|\partial_x \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(v)(\tau))\,d\tau\right\|_{\mathbf{L}^\infty} \\
&\le C\int_0^t \|\mathcal{N}(v(\tau))\|_{\mathbf{L}^\infty}(t-\tau)^{-\frac{1}{\alpha}}\,d\tau \\
&\le C(\|v\|_{\mathbf{X}}^{\sigma+1}+\|v\|_{\mathbf{X}}^{\rho+1})\int_0^t\left(1+\tau^{-\frac{1}{\alpha}}\right)(t-\tau)^{-\frac{1}{\alpha}}\,d\tau \\
&\le C(\|v\|_{\mathbf{X}}^{\sigma+1}+\|v\|_{\mathbf{X}}^{\rho+1})t^{-\frac{1}{\alpha}}.
\end{aligned}$$

Combining these estimates we get

$$\left\|\partial_x \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(v)(\tau))\,d\tau\right\|_{\mathbf{L}^\infty} \le C(\|v\|_{\mathbf{X}}^{\sigma+1}+\|v\|_{\mathbf{X}}^{\rho+1})t^{-\frac{1}{\alpha}}\langle t\rangle^{-\frac{1}{\alpha}} \tag{2.35}$$

for all $t>0$. Similarly we estimate

$$\begin{aligned}
&\left\|\partial_x \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(v)(\tau))\,d\tau\right\|_{\mathbf{L}^{\infty,1+a}} \\
&\le \int_0^t \|\mathcal{N}(v(\tau))\|_{\mathbf{L}^1}\langle t-\tau\rangle(t-\tau)^{-\frac{1}{\alpha}}\,d\tau \\
&+ \int_0^t \|\mathcal{N}(v(\tau))\|_{\mathbf{L}^{\infty,\alpha+1}}(t-\tau)^{-\frac{1}{\alpha}}\,d\tau \\
&\le C(\|v\|_{\mathbf{X}}^{\sigma+1}+\|v\|_{\mathbf{X}}^{\rho+1})\left(\int_0^t\left(\langle\tau\rangle^{-\frac{\sigma}{\alpha}}+\tau^{-\frac{1}{\alpha}}\langle\tau\rangle^{-\frac{\rho}{\alpha}}\right)\langle t-\tau\rangle(t-\tau)^{-\frac{1}{\alpha}}\,d\tau\right. \\
&\left.+\int_0^t\left(\langle\tau\rangle^{1-\frac{\sigma}{\alpha}}+\tau^{-\frac{1}{\alpha}}\langle\tau\rangle^{1-\frac{\rho}{\alpha}}\right)(t-\tau)^{-\frac{1}{\alpha}}\,d\tau\right) \\
&\le C(\|v\|_{\mathbf{X}}^{\sigma+1}+\|v\|_{\mathbf{X}}^{\rho+1})t^{-\frac{1}{\alpha}}\langle t\rangle
\end{aligned}$$

for all $t > 0$, since $\sigma > \alpha$ and $\rho > \alpha - 1$. Thus we get from (2.33) through (2.35)

$$\left\| \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(v)(\tau))\, d\tau \right\|_{\mathbf{X}} \leq C(\|v\|_{\mathbf{X}}^{\sigma+1} + \|v\|_{\mathbf{X}}^{\rho+1}).$$

In the same manner we estimate the difference $\mathcal{N}(w) - \mathcal{N}(v)$. Therefore due to Theorem 1.17 there exists a unique solution

$$u(t,x) \in \mathbf{C}\left([0,\infty); \mathbf{L}^{\infty,\alpha+1}(\mathbf{R})\right) \cap \mathbf{C}\left((0,\infty); \mathbf{W}_\infty^1(\mathbf{R})\right)$$

to the problem (2.26).

Now we apply Theorem 2.4 for obtaining the large time asymptotics of solutions. By Lemma 1.40 we get the following asymptotic representation

$$\mathcal{G}(t)\, u_0 = \theta t^{-\frac{1}{\alpha}} G(\xi) + O\left(t^{-\frac{1}{\alpha}-\gamma} \langle \xi \rangle^{-1-\alpha}\right)$$

for large time $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $\xi = t^{-\frac{1}{\alpha}} x$, $\gamma = \min\left(1, \frac{1}{\alpha}\right)$ and

$$\theta = \int_{\mathbf{R}} u_0(x)\, dx.$$

Now we consider the difference

$$\begin{aligned}
&\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\, d\tau - t^{-\frac{1}{\alpha}} G\left(t^{-\frac{1}{\alpha}} x\right) \int_0^\infty \vartheta(\tau)\, d\tau \\
&= \int_0^{\frac{t}{2}} \left(\mathcal{G}(t-\tau)\mathcal{N}(u)(\tau) - \vartheta(\tau)(t-\tau)^{-\frac{1}{\alpha}} G\left((t-\tau)^{-\frac{1}{\alpha}} x\right)\right) d\tau \\
&+ \int_0^{\frac{t}{2}} \left((t-\tau)^{-\frac{1}{\alpha}} G\left((t-\tau)^{-\frac{1}{\alpha}} x\right) - t^{-\frac{1}{\alpha}} G\left(t^{-\frac{1}{\alpha}} x\right)\right) \vartheta(\tau)\, d\tau \\
&+ t^{-\frac{1}{\alpha}} G\left(t^{-\frac{1}{\alpha}} x\right) \int_{\frac{t}{2}}^\infty \vartheta(\tau)\, d\tau, \qquad (2.36)
\end{aligned}$$

where

$$\vartheta(\tau) = \int_{\mathbf{R}} \mathcal{N}(u)(\tau,x)\, dx = a \int_{\mathbf{R}} |u(\tau,x)|^\sigma\, dx.$$

We have

$$\vartheta(\tau) \leq C \|u\|_{\mathbf{X}}^\sigma \{t\}^{-\alpha_1} \langle t \rangle^{-1-\gamma},$$

where $0 < \alpha_1 < 1$. By Lemma 1.40

$$\begin{aligned}
&\left\| \left| t^{-\frac{1}{\alpha}} (\cdot) \right|^{\alpha+1} \left(\mathcal{G}(t-\tau)\mathcal{N}(u)(\tau) - \vartheta(\tau)(t-\tau)^{-\frac{1}{\alpha}} G\left((t-\tau)^{-\frac{1}{\alpha}} (\cdot)\right)\right) \right\|_{\mathbf{L}^\infty} \\
&\leq C t^{-\frac{2}{\alpha}} \|\mathcal{N}(u)(\tau)\|_{\mathbf{L}^{1,1}} + C t^{-\frac{\alpha+1}{\alpha}} \|\mathcal{N}(u)(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}}
\end{aligned}$$

for all $0 < \tau < \frac{t}{2}$ and

$$\left\| \left| t^{-\frac{1}{\alpha}} (\cdot) \right|^{\alpha+1} \mathcal{G}(t-\tau) \mathcal{N}(u)(\tau) \right\|_{\mathbf{L}^\infty}$$
$$\leq C \|\mathcal{N}(u)(\tau)\|_{\mathbf{L}^\infty} + C\tau^{-\frac{\alpha+1}{\alpha}} \|\mathcal{N}(u)(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}}$$

for all $\frac{t}{2} \leq \tau < t$. Therefore we have by virtue of estimates (2.31), (2.32) and (2.34)

$$\left\| \left\langle t^{-\frac{1}{\alpha}} (\cdot) \right\rangle^{\alpha+1} \int_0^{\frac{t}{2}} \left(\mathcal{G}(t-\tau) \mathcal{N}(u)(\tau) \right.\right.$$
$$\left.\left. -\vartheta(\tau)(t-\tau)^{-\frac{1}{\alpha}} G\left((t-\tau)^{-\frac{1}{\alpha}} x\right)\right) d\tau \right\|_{\mathbf{L}^\infty}$$
$$\leq C \int_0^{\frac{t}{2}} t^{-\frac{2}{\alpha}} \|\mathcal{N}(u)(\tau)\|_{\mathbf{L}^{1,1}} d\tau + C \int_0^{\frac{t}{2}} t^{-\frac{\alpha+1}{\alpha}} \|\mathcal{N}(u)(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} d\tau$$
$$\leq C \|u\|_{\mathbf{X}}^\sigma \left(t^{-\frac{2}{\alpha}} \int_0^{\frac{t}{2}} \left(\langle\tau\rangle^{\frac{1-\sigma}{\alpha}} + \tau^{-\frac{1}{\alpha}} \langle\tau\rangle^{\frac{1-\rho}{\alpha}} \right) d\tau \right.$$
$$\left. + t^{-1-\frac{1}{\alpha}} \int_0^{\frac{t}{2}} \left(\langle\tau\rangle^{1-\frac{\sigma}{\alpha}} + \tau^{-\frac{1}{\alpha}} \langle\tau\rangle^{1-\frac{\rho}{\alpha}} \right) d\tau \right) \leq C \langle t\rangle^{-\frac{1}{\alpha}-\gamma} \|u\|_{\mathbf{X}}^\sigma$$

and

$$\left\| \left\langle t^{-\frac{1}{\alpha}} (\cdot) \right\rangle^{\alpha+1} \int_{\frac{t}{2}}^{t} \mathcal{G}(t-\tau) \mathcal{N}(u)(\tau)\, d\tau \right\|_{\mathbf{L}^\infty}$$
$$\leq C \|u\|_{\mathbf{X}}^\sigma \int_{\frac{t}{2}}^{t} \|\mathcal{N}(u)(\tau)\|_{\mathbf{L}^\infty} d\tau + C \int_{\frac{t}{2}}^{t} \tau^{-\frac{\alpha+1}{\alpha}} \|\mathcal{N}(u)(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} d\tau$$
$$\leq C \|u\|_{\mathbf{X}}^\sigma \int_{\frac{t}{2}}^{t} \left(\langle\tau\rangle^{-\frac{\sigma+1}{\alpha}} + \tau^{-\frac{1}{\alpha}} \langle\tau\rangle^{-\frac{\rho+1}{\alpha}} \right) d\tau \leq C \langle t\rangle^{-\frac{1}{\alpha}-\gamma} \|u\|_{\mathbf{X}}^\sigma,$$

where $\gamma = \min\left(\frac{\sigma}{\alpha}, \frac{\rho+1}{\alpha}\right) - 1 > 0$.

We now estimate the difference

$$\left\| \left\langle t^{-\frac{1}{\alpha}} x \right\rangle^{\alpha+1} \int_0^{\frac{t}{2}} \left((t-\tau)^{-\frac{1}{\alpha}} G\left((t-\tau)^{-\frac{1}{\alpha}} x\right) - t^{-\frac{1}{\alpha}} G\left(t^{-\frac{1}{\alpha}} x\right)\right) \vartheta(\tau)\, d\tau \right\|_{\mathbf{L}^\infty}$$
$$\leq C \int_0^{\frac{t}{2}} \left\| \left\langle t^{-\frac{1}{\alpha}} (\cdot) \right\rangle^{\alpha+1} \left((t-\tau)^{-\frac{1}{\alpha}} G\left((t-\tau)^{-\frac{1}{\alpha}} x\right) - t^{-\frac{1}{\alpha}} G\left(t^{-\frac{1}{\alpha}} x\right)\right) \right\|_{\mathbf{L}^\infty}$$
$$\times |\vartheta(\tau)|\, d\tau \leq C \|u\|_{\mathbf{X}}^\sigma \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{\alpha}-\gamma} \langle\tau\rangle^{\gamma-\frac{\sigma}{\alpha}} d\tau \leq C \langle t\rangle^{-\frac{1}{\alpha}-\gamma} \|u\|_{\mathbf{X}}^\sigma.$$

For the last summand in (2.36) we have

$$\left\| \left\langle t^{-\frac{1}{\alpha}} (\cdot) \right\rangle^{\alpha+1} t^{-\frac{1}{\alpha}} G\left(t^{-\frac{1}{\alpha}} x\right) \int_{\frac{t}{2}}^{\infty} \vartheta(\tau)\, d\tau \right\|_{\mathbf{L}^\infty}$$
$$\leq Ct^{-\frac{1}{\alpha}} \|u\|_{\mathbf{X}}^{\sigma} \int_{\frac{t}{2}}^{\infty} \langle\tau\rangle^{-\frac{\sigma}{\alpha}} d\tau \leq C \langle t\rangle^{-\frac{1}{\alpha}-\gamma} \|u\|_{\mathbf{X}}^{\sigma}.$$

Thus in the same way as in the proof of Theorem 2.4 we see that there exists a constant

$$A = \theta + \int_0^\infty \vartheta(\tau)\, d\tau$$

such that asymptotics (2.28) is valid. Theorem 2.22 is proved.

2.2.2 Large initial data

When the coefficient of the nonlinearity $a > 0$ and the order of derivative $\alpha \in (1, 2]$, due to the special form of the nonlinearity $\mathcal{N}(u) = a|u|^{\sigma-1} u + b|u|^{\rho} u_x$, we can prove global existence of solutions to the Cauchy problem (2.26) without any restriction to the size of the initial data.

Theorem 2.25. *Let $\sigma > \alpha$, $\rho > \alpha - 1$, $\alpha \in (1, 2)$ and $a > 0$. Suppose that the initial data $u_0 \in \mathbf{L}^{\infty,\alpha+1}(\mathbf{R})$. Then there exists a unique solution*

$$u \in \mathbf{C}\left([0, \infty); \mathbf{L}^{\infty,\alpha+1}(\mathbf{R})\right) \cap \mathbf{C}\left((0, \infty); \mathbf{W}_\infty^1(\mathbf{R})\right)$$

to the Cauchy problem for equation (2.26). Moreover the asymptotics (2.28) is true.

To prove Theorem 2.25 we first apply the so-called energy method to estimate the $\mathbf{L}^2(\mathbf{R})$ norm: that is we multiply equation (2.26) by $2u$ and integrate with respect to $x \in \mathbf{R}$, to get

$$\frac{d}{dt} \|u(t)\|_{\mathbf{L}^2}^2 \leq -2 \left\| |\partial_x|^{\frac{\alpha}{2}} u(t) \right\|_{\mathbf{L}^2}^2; \tag{2.37}$$

hence integrating with respect to time we see that

$$\|u(t)\|_{\mathbf{L}^2}^2 + 2\int_0^t \left\| |\partial_x|^{\frac{\alpha}{2}} u(\tau) \right\|_{\mathbf{L}^2}^2 d\tau \leq \|u_0\|_{\mathbf{L}^2}^2$$

for all $t \geq 0$. In particular we have

$$\sup_{t\geq 0} \|u(t)\|_{\mathbf{L}^2} \leq \|u_0\|_{\mathbf{L}^2}. \tag{2.38}$$

Note that time decay estimate (2.38) is not optimal. To get an optimal time decay estimate we need to show that the $\mathbf{L}^1$ norm of the solution does not grow with time. The important point is that we can easily show that the norm $\|u(t)\|_{\mathbf{L}^1}$ is bounded. Using the ideas of papers Bardos et al. [1979], Biler et al.

[1998], Lax [1971] we multiply equation (2.26) by $S(t,x) \equiv \text{sign}\, u \equiv \frac{u}{|u|}$ and integrate with respect to x over $\mathbf{R}$ to get

$$\int_{\mathbf{R}} u_t(t,x) S(t,x)\, dx + \int_{\mathbf{R}} \mathcal{N}(u)(t,x) |S(t,x)\, dx$$
$$= -\int_{\mathbf{R}} S(t,x) |\partial_x|^{\alpha} u(t,x)\, dx. \qquad (2.39)$$

We have

$$\int_{\mathbf{R}} u_t(t,x) S(t,x)\, dx = \int_{\mathbf{R}} \frac{\partial}{\partial t} |u(t,x)|\, dx = \frac{d}{dt} \|u(t)\|_{\mathbf{L}^1},$$

$$\int_{\mathbf{R}} \mathcal{N}(u)(t,x) |S(t,x)\, dx$$
$$= a\int_{\mathbf{R}} |u|^{\sigma+1} dx + \frac{b}{\rho+1} \int_{\mathbf{R}} \frac{\partial}{\partial x} \left(|u(t,x)|^{\rho} u(t,x)\right) dx \geq 0.$$

Representing the operator $|\partial_x|^{\alpha}$ via the Riesz potential (see Stein [1970]) let us show that

$$\int_{\mathbf{R}} S(t,x) |\partial_x|^{\alpha} u(t,x)\, dx \leq 0. \qquad (2.40)$$

We have by Erdélyi et al. [1954]

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} |x|^{-\nu} e^{ixy} dx = \frac{\pi}{\sqrt{2\pi}\Gamma(\nu)\cos\left(\frac{\pi\nu}{2}\right)} |y|^{\nu-1}$$

for $\nu \in (0,1)$, then

$$|\partial_x|^{-\nu} \phi = \mathcal{F}^{-1} |\eta|^{-\nu} \widehat{\phi}(\eta) = C\int_{\mathbf{R}} |x-y|^{\nu-1} \phi(y)\, dy.$$

Thus for $\alpha \in (1,2)$ we get with $\nu = 2-\alpha \in (0,1)$ integrating by parts with respect to $y \in \mathbf{R}$ two times

$$|\partial_x|^{\alpha} \phi = \mathcal{F}^{-1} |\eta|^{\alpha} \widehat{\phi}(\eta) = -\partial_x^2 |\partial_x|^{-\nu} \phi = -C\partial_x^2 \int_{\mathbf{R}} |x-y|^{1-\alpha} \phi(y)\, dy.$$

Denote $S(t,x) = \text{sign}(u(t,x))$ and represent $u(t,x) = S(t,x)|u(t,x)|$. We make a regularization

$$K_{\varepsilon}''(x) = \begin{cases} \partial_x^2 |x|^{1-\alpha}, \text{ for } |x| \geq \varepsilon \\ 0, \text{ for } |x| < \varepsilon, x \neq 0. \end{cases}$$

Note that $K_{\varepsilon}''(x) \geq 0$ for all $x \in \mathbf{R}\backslash\{0\}$. We can easily see that

$$\partial_x^2 \int_{\mathbf{R}} |x-y|^{1-\alpha} u(t,y)\, dy = \lim_{\varepsilon\to 0} \partial_x^2 \int_{\mathbf{R}} K_{\varepsilon}(x-y) u(t,y)\, dy.$$

(To justify our calculations we note that the linear operator $\mathcal{L}$ in equation (2.26) is strongly dissipative, therefore by the smoothing effect the solutions obtain regularity $u \in \mathbf{C}\left((0,\infty);\mathbf{C}^2(\mathbf{R})\right)$ (see Naumkin and Shishmarev [1994b]).) Then via identity $S(t,y)S(t,x) = 1 - \frac{1}{2}\left(S(t,x) - S(t,y)\right)^2$ we get

$$\begin{aligned}
&\int_{\mathbf{R}} dx S(t,x)\,\partial_x^2 \int_{\mathbf{R}} K_\varepsilon(x-y)\,u(t,y)\,dy \\
&= \int_{\mathbf{R}} dx S(t,x)\,\partial_x \int_{\mathbf{R}} dy K_\varepsilon(x-y)\,S(t,y)\,\partial_y |u(t,y)| \\
&= \int_{\mathbf{R}} dy \partial_y |u(t,y)| \int_{\mathbf{R}} dx S(t,y)\,S(t,x)\,\partial_x K_\varepsilon(x-y) \\
&= \int_{\mathbf{R}} dy \partial_y |u(t,y)| \int_{\mathbf{R}} dx \partial_x K_\varepsilon(x-y) \\
&- \frac{1}{2}\int_{\mathbf{R}} dy \partial_y |u(t,y)| \int_{\mathbf{R}} dx \left(S(t,x) - S(t,y)\right)^2 \partial_x K_\varepsilon(x-y) \\
&= -\frac{1}{2}\int_{\mathbf{R}} dy \partial_y |u(t,y)| \int_{\mathbf{R}} dx \left(S(t,x) - S(t,y)\right)^2 \partial_x K_\varepsilon(x-y) \\
&= -\frac{1}{2}\int_{\mathbf{R}} dy\, |u(t,y)| \int_{\mathbf{R}} dx K_\varepsilon''(x-y)\left(S(t,x) - S(t,y)\right)^2 \le 0;
\end{aligned}$$

hence we have

$$\begin{aligned}
&\int_{\mathbf{R}} S(t,x)\,|\partial_x|^\alpha u(t,x)\,dx = -C\int_{\mathbf{R}} dx S(t,x)\,\partial_x^2 \int_{\mathbf{R}} dy\,|x-y|^{1-\alpha} u(t,y) \\
&= \frac{C}{2}\lim_{\varepsilon\to 0}\int_{\mathbf{R}} dy\,|u(t,y)| \int_{\mathbf{R}} dx K_\varepsilon''(x-y)\left(S(t,x) - S(t,y)\right)^2 \ge 0.
\end{aligned}$$

Thus (2.40) is true, and from (2.39) we find

$$\frac{d}{dt}\|u(t)\|_{\mathbf{L}^1} \le 0. \tag{2.41}$$

By (2.41) we see that the norm $\|u(t)\|_{\mathbf{L}^1}$ is bounded for all $t \ge 0$. We now prove that the norm $\|u(t)\|_{\mathbf{L}^2} \to 0$ as $t \to \infty$. Taking $\varrho \in (0,1)$ from the Plancherel theorem we get

$$\begin{aligned}
\left\| |\partial_x|^{\frac{\alpha}{2}} u(t)\right\|_{\mathbf{L}^2}^2 &= \int_{\mathbf{R}} |\eta|^\alpha |\widehat{u}(t,\eta)|^2\,d\eta \ge \int_{|\eta|\ge\varrho} |\eta|^\alpha |\widehat{u}(t,\eta)|^2\,d\eta \\
&\ge \varrho^\alpha \|u(t)\|_{\mathbf{L}^2}^2 - \varrho^{\alpha+1}\sup_{|\eta|<\varrho}|\widehat{u}(t,\eta)|^2 \\
&\ge \varrho^\alpha \|u(t)\|_{\mathbf{L}^2}^2 - \varrho^{\alpha+1}\|u(t)\|_{\mathbf{L}^1}^2.
\end{aligned}$$

Since the norm $\|u(t)\|_{\mathbf{L}^1}$ is bounded, by choosing $\varrho(t) = (1+t)^{-\frac{1}{\alpha}}$ we obtain

$$\frac{d}{dt}\|u(t)\|_{\mathbf{L}^2}^2 \le -2(1+t)^{-1}\|u(t)\|_{\mathbf{L}^2}^2 + C(1+t)^{-1-\frac{1}{\alpha}}.$$

We substitute $\|u(t)\|_{\mathbf{L}^2}^2 = h(t)(1+t)^{-2}$, then for $h(t)$ we have

$$h'(t) \le C(1+t)^{1-\frac{1}{\alpha}};$$

hence integration with respect to time yields

$$h(t) \le C(1+t)^{2-\frac{1}{\alpha}}.$$

Therefore we get the optimal time decay estimate

$$\|u(t)\|_{\mathbf{L}^2} \le C(1+t)^{-\frac{1}{2\alpha}}.$$

Now we estimate the $\mathbf{L}^\infty(\mathbf{R})$ norm of the solutions. Since

$$u \in \mathbf{C}\left([0,\infty);\mathbf{L}^{\infty,\alpha+1}(\mathbf{R})\right) \cap \mathbf{C}\left((0,\infty);\mathbf{W}_\infty^1(\mathbf{R})\right)$$

we see that $\sup_{x\in\mathbf{R}} u(t,x) \ge 0$ and $\inf_{x\in\mathbf{R}} u(t,x) \le 0$ for all $t \in (0,\infty)$.

By the method of paper Constantin and Escher [1998] we obtain the following result.

Lemma 2.26. *Let $u \in \mathbf{C}^1\left((T_1,T_2);\mathbf{L}^{\infty,\alpha+1}(\mathbf{R}) \cap \mathbf{C}(\mathbf{R})\right)$ and*

$$\widetilde{u}(t) = \inf_{x\in\mathbf{R}} u(t,x) < 0$$

for all $t \in (T_1,T_2)$. Then there exists a point $\zeta(t) \in \mathbf{R}$ such that $\widetilde{u}(t) = u(t,\zeta(t))$; moreover $\widetilde{u}'(t) = u_t(t,\zeta(t))$ almost everywhere on (T_1,T_2).

Proof. Since $\widetilde{u}(t) = \inf_{x\in\mathbf{R}} u(t,x) < 0$ and $u(t,x) \in \mathbf{L}^{\infty,\alpha+1}(\mathbf{R}) \cap \mathbf{C}(\mathbf{R})$ for every $t \in (T_1,T_2)$, we see that there exists a point $\zeta(t) \in \mathbf{R}$ such that $\widetilde{u}(t) = u(t,\zeta(t))$ for all $t \in (T_1,T_2)$. By virtue of the estimate

$$\begin{aligned}\widetilde{u}(s) - \widetilde{u}(t) &\le u(s,\zeta(t)) - u(t,\zeta(t)) \le \|u(s) - u(t)\|_{\mathbf{L}^\infty} \\ &\le \left\| \int_t^s u_t(\tau)\,d\tau \right\|_{\mathbf{L}^\infty} \le |t-s| \sup_{\tau\in(s,t)} \|u_\tau(\tau)\|_{\mathbf{L}^\infty}\end{aligned}$$

for all $s,t \in (T_1,T_2)$ we see that $\widetilde{u}(t)$ has a bounded variation on (T_1,T_2) and consequently is almost everywhere differentiable on (T_1,T_2) (see Kolmogorov and Fomin [1957]). Then we have

$$\widetilde{u}'(t) = \lim_{s\to t+0} \frac{\widetilde{u}(s) - \widetilde{u}(t)}{s-t} \le \lim_{s\to t+0} \frac{u(s,\zeta(t)) - u(t,\zeta(t))}{s-t} = u_t(t,\zeta(t))$$

and

$$\widetilde{u}'(t) = \lim_{s\to t-0} \frac{\widetilde{u}(t) - \widetilde{u}(s)}{t-s} \ge \lim_{s\to t-0} \frac{u(t,\zeta(t)) - u(s,\zeta(t))}{t-s} = u_t(t,\zeta(t))$$

almost everywhere on (T_1,T_2), since

$$u_t(t,x) = \lim_{s\to t} \frac{1}{t-s}\left(u(t,x) - u(s,x)\right)$$

uniformly with respect to $x \in \mathbf{R}$. Therefore $\widetilde{u}'(t) = u_t(t,\zeta(t))$ almost everywhere on (T_1,T_2). Lemma 2.26 is proved.

Thus taking the point $\zeta(t)$ in equation (2.26) we get for the function $\widetilde{u}(t) = \inf_{x\in\mathbf{R}} u(t,x)$

$$\frac{d}{dt}\widetilde{u}(t) \geq 0,$$

it then follows that $\widetilde{u}(t) \geq \widetilde{u}(0)$ for all $t > 0$. In the same manner we have $\sup_{x\in\mathbf{R}} u(t,x) \leq \sup_{x\in\mathbf{R}} u_0(x)$ for all $t > 0$. Thus the $\mathbf{L}^\infty$ norm is bounded

$$\|u(t)\|_{\mathbf{L}^\infty} \leq \|u_0\|_{\mathbf{L}^\infty}$$

for all $t \geq 0$. To get an optimal time decay estimate for the $\mathbf{L}^\infty(\mathbf{R})$ norm we rewrite the integral equation (2.30) in the form

$$u(x,t) = \mathcal{G}(t)u_0 + a\int_0^t d\tau \int_{\mathbf{R}} G(t-\tau, x-y)\,|u(\tau,y)|^\sigma u(\tau,y)\,dy$$
$$+ \frac{b}{\rho+1}\int_0^t d\tau \int_{\mathbf{R}} G_x(t-\tau, x-y)\,|u(\tau,y)|^\rho u(\tau,y)\,dy.$$

Applying the Young inequality for convolutions we obtain with $\frac{1}{q} = 1 - \frac{1}{p} = \frac{2}{3}(\alpha - 1)$, and $1 - \frac{\alpha-1}{3} < \delta < \min\left(1, \rho + 1 - \frac{2}{q}\right)$

$$\begin{aligned}
\|u\|_{\mathbf{L}^\infty} &\leq \min\left(\|u_0\|_{\mathbf{L}^1}\|G(t)\|_{\mathbf{L}^\infty}, \|u_0\|_{\mathbf{L}^\infty}\|G(t)\|_{\mathbf{L}^1}\right) \\
&+ C\int_0^t \|u(\tau)\|_{\mathbf{L}^\infty}^{\sigma-1}\|u(\tau)\|_{\mathbf{L}^2}^2\|G(t-\tau)\|_{\mathbf{L}^\infty}\,d\tau \\
&+ C\int_0^t \left\||u(\tau)|^{\rho+1}\right\|_{\mathbf{L}^q}\|G_x(t-\tau)\|_{\mathbf{L}^p}\,d\tau \\
&\leq C\langle t\rangle^{-\frac{1}{\alpha}} + C\int_0^t \|u(\tau)\|_{\mathbf{L}^\infty}^{\sigma-1}\langle\tau\rangle^{-\frac{1}{\alpha}}(t-\tau)^{-\frac{1}{\alpha}}\,d\tau \\
&+ C\int_0^t \|u(\tau)\|_{\mathbf{L}^\infty}^{\rho+1-\frac{2}{q}}\|u(\tau)\|_{\mathbf{L}^2}^{\frac{2}{q}}(t-\tau)^{-\frac{1}{\alpha}\left(2-\frac{1}{p}\right)}\,d\tau \\
&\leq C\langle t\rangle^{-\frac{1}{\alpha}} + C\int_0^t \|u(\tau)\|_{\mathbf{L}^\infty}^{\delta}\langle\tau\rangle^{-\frac{1}{\alpha q}}(t-\tau)^{-\frac{1}{\alpha}\left(1+\frac{1}{q}\right)}\,d\tau;
\end{aligned}$$

therefore the optimal time decay estimate follows

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-\frac{1}{\alpha}} \tag{2.42}$$

for all $t > 0$.

Now we can estimate the norm $\mathbf{L}^{\infty,\alpha+1}(\mathbf{R})$

$$\|u(t)\|_{\mathbf{L}^{\infty,\alpha+1}} \le \|u_0\|_{\mathbf{L}^1} \|G(t)\|_{\mathbf{L}^{\infty,\alpha+1}} + \|u_0\|_{\mathbf{L}^{\infty,\alpha+1}} \|G(t)\|_{\mathbf{L}^1}$$
$$+ C\int_0^t \left\| |u(\tau)|^{\sigma+1} \right\|_{\mathbf{L}^1} \|G(t-\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} d\tau$$
$$+ C\int_0^t \left\| |u(\tau)|^{\sigma+1} \right\|_{\mathbf{L}^{\infty,\alpha+1}} \|G(t-\tau)\|_{\mathbf{L}^1} d\tau$$
$$+ C\int_0^t \left\| |u(\tau)|^{\rho+1} \right\|_{\mathbf{L}^1} \|G_x(t-\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} d\tau$$
$$+ C\int_0^t \left\| |u(\tau)|^{\rho+1} \right\|_{\mathbf{L}^{\infty,\alpha+1}} \|G_x(t-\tau)\|_{\mathbf{L}^1} d\tau.$$

Consequently

$$\|u(t)\|_{\mathbf{L}^{\infty,\alpha+1}} \le C\langle t\rangle + C\int_0^t \|u(\tau)\|_{\mathbf{L}^\infty}^\sigma \langle t-\tau\rangle d\tau$$
$$+ C\int_0^t \|u(\tau)\|_{\mathbf{L}^\infty}^\rho \langle t-\tau\rangle (t-\tau)^{-\frac{1}{\alpha}} d\tau$$
$$+ C\int_0^t \|u(\tau)\|_{\mathbf{L}^\infty}^\sigma \|u(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} d\tau$$
$$+ C\int_0^t \|u(\tau)\|_{\mathbf{L}^\infty}^\rho \|u(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} (t-\tau)^{-\frac{1}{\alpha}} d\tau$$
$$\le C\langle t\rangle + C\int_0^t \|u(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} \left(\langle\tau\rangle^{-\frac{\sigma}{\alpha}} + \langle\tau\rangle^{-\frac{\rho}{\alpha}} (t-\tau)^{-\frac{1}{\alpha}}\right) d\tau.$$

Thus

$$\sup_{0\le\tau\le t} \|u(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} \le C\langle t\rangle + C\int_0^t \langle\tau\rangle^{-\frac{\sigma}{\alpha}} \|u(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} d\tau$$
$$+ C\int_{t-\varepsilon t}^t \|u(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} \langle\tau\rangle^{-\frac{\rho}{\alpha}} (t-\tau)^{-\frac{1}{\alpha}} d\tau$$
$$+ C\int_0^{t-\varepsilon t} \|u(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} \langle\tau\rangle^{-\frac{\rho}{\alpha}} (t-\tau)^{-\frac{1}{\alpha}} d\tau$$
$$\le C\langle t\rangle + C\int_0^t \langle\tau\rangle^{-\frac{\sigma}{\alpha}} \|u(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} d\tau + C\varepsilon^\gamma \sup_{0\le\tau\le t} \|u(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}}$$
$$+ C(\varepsilon t)^{-\frac{1}{\alpha}} \int_0^{t-\varepsilon t} \langle\tau\rangle^{-\frac{\rho}{\alpha}} \|u(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} d\tau.$$

Hence for the function $h(t) = \sup_{0\le\tau\le t} \|u(\tau)\|_{\mathbf{L}^{1,\alpha+1}}$ we get the inequality

$$h(t) \le C\langle t\rangle + C\int_0^t \langle\tau\rangle^{-\frac{\sigma}{\alpha}} h(\tau) d\tau + C(\varepsilon t)^{-\frac{1}{\alpha}} \int_0^{t-\varepsilon t} \langle\tau\rangle^{-\frac{\rho}{\alpha}} h(\tau) d\tau$$

and by Gronwall's Lemma it follows that

$$\|u(t)\|_{\mathbf{L}^{\infty,\alpha+1}} \le C\langle t\rangle \tag{2.43}$$

for all $t > 0$.

By a priori estimates (2.41) and (2.43), due to Theorem 1.20 there exists a unique solution

$$u(t,x) \in \mathbf{C}\left([0,\infty);\mathbf{L}^{\infty,\alpha+1}(\mathbf{R})\right) \cap \mathbf{C}\left((0,\infty);\mathbf{W}^1_\infty(\mathbf{R})\right)$$

to the problem (2.26), such that

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\langle t\rangle^{-\frac{1}{\alpha}}, \|u(t)\|_{\mathbf{L}^{\infty,\alpha+1}} \le C\langle t\rangle .$$

Then via Theorem 2.4 the asymptotic representation (2.28) for the solutions is valid.

2.2.3 Nonlinear heat equation

In this section we consider the case of the nonlinear heat equation (2.26) with $\alpha = 2$. We prove the following result.

Theorem 2.27. *Let $\sigma > 2$, $\rho > 1$, $\alpha = 2$ and $a > 0$. Suppose that the initial data are such that $u_0(x)e^{\frac{x^2}{4}} \in \mathbf{L}^\infty(\mathbf{R})$. Then there exists a unique solution $u \in \mathbf{C}([0,\infty);\mathbf{L}^\infty(\mathbf{R})) \cap \mathbf{C}\left((0,\infty);\mathbf{W}^1_\infty(\mathbf{R})\right)$ to the Cauchy problem (2.26). Moreover there exists a function A such that the asymptotics*

$$u(t,x) = t^{-\frac{1}{2}}e^{-\frac{\xi^2}{4}}A\left(\frac{\xi}{2\sqrt{t}}\right) + O\left(t^{-\frac{1}{2}-\gamma}e^{-\frac{\xi^2}{4}}\right) \tag{2.44}$$

is true for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $\xi = t^{-\frac{1}{2}}x$, $\gamma \in \left(0,\frac{1}{2}\right)$.

For our convenience in the case $\alpha = 2$ we put the initial data at $t = 1$. Thus we consider the Cauchy problem

$$\begin{cases} u_t - u_{xx} + \mathcal{N}(u) = 0, \ x \in \mathbf{R}, t > 1, \\ \qquad u(1,x) = u_1(x), \ x \in \mathbf{R}. \end{cases} \tag{2.45}$$

We now make a change of the dependent and independent variables $u(t,x) = t^{-\frac{1}{2}}e^{-t\xi^2}v(t,\xi)$, where $\xi = \frac{x}{2t}$, $E = e^{-t\xi^2}$. Then we get from (2.45)

$$\begin{cases} v_t - \frac{1}{4t^2}v_{\xi\xi} + \mathcal{P}(v) = 0, \ \xi \in \mathbf{R}, t > 1, \\ \qquad v(1,\xi) = e^{\xi^2}u_1(2\xi), \ \xi \in \mathbf{R}, \end{cases} \tag{2.46}$$

where the nonlinearity is

$$\mathcal{P}(v) = aE^\sigma t^{-\frac{\sigma}{2}}|v|^\sigma v + bt^{-\frac{\rho}{2}}E^\rho |v|^\rho \left(\frac{1}{2t}v_\xi - \xi v\right).$$

As in the previous section we get estimate

$$E\left|v\right| \leq C.$$

Then by Lemma 2.26 we get for the function $\widetilde{v}\left(t\right) = \max_{\xi\in\mathbf{R}}\left|v\left(t,\xi\right)\right|$

$$\widetilde{v}'\left(t\right) \leq C\left(t^{-\frac{\sigma}{2}} + t^{-\frac{\rho+1}{2}}\right)\widetilde{v}\left(t\right)$$

for all $t > 1$, where $\sigma > 2$ and $\rho > 1$. Hence by integrating with respect to time we obtain

$$\left\|v\left(t\right)\right\|_{\mathbf{L}^\infty} \leq C$$

for all $t > 1$. In the same manner we get the estimate

$$\left\|v_\xi\left(t\right)\right\|_{\mathbf{L}^\infty} \leq C\left\langle t\right\rangle^\gamma$$

for all $t > 1$, where $\gamma \in \left(0, \frac{1}{2}\right)$. To calculate the asymptotics as $t \to \infty$ we rewrite (2.46) in the integral form

$$\begin{aligned}
&v\left(t,\xi\right) = \frac{\sqrt{t}}{\sqrt{\pi\left(t-1\right)}}\int_{\mathbf{R}} e^{-\frac{t}{t-1}\left|\xi-\eta\right|^2} v\left(1,\eta\right)d\eta \\
&-\int_0^t d\tau \frac{1}{\sqrt{\pi\left(\frac{1}{\tau}-\frac{1}{t}\right)}}\int_{\mathbf{R}} e^{-\left|\xi-\eta\right|^2/\left(\frac{1}{\tau}-\frac{1}{t}\right)}\mathcal{P}\left(v\left(\tau,\eta\right)\right)d\eta.
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{\sqrt{t}}{\sqrt{\pi\left(t-1\right)}}\int_{\mathbf{R}} e^{-\frac{t}{t-1}\left|\xi-\eta\right|^2} v\left(1,\eta\right)d\eta \\
&= \frac{1}{\sqrt{\pi}}\int_{\mathbf{R}} e^{-\left|\xi-\eta\right|^2} v\left(1,\eta\right)d\eta \\
&+ \left(\frac{\sqrt{t}}{\sqrt{\pi\left(t-1\right)}} - \frac{1}{\sqrt{\pi}}\right)\int_{\mathbf{R}} e^{-\frac{t}{t-1}\left|\xi-\eta\right|^2} v\left(1,\eta\right)d\eta \\
&+ \frac{1}{\sqrt{\pi}}\int_{\mathbf{R}} e^{-\left|\xi-\eta\right|^2}\left(e^{-\frac{1}{t-1}\left|\xi-\eta\right|^2} - 1\right)v\left(1,\eta\right)d\eta \\
&= \frac{1}{\sqrt{\pi}}\int_{\mathbf{R}} e^{-\left|\xi-\eta\right|^2} v\left(1,\eta\right)d\eta + O\left(t^{-1}\left\|v\left(1\right)\right\|_{\mathbf{L}^\infty}\right). \qquad (2.47)
\end{aligned}$$

Then by the decay properties of the nonlinearity $\mathcal{P}\left(v\left(\tau,\eta\right)\right)$ we have the estimate

$$\int_{\frac{t}{2}}^t d\tau \frac{1}{\sqrt{\pi\left(\frac{1}{\tau}-\frac{1}{t}\right)}}\int_{\mathbf{R}} e^{-\left(\xi-\eta\right)^2/\left(\frac{1}{\tau}-\frac{1}{t}\right)}\mathcal{P}\left(v\left(\tau,\eta\right)\right)d\eta = O\left(t^{-\gamma}\right).$$

Finally we get

$$\int_1^{\frac{t}{2}} d\tau \frac{1}{\sqrt{\pi\left(\frac{1}{\tau}-\frac{1}{t}\right)}} \int_{\mathbf{R}} d\eta e^{-(\xi-\eta)^2/\left(\frac{1}{\tau}-\frac{1}{t}\right)} \mathcal{P}\left(v\left(\tau,\eta\right)\right)$$
$$= \int_1^{\frac{t}{2}} d\tau \mathcal{P}\left(v\left(\tau,\xi\right)\right) \frac{1}{\sqrt{\pi\left(\frac{1}{\tau}-\frac{1}{t}\right)}} \int_{\mathbf{R}} d\eta e^{-(\xi-\eta)^2/\left(\frac{1}{\tau}-\frac{1}{t}\right)}$$
$$+ \int_1^{\frac{t}{2}} d\tau \frac{1}{\sqrt{\pi\left(\frac{1}{\tau}-\frac{1}{t}\right)}} \int_{\mathbf{R}} d\eta e^{-(\xi-\eta)^2/\left(\frac{1}{\tau}-\frac{1}{t}\right)} \left(\mathcal{P}\left(v\left(\tau,\eta\right)\right) - \mathcal{P}\left(v\left(\tau,\xi\right)\right)\right)$$
$$= \int_1^{\frac{t}{2}} d\tau \mathcal{P}\left(v\left(\tau,\xi\right)\right) \frac{1}{\sqrt{\pi\left(\frac{1}{\tau}-\frac{1}{t}\right)}} \int_{\mathbf{R}} d\eta e^{-(\xi-\eta)^2/\left(\frac{1}{\tau}-\frac{1}{t}\right)} + O\left(t^{-\gamma}\right)$$
$$= \int_1^{\frac{t}{2}} \mathcal{P}\left(v\left(\tau,\xi\right)\right) d\tau + O\left(t^{-\gamma}\right) = \int_1^{\infty} \mathcal{P}\left(v\left(\tau,\xi\right)\right) d\tau + O\left(t^{-\gamma}\right). \tag{2.48}$$

Thus by (2.47) and (2.48) the asymptotics (2.44) is valid with a function

$$A\left(\xi\right) = \int_{\mathbf{R}} e^{-(\xi-\eta)^2} v\left(1,\eta\right) d\eta + \int_1^{\infty} \mathcal{P}\left(v\left(\tau,\xi\right)\right) d\tau.$$

Theorem 2.27 is proved.

2.3 Damped wave equation

This section studies the global existence to the Cauchy problem for the nonlinear damped wave equation

$$\begin{cases} \mathcal{L}u = \mathcal{N}\left(u\right), \ x \in \mathbf{R}^n, \ t > 0 \\ u\left(0,x\right) = u_0\left(x\right), \ \partial_t u\left(0,x\right) = u_1\left(x\right), \ x \in \mathbf{R}^n, \end{cases} \tag{2.49}$$

where $\mathcal{L} = \partial_t^2 + \partial_t - \Delta$. We suppose that the nonlinearity $\mathcal{N}\left(u\right) \in \mathbf{C}^k\left(\mathbf{R}\right)$ satisfies the estimate

$$\left|\frac{d^j}{du^j}\mathcal{N}\left(u\right)\right| \le C\left|u\right|^{\rho-j}, \ \ 0 \le j \le k \le \rho,$$

thus considering as a typical example $\mathcal{N}\left(u\right) = \left|u\right|^{\rho}$. In this section we restrict our attention to the supercritical power $\rho > 1 + \frac{2}{n}$ of the nonlinearity. Note that by the Duhamel principle the Cauchy problem (2.49) can be rewritten in the form of the integral equation

$$u\left(t\right) = \widetilde{\mathcal{G}}\left(t\right) u_0 + \mathcal{G}\left(t\right) u_1 + \int_0^t \mathcal{G}\left(t-\tau\right) \mathcal{N}\left(u\left(\tau\right)\right) d\tau, \tag{2.50}$$

where the Green operators are

$$\mathcal{G}(t) = e^{-\frac{t}{2}} \overline{\mathcal{F}}_{\xi \to x} L(t, \xi) \mathcal{F}_{x \to \xi},$$

$$\widetilde{\mathcal{G}}(t) = (\partial_t + 1) \mathcal{G}(t)$$

and the symbol

$$L(t, \xi) = \frac{\sin\left(t\sqrt{|\xi|^2 - \frac{1}{4}}\right)}{\sqrt{|\xi|^2 - \frac{1}{4}}}.$$

Therefore, we see that general results of Section 2.1 can be applied with obvious modifications. Denote the heat kernel by

$$G_0(t, x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$

We then prove the following result.

Theorem 2.28. *Let $\rho > 1 + \frac{2}{n}$. Suppose that the initial data are sufficiently small in space*

$$u_0 \in \mathbf{H}^{\alpha,0}(\mathbf{R}^n) \cap \mathbf{H}^{0,\delta}(\mathbf{R}^n), \ u_1 \in \mathbf{H}^{\alpha-1,0}(\mathbf{R}^n) \cap \mathbf{H}^{0,\delta}(\mathbf{R}^n),$$

where $\delta > \frac{n}{2}$, $[\alpha] \le \rho$; $\alpha \ge \frac{n}{2} - \frac{1}{\rho-1}$ for $n \ge 2$ and $\alpha \in \left[\frac{1}{2} - \frac{1}{2(\rho-1)}, 1\right)$ for $n = 1$. Then the Cauchy problem (2.49) has a unique global solution

$$u \in \mathbf{C}\left([0, \infty); \mathbf{H}^{\alpha,0}(\mathbf{R}^n) \cap \mathbf{H}^{0,\delta}(\mathbf{R}^n)\right).$$

Moreover the following asymptotics is valid

$$\|u(t) - AG_0(t)\|_{\mathbf{L}^p} \le Ct^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\min\left(1, \frac{\delta}{2}-\frac{n}{4}, \frac{n}{2}(\rho-1)-1\right)} \qquad (2.51)$$

for all $t > 0$, where

$$A = \theta_1 + \theta_0 + \int_0^\infty \int_{\mathbf{R}^n} \mathcal{N}(u(t, x))\, dx dt,$$

$$\theta_j = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} u_j(x)\, dx, \ j = 0, 1,$$

$2 \le p \le \frac{2n}{n-2\alpha}$ if $\alpha < \frac{n}{2}$, $2 \le p < \infty$ if $\alpha = \frac{n}{2}$, and $2 \le p \le \infty$ if $\alpha > \frac{n}{2}$.

Preliminary Lemmas

We consider the linear Cauchy problem

$$\begin{cases} \mathcal{L}u = f(t, x), \ x \in \mathbf{R}^n, \ t > 0, \\ u(0, x) = u_0(x), \ \partial_t u(0, x) = u_1(x), \ x \in \mathbf{R}^n, \end{cases} \qquad (2.52)$$

where $\mathcal{L} = \partial_t^2 + \partial_t - \Delta$. The solution of (2.52) can be written by the Duhamel formula

$$u(t) = \widetilde{\mathcal{G}}(t) u_0 + \mathcal{G}(t) u_1 + \int_0^t \mathcal{G}(t-\tau) f(\tau) d\tau. \tag{2.53}$$

Note that the symbol

$$L(t,\xi) = \frac{\sin\left(t\sqrt{|\xi|^2 - \frac{1}{4}}\right)}{\sqrt{|\xi|^2 - \frac{1}{4}}}.$$

is smooth and bounded: $L(t,\xi) \in \mathbf{C}^\infty(\mathbf{R}^n)$. Moreover the symbol $L(t,\xi)$ decays like $\frac{1}{|\xi|}$ for $|\xi| \to \infty$. This implies a gain of one derivative concerning the initial datum u_1 and a forcing term f. By Lemma 1.35 we obtain the following result.

Lemma 2.29. *The estimates*

$$\begin{aligned} \left\| |\nabla|^\alpha \mathcal{G}(t) \psi \right\|_{\mathbf{L}^2} &\le C \langle t \rangle^{-\frac{\alpha-\beta}{2} - \frac{n}{2}\left(\frac{1}{q} - \frac{1}{2}\right)} \left\| |\nabla|^\beta \psi \right\|_{\mathbf{L}^q} \\ &+ Ce^{-\frac{t}{4}} \left\| |\nabla|^\alpha \langle \Delta \rangle^{-\frac{1}{2}} \psi \right\|_{\mathbf{L}^2} \end{aligned}$$

and

$$\begin{aligned} \left\| |\cdot|^\delta \mathcal{G}(t) \psi \right\|_{\mathbf{L}^2} &\le C \langle t \rangle^{\frac{\delta}{2} - \frac{n}{4}} \|\psi\|_{\mathbf{L}^1} + C \langle t \rangle^{-\frac{n}{2}\left(\frac{1}{q} - \frac{1}{2}\right)} \left\| |\cdot|^\delta \psi \right\|_{\mathbf{L}^q} \\ &+ Ce^{-\frac{t}{4}} \left\| \langle \Delta \rangle^{-\frac{1}{2}} \langle \cdot \rangle^\delta \psi \right\|_{\mathbf{L}^2} \end{aligned}$$

are true for all $t > 0$, *where* $1 \le q \le 2$, $\delta > \frac{n}{2}$, $\alpha \ge \beta \ge 0$, *provided that the right-hand sides are finite.*

As above we estimate the operator

$$\widetilde{\mathcal{G}}(t) = (\partial_t + 1) \mathcal{G}(t) = \overline{\mathcal{F}}_{\xi \to x} \widetilde{L}(t,\xi) \mathcal{F}_{x \to \xi}$$

with a symbol

$$\widetilde{L}(t,\xi) = e^{-\frac{t}{2}} \cos\left(t\sqrt{|\xi|^2 - \frac{1}{4}}\right)$$

to obtain the following result.

Lemma 2.30. *The estimates*

$$\left\| |\nabla|^\alpha \widetilde{\mathcal{G}}(t) \psi \right\|_{\mathbf{L}^2} \le C \langle t \rangle^{-\frac{\alpha}{2} - \frac{n}{4}} \|\psi\|_{\mathbf{L}^1} + Ce^{-\frac{t}{4}} \left\| |\nabla|^\alpha \psi \right\|_{\mathbf{L}^2}$$

and

$$\left\| |\cdot|^\delta \widetilde{\mathcal{G}}(t) \psi \right\|_{\mathbf{L}^2} \le C \langle t \rangle^{\frac{\delta}{2} - \frac{n}{4}} \|\psi\|_{\mathbf{L}^1} + C \left\| \langle \cdot \rangle^\delta \psi \right\|_{\mathbf{L}^2}$$

are true for all $t > 0$, *where* $\delta > \frac{n}{2}$, $\alpha \ge 0$, *provided that the right-hand sides are finite.*

We now define two norms

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0}\left(\langle t\rangle^{\frac{n}{4}}\|\phi(t)\|_{\mathbf{L}^2} + \langle t\rangle^{\frac{n}{4}+\frac{\alpha}{2}}\||\nabla|^{\alpha}\phi(t)\|_{\mathbf{L}^2} + \langle t\rangle^{\frac{n}{4}-\frac{\delta}{2}}\left\||\cdot|^{\delta}\phi(t)\right\|_{\mathbf{L}^2}\right),$$

$$\|\psi\|_{\mathbf{Y}} = \sup_{t>0}\Bigg(\langle t\rangle^{\eta}\left\||\nabla|^{[\alpha]}\psi(t)\right\|_{\mathbf{L}^{\widetilde{q}}} + \sup_{1\le r\le\widetilde{q}}\langle t\rangle^{\frac{n}{2}\left(\rho-\frac{1}{r}\right)}\|\psi(t)\|_{\mathbf{L}^r}$$
$$+ \langle t\rangle^{(\rho-1)\left(\frac{\mu}{2}+\frac{n}{4}\right)+\frac{n}{4}-\frac{\delta}{2}}\left\|\langle\cdot\rangle^{\delta}\psi(t)\right\|_{\mathbf{L}^q}\Bigg),$$

where $\delta > \frac{n}{2}$, $\alpha \ge \mu$, $[\alpha] \le \rho$; $\mu = \frac{n}{2} - \frac{1}{\rho-1}$, $\widetilde{q} = \frac{2n}{n+2+2[\alpha]-2\alpha}$, $q = \frac{2n}{n+2}$, $\eta = \frac{\alpha}{2} + \frac{n}{2}\rho - \frac{n}{4} - \frac{1}{2}$ for $n \ge 2$ and $\mu = \frac{1}{2} - \frac{1}{2(\rho-1)}$, $\widetilde{q} = 2$, $q = 1$, $\eta = \frac{\rho}{2} - \frac{1}{4}$ for $n = 1$.

The following lemma states the interpolation inequalities.

Lemma 2.31. *The estimates*

$$\sup_{t>0}\left(\sup_{0\le\beta\le\alpha}\langle t\rangle^{\frac{n}{4}+\frac{\beta}{2}}\left\||\nabla|^{\beta}\phi(t)\right\|_{\mathbf{L}^2} + \sup_{0\le\sigma\le\delta}\langle t\rangle^{\frac{n}{4}-\frac{\sigma}{2}}\||\cdot|^{\sigma}\phi(t)\|_{\mathbf{L}^2}\right)$$
$$\le C\|\phi\|_{\mathbf{X}}$$

and

$$\sup_{t>0}\sup_{1\le r\le\frac{2n}{n-2\alpha}}\langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{r}\right)}\|\phi(t)\|_{\mathbf{L}^r} \le C\|\phi\|_{\mathbf{X}}$$

are valid, provided that the right-hand sides are bounded.

Proof. By the Hölder inequality we have

$$\left\||\nabla|^{\beta}\phi\right\|_{\mathbf{L}^2} = \left\||\xi|^{\beta}\widehat{\phi}\right\|_{\mathbf{L}^2}$$
$$\le \left\|\widehat{\phi}\right\|_{\mathbf{L}^2}^{1-\frac{\beta}{\alpha}}\left\||\xi|^{\alpha}\widehat{\phi}\right\|_{\mathbf{L}^2}^{\frac{\beta}{\alpha}} = \|\phi\|_{\mathbf{L}^2}^{1-\frac{\beta}{\alpha}}\||\nabla|^{\alpha}\phi\|_{\mathbf{L}^2}^{\frac{\beta}{\alpha}};$$

therefore,

$$\langle t\rangle^{\frac{n}{4}+\frac{\beta}{2}}\left\||\nabla|^{\beta}\phi\right\|_{\mathbf{L}^2}$$
$$\le \left(\langle t\rangle^{\frac{n}{4}}\|\phi(t)\|_{\mathbf{L}^2}\right)^{1-\frac{\beta}{\alpha}}\left(\langle t\rangle^{\frac{n}{4}+\frac{\alpha}{2}}\||\nabla|^{\alpha}\phi(t)\|_{\mathbf{L}^2}\right)^{\frac{\beta}{\alpha}} \le \|\phi\|_{\mathbf{X}}.$$

In the same manner we obtain

$$\langle t\rangle^{\frac{n}{4}-\frac{\sigma}{2}}\||\cdot|^{\sigma}\phi(t)\|_{\mathbf{L}^2}$$
$$\le \left(\langle t\rangle^{\frac{n}{4}}\|\phi(t)\|_{\mathbf{L}^2}\right)^{1-\frac{\sigma}{\delta}}\left(\langle t\rangle^{\frac{n}{4}-\frac{\delta}{2}}\left\||\cdot|^{\delta}\phi(t)\right\|_{\mathbf{L}^2}\right)^{\frac{\sigma}{\delta}} \le \|\phi\|_{\mathbf{X}}.$$

Thus the first estimate of the lemma is true.

If $2 \leq r \leq \frac{2n}{n-2\alpha}$ we apply the Sobolev Imbedding Theorem 1.4 with $\beta = \frac{n}{2} - \frac{n}{r} \in [0, \alpha]$

$$\|\phi\|_{\mathbf{L}^r} \leq C \left\| |\nabla|^\beta \phi \right\|_{\mathbf{L}^2};$$

therefore,

$$\langle t \rangle^{\frac{n}{2}\left(1-\frac{1}{r}\right)} \|\phi(t)\|_{\mathbf{L}^r} \leq C \langle t \rangle^{\frac{n}{2}\left(1-\frac{1}{r}\right)} \left\| |\nabla|^\beta \phi(t) \right\|_{\mathbf{L}^2} \leq C \|u\|_{\mathbf{X}}.$$

For the case of $1 \leq r \leq 2$ we also have

$$\langle t \rangle^{\frac{n}{2}\left(1-\frac{1}{r}\right)} \|\phi(t)\|_{\mathbf{L}^r}$$
$$\leq C \left(\langle t \rangle^{\frac{n}{4}-\frac{\delta}{2}} \left\| |\cdot|^\delta \phi \right\|_{\mathbf{L}^2} \right)^{\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{2}\right)} \left(\langle t \rangle^{\frac{n}{4}} \|\phi\|_{\mathbf{L}^2} \right)^{1-\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{2}\right)} \leq C \|u\|_{\mathbf{X}}.$$

The second estimate of the lemma is true, and Lemma 2.31 is thus proved.

In the next lemma we estimate the integral in the Duhamel formula (2.53).

Lemma 2.32. *Suppose that $\delta > \frac{n}{2}$ and $\rho > 1 + \frac{2}{n}$. Then the estimate*

$$\left\| \int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau \right\|_{\mathbf{X}} \leq C \|\psi\|_{\mathbf{Y}}$$

is true.

Proof. By the Sobolev Imbedding Theorem 1.4 with $\phi = |\nabla|^{\alpha-1}\psi$ we have

$$\left\| |\nabla|^\alpha \langle \Delta \rangle^{-\frac{1}{2}} \psi \right\|_{\mathbf{L}^2} \leq C \left\| |\nabla|^{\alpha-1} \psi \right\|_{\mathbf{L}^2} \leq C \left\| |\nabla|^{[\alpha]} \psi \right\|_{\mathbf{L}^{\widetilde{q}}},$$

where $\widetilde{q} = \frac{2n}{n+2+2[\alpha]-2\alpha}$ for $n \geq 2$ and $\widetilde{q} = 2$, $[\alpha] = 0$ for $n = 1$. Applying the first estimate of Lemma 2.29 with $\beta = 0$, $q = 1$ in the case $0 < \tau < \frac{t}{2}$ and $\beta = [\alpha]$, $q = \widetilde{q}$ in the case $\frac{t}{2} < \tau < t$, we obtain, by taking $\zeta = \frac{1}{2}$ for $n \geq 2$ and $\zeta = \frac{\alpha}{2}$ for $n = 1$

$$\left\| |\nabla|^\alpha \int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau \right\|_{\mathbf{L}^2}$$
$$\leq C \int_0^{\frac{t}{2}} \langle t-\tau \rangle^{-\frac{\alpha}{2}-\frac{n}{4}} \left(\|\psi(\tau)\|_{\mathbf{L}^1} + \left\| |\nabla|^{[\alpha]} \psi(\tau) \right\|_{\mathbf{L}^{\widetilde{q}}} \right) d\tau$$
$$+ C \int_{\frac{t}{2}}^t \langle t-\tau \rangle^{-\frac{\alpha-[\alpha]}{2}-\frac{n}{2}\left(\frac{1}{\widetilde{q}}-\frac{1}{2}\right)} \left\| |\nabla|^{[\alpha]} \psi(\tau) \right\|_{\mathbf{L}^{\widetilde{q}}} d\tau$$
$$\leq C \|\psi\|_{\mathbf{Y}} \int_0^{\frac{t}{2}} \langle t-\tau \rangle^{-\frac{\alpha}{2}-\frac{n}{4}} \langle \tau \rangle^{-\frac{n}{2}(\rho-1)}\,d\tau$$
$$+ C \|\psi\|_{\mathbf{Y}} \int_{\frac{t}{2}}^t \langle t-\tau \rangle^{-\zeta} \langle \tau \rangle^{-\eta}\,d\tau \leq C \langle t \rangle^{-\frac{\alpha}{2}-\frac{n}{4}} \|\psi\|_{\mathbf{Y}}$$

for all $t > 0$, since $\rho > 1 + \frac{2}{n}$. Similarly we find

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}(t-\tau)\,\psi(\tau)\,d\tau \right\|_{\mathbf{L}^2} \\
&\leq C \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{4}} \left(\|\psi(\tau)\|_{\mathbf{L}^1} + \|\psi(\tau)\|_{\mathbf{L}^{\widetilde{q}}} \right) d\tau \\
&+ C \int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\widetilde{q}}-\frac{1}{2}\right)} \|\psi(\tau)\|_{\mathbf{L}^{\widetilde{q}}}\, d\tau \\
&\leq C \|\psi\|_{\mathbf{Y}} \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{4}} \langle\tau\rangle^{-\frac{n}{2}(\rho-1)}\, d\tau \\
&+ C \|\psi\|_{\mathbf{Y}} \int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\widetilde{q}}-\frac{1}{2}\right)} \langle\tau\rangle^{-\frac{n}{2}\left(\rho-\frac{1}{\widetilde{q}}\right)}\, d\tau \leq C\langle t\rangle^{-\frac{n}{4}} \|\psi\|_{\mathbf{Y}}.
\end{aligned}$$

Finally by the second estimate of Lemma 2.29, and using the Sobolev Imbedding Theorem 1.4

$$\left\| \langle \Delta\rangle^{-\frac{1}{2}} \langle\cdot\rangle^{\delta} \psi \right\|_{\mathbf{L}^2} \leq C \left\| \langle\cdot\rangle^{\delta} \psi \right\|_{\mathbf{L}^q}$$

with $q = \max\left(1, \frac{2n}{n+2}\right)$, we have

$$\begin{aligned}
&\left\| |\cdot|^{\delta} \int_0^t \mathcal{G}(t-\tau)\,\psi(\tau)\,d\tau \right\|_{\mathbf{L}^2} \leq C \int_0^t \langle t-\tau\rangle^{\frac{\delta}{2}-\frac{n}{4}} \|\psi(\tau)\|_{\mathbf{L}^1}\, d\tau \\
&+ C \int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \left\| \langle\cdot\rangle^{\delta} \psi(\tau) \right\|_{\mathbf{L}^q} d\tau \\
&\leq C \|\psi\|_{\mathbf{Y}} \int_0^t \langle t-\tau\rangle^{\frac{\delta}{2}-\frac{n}{4}} \langle\tau\rangle^{-\frac{n}{2}(\rho-1)}\, d\tau \\
&+ C \|\psi\|_{\mathbf{Y}} \int_0^t \langle t-\tau\rangle^{-\frac{\min(2,n)}{4}} \langle\tau\rangle^{-(\rho-1)\left(\frac{\mu}{2}+\frac{n}{4}\right)+\frac{\delta}{2}-\frac{n}{4}}\, d\tau \\
&\leq C\langle t\rangle^{\frac{\delta}{2}-\frac{n}{4}} \|\psi\|_{\mathbf{Y}}.
\end{aligned}$$

Lemma 2.32 is proved.

Lemma 2.33. *The estimate*

$$\|\mathcal{N}(u)\|_{\mathbf{Y}} \leq C \|u\|_{\mathbf{X}}^{\rho}$$

is true, provided that the right-hand side is bounded.

Proof. As above we choose $\delta > \frac{n}{2}$, $\mu = \frac{n}{2} - \frac{1}{\rho-1}$, $\widetilde{q} = \frac{2n}{n+2+2[\alpha]-2\alpha}$, $q = \frac{2n}{n+2}$ for $n \geq 2$ and $\mu = \frac{1}{2} - \frac{1}{2(\rho-1)} > 0$, $\widetilde{q} = 2$, $q = 1$ for $n = 1$.

By the Sobolev Imbedding Theorem 1.4 we have with $r = n(\rho-1)$ for $n \geq 2$ and $r = 2(\rho-1)$ for $n = 1$

$$\|u\|_{\mathbf{L}^r} \le C \, \||\nabla|^{\mu} u\|_{\mathbf{L}^2};$$

as a result of the Hölder inequality we have by taking $q = \max\left(1, \frac{2n}{2+n}\right)$

$$\begin{aligned}\left\|\langle\cdot\rangle^{\delta} \mathcal{N}(u)\right\|_{\mathbf{L}^q} &\le C \left\|\langle\cdot\rangle^{\delta} |u|^{\rho}\right\|_{\mathbf{L}^q} \le C \left\|\langle\cdot\rangle^{\delta} u\right\|_{\mathbf{L}^2} \|u\|_{\mathbf{L}^r}^{\rho-1} \\ &\le C \left\|\langle\cdot\rangle^{\delta} u\right\|_{\mathbf{L}^2} \||\nabla|^{\mu} u\|_{\mathbf{L}^2}^{\rho-1}. \end{aligned} \tag{2.54}$$

Using (2.54) we get

$$\begin{aligned}&\langle t\rangle^{(\rho-1)\left(\frac{\mu}{2}+\frac{n}{4}\right)+\frac{n}{4}-\frac{\delta}{2}} \left\|\langle\cdot\rangle^{\delta} \mathcal{N}(u)\right\|_{\mathbf{L}^q} \\ &\le C \langle t\rangle^{\frac{n}{4}-\frac{\delta}{2}} \left\|\langle\cdot\rangle^{\delta} u\right\|_{\mathbf{L}^2} \left(\langle t\rangle^{\frac{\mu}{2}+\frac{n}{4}} \||\nabla|^{\mu} u\|_{\mathbf{L}^2}\right)^{\rho-1} \le C \|u\|_{\mathbf{X}}^{\rho}. \end{aligned} \tag{2.55}$$

We now consider the estimates of the norm $\left\||\nabla|^{[\alpha]} \mathcal{N}(u)\right\|_{\mathbf{L}^{\widetilde{q}}}$, where $\alpha \ge \mu$; $\widetilde{q} = \frac{2n}{n+2+2[\alpha]-2\alpha}$ for $n \ge 2$ and $\widetilde{q} = 2$ for $n = 1$.

First let us consider the case $[\alpha] = 0$, then via Lemma 2.31 we have by Sobolev Imbedding Theorem 1.4 with $\beta = \frac{n}{2} - \frac{n}{\rho\widetilde{q}}$

$$\begin{aligned}\langle t\rangle^{\eta} \left\||\nabla|^{[\alpha]} \mathcal{N}(u)\right\|_{\mathbf{L}^{\widetilde{q}}} &= \langle t\rangle^{\eta} \|u\|_{\mathbf{L}^{\rho\widetilde{q}}}^{\rho} \\ &\le C \left(\langle t\rangle^{\frac{\beta}{2}+\frac{n}{4}} \left\||\nabla|^{\beta} u\right\|_{\mathbf{L}^2}\right)^{\rho} \le C \|u\|_{\mathbf{X}}^{\rho}. \end{aligned} \tag{2.56}$$

Now we consider the case $[\alpha] \ge 1$, $n \ge 2$. By the Leibnitz rule and by the Hölder inequality with $1 \le q_j \le \infty$ such that $\sum_{j=0}^{[\alpha]} \frac{1}{q_j} = \frac{1}{\widetilde{q}}$, we get (provided that $[\alpha] \le \rho$)

$$\left\||\nabla|^{[\alpha]} \mathcal{N}(u)\right\|_{\mathbf{L}^{\widetilde{q}}} \le C \left\|u^{\rho-[\alpha]}\right\|_{\mathbf{L}^{q_0}} \sum_{k_j \ge 0,\ k_1+\ldots+k_{[\alpha]}=[\alpha]} \prod_{j=1}^{[\alpha]} \left\||\nabla|^{k_j} u\right\|_{\mathbf{L}^{q_j}}.$$

We now choose $\frac{1}{q_j} = \frac{1}{2} - \frac{\mu+\beta_j-k_j}{n}$, $\frac{1}{(\rho-[\alpha])q_0} = \frac{1}{2} - \frac{\mu}{n}$, so that $0 \le \beta_j < k_j + \frac{1}{\rho-1}$ are such that $\sum_{j=1}^{[\alpha]} \beta_j = \alpha - \mu$, then we get

$$\sum_{j=0}^{[\alpha]} \frac{1}{q_j} = \frac{[\alpha]-\alpha}{n} + (\rho-1)\left(\frac{1}{2} - \frac{\mu}{n}\right) + \frac{1}{2} = \frac{1}{\widetilde{q}},$$

since $\frac{1}{2} - \frac{\mu}{n} = \frac{1}{n(\rho-1)}$. Then by the Sobolev Imbedding Theorem 1.4 we have

$$\left\||\nabla|^{k_j} u\right\|_{\mathbf{L}^{q_j}} \le C \left\||\nabla|^{\mu+\beta_j} u\right\|_{\mathbf{L}^2}$$

and

$$\left\| u^{\rho-[\alpha]} \right\|_{\mathbf{L}^{q_0}} \le C \left\| u \right\|^{\rho-[\alpha]}_{\mathbf{L}^{(\rho-[\alpha])q_0}} \le C \left\| |\nabla|^{\mu} u \right\|^{\rho-[\alpha]}_{\mathbf{L}^2}.$$

Therefore we obtain

$$\left\| |\nabla|^{[\alpha]} \mathcal{N}(u) \right\|_{\mathbf{L}^{\widetilde{q}}} \le C \left\| |\nabla|^{\mu} u \right\|^{\rho-[\alpha]}_{\mathbf{L}^2} \prod_{j=1}^{[\alpha]} \left\| |\nabla|^{\mu+\beta_j} u \right\|_{\mathbf{L}^2};$$

hence by Lemma 2.30

$$\langle t \rangle^{\eta} \left\| |\nabla|^{[\alpha]} \mathcal{N}(u) \right\|_{\mathbf{L}^{\widetilde{q}}} \le C \left(\langle t \rangle^{\frac{\mu}{2}+\frac{n}{4}} \left\| |\nabla|^{\mu} u \right\|_{\mathbf{L}^2} \right)^{\rho-[\alpha]}$$
$$\times \prod_{j=1}^{[\alpha]} \langle t \rangle^{\frac{\mu}{2}+\frac{\beta_j}{2}+\frac{n}{4}} \left\| |\nabla|^{\mu+\beta_j} u \right\|_{\mathbf{L}^2} \le C \left\| u \right\|^{\rho}_{\mathbf{X}}. \tag{2.57}$$

Now the second estimate of Lemma 2.31 yields

$$\langle t \rangle^{\frac{n}{2}\left(\rho-\frac{1}{r}\right)} \left\| \mathcal{N}(u(t)) \right\|_{\mathbf{L}^r} \le \left(\langle t \rangle^{\frac{n}{2}\left(1-\frac{1}{\rho r}\right)} \left\| u(t) \right\|_{\mathbf{L}^{\rho r}} \right)^{\rho} \le C \left\| u \right\|^{\rho}_{\mathbf{X}} \tag{2.58}$$

for all $1 \le r \le \widetilde{q}$. Collecting estimates (2.54) through (2.58) we obtain the result of the lemma which is then proved.

The following lemma says that the asymptotic behavior of solutions to the linear Cauchy problem (2.52) is similar to that of the heat equation. Denote the heat kernel

$$G_0(t,x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}},$$
$$\theta = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \psi(x)\, dx$$

and

$$\widetilde{\mathcal{G}}(t) = (\partial_t + 1)\mathcal{G}(t).$$

By application of Lemma 1.37 we find.

Lemma 2.34. *The estimates*

$$\left\| |\nabla|^{\alpha} \left(\mathcal{G}(t)\psi - \theta G_0(t) \right) \right\|_{\mathbf{L}^2} \le C \langle t \rangle^{-\frac{\alpha}{2}-\frac{n}{4}-1} \left\| \psi \right\|_{\mathbf{L}^1}$$
$$+ C \langle t \rangle^{-\frac{\alpha}{2}-\frac{\delta}{2}} \left\| \langle \cdot \rangle^{\delta} \psi \right\|_{\mathbf{L}^2} + C e^{-\frac{t}{2}} \left\| |\nabla|^{\alpha} \langle \Delta \rangle^{-\frac{1}{2}} \psi \right\|_{\mathbf{L}^2}$$

and

$$\left\| |\nabla|^{\alpha} \left(\widetilde{\mathcal{G}}(t)\psi - \theta G_0(t) \right) \right\|_{\mathbf{L}^2} \le C \langle t \rangle^{-\frac{\alpha}{2}-\frac{n}{4}-1} \left\| \psi \right\|_{\mathbf{L}^1}$$
$$+ C \langle t \rangle^{-\frac{\alpha}{2}-\frac{\delta}{2}} \left\| \langle \cdot \rangle^{\delta} \psi \right\|_{\mathbf{L}^2} + C e^{-\frac{t}{4}} \left\| |\nabla|^{\alpha} \psi \right\|_{\mathbf{L}^2}$$

are true for all $t > 0$, *where* $\frac{n}{2} < \delta < n$, $\alpha \ge 0$, *provided that the right-hand sides are finite.*

Proof of Theorem 2.28

We rewrite the Cauchy problem (2.49) in the form of the integral equation (2.50). We apply Theorem 1.17 to prove the existence of the global solution. Denote

$$\mathbf{X} = \left\{ v \in \mathbf{L}^{\infty}\left(0, \infty; \mathbf{L}^{2}\left(\mathbf{R}\right)\right) : \|v\|_{\mathbf{X}} \leq C \right\}.$$

Applying estimates of Lemma 2.29 and Lemma 2.30 we get

$$\begin{aligned}
\left\| |\nabla|^{\alpha} \mathcal{G}\left(t\right) u_{1} \right\|_{\mathbf{L}^{2}} &\leq C \left\langle t \right\rangle^{-\frac{\alpha}{2}-\frac{n}{4}} \left(\|u_{1}\|_{\mathbf{H}^{0,\delta}} + \|u_{1}\|_{\mathbf{H}^{\alpha-1,0}} \right), \\
\left\| |\nabla|^{\alpha} \widetilde{\mathcal{G}}\left(t\right) u_{0} \right\|_{\mathbf{L}^{2}} &\leq C \left\langle t \right\rangle^{-\frac{\alpha}{2}-\frac{n}{4}} \left(\|u_{0}\|_{\mathbf{H}^{0,\delta}} + \|u_{0}\|_{\mathbf{H}^{\alpha,0}} \right), \\
\left\| |\cdot|^{\delta} \mathcal{G}\left(t\right) u_{1} \right\|_{\mathbf{L}^{2}} &\leq C \left\langle t \right\rangle^{\frac{\delta}{2}-\frac{n}{4}} \|u_{1}\|_{\mathbf{H}^{0,\delta}}, \\
\left\| |\cdot|^{\delta} \widetilde{\mathcal{G}}\left(t\right) u_{0} \right\|_{\mathbf{L}^{2}} &\leq C \left\langle t \right\rangle^{\frac{\delta}{2}-\frac{n}{4}} \|u_{0}\|_{\mathbf{H}^{0,\delta}}.
\end{aligned}$$

Applying Lemma 2.32 and Lemma 2.33 we have

$$\begin{aligned}
&\left\| \int_{0}^{t} \mathcal{G}\left(t-\tau\right) \left(\mathcal{N}\left(w\right)\left(\tau\right) - \mathcal{N}\left(v\right)\left(\tau\right) \right) d\tau \right\|_{\mathbf{X}} \leq C \left\| \mathcal{N}\left(w\right) - \mathcal{N}\left(v\right) \right\|_{\mathbf{Y}} \\
&\leq C \left(\|w\|_{\mathbf{X}}^{\rho-1} + \|v\|_{\mathbf{X}}^{\rho-1} \right) \|w - v\|_{\mathbf{X}}.
\end{aligned}$$

Hence due to Theorem 1.17 there exists a unique solution

$$u \in \mathbf{C}\left([0, \infty); \mathbf{H}^{\alpha,0}(\mathbf{R}^{n}) \cap \mathbf{H}^{\delta,0}(\mathbf{R}^{n})\right)$$

to the Cauchy problem (2.49).

We apply Theorem2.4 to prove asymptotics (2.51). By Lemma 2.34 and the Sobolev Imbedding Theorem 1.4 we have for $2 \leq p \leq \frac{2n}{n-2\alpha}$ if $\alpha < \frac{n}{2}$, $2 \leq p < \infty$ if $\alpha = \frac{n}{2}$, and $2 \leq p \leq \infty$ if $\alpha > \frac{n}{2}$

$$\begin{aligned}
&\left\| \mathcal{G}\left(t\right) u_{1} - \theta_{1} G_{0}\left(t\right) \right\|_{\mathbf{L}^{p}} \\
&\leq C \left\| |\nabla|^{\alpha} \left(\mathcal{G}\left(t\right) u_{1} - \theta_{1} G_{0}\left(t\right) \right) \right\|_{\mathbf{L}^{2}} \\
&\leq C \left\langle t \right\rangle^{-\frac{n}{2}\left(1-\frac{2}{p}\right)} \left(\left\langle t \right\rangle^{-\frac{n}{4}-1} \|u_{1}\|_{\mathbf{L}^{1}} + C \left\langle t \right\rangle^{-\frac{\delta}{2}} \|u_{1}\|_{\mathbf{H}^{0,\delta}} \right. \\
&\left. + \left\langle t \right\rangle^{-\max\left(\frac{n}{4}+1, \frac{\delta}{2}\right)} \|u_{1}\|_{\mathbf{H}^{\alpha-1,0}} \right) \\
&\leq C \left\langle t \right\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)} \left(\left\langle t \right\rangle^{-1} \|u_{1}\|_{\mathbf{L}^{1}} + C \left\langle t \right\rangle^{-\frac{\delta}{2}+\frac{n}{4}} \|u_{1}\|_{\mathbf{H}^{0,\delta}} \right. \\
&\left. + \left\langle t \right\rangle^{-\max\left(1, \frac{\delta}{2}-\frac{n}{4}\right)} \|u_{1}\|_{\mathbf{H}^{\alpha-1,0}} \right)
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \widetilde{\mathcal{G}}\left(t\right) u_{0} - \theta_{0} G_{0}\left(t\right) \right\|_{\mathbf{L}^{p}} \\
&\leq C \left\langle t \right\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)} \left(\left\langle t \right\rangle^{-1} \|u_{1}\|_{\mathbf{L}^{1}} + C \left\langle t \right\rangle^{-\frac{\delta}{2}+\frac{n}{4}} \|u_{1}\|_{\mathbf{H}^{0,\delta}} \right. \\
&\left. + \left\langle t \right\rangle^{-\max\left(1, \frac{\delta}{2}-\frac{n}{4}\right)} \|u_{0}\|_{\mathbf{H}^{\alpha,0}} \right)
\end{aligned}$$

for all $t > 0$, where

$$\theta_j = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} u_j(x)\, dx,$$

for $j = 0, 1$ and $G_0(t, x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$. We use Lemma 2.29 with $\beta = [\alpha]$, $q = \widetilde{q}$ to obtain

$$\begin{aligned}
&\left\| \int_{\frac{t}{2}}^{t} \mathcal{G}(t-\tau)\, \mathcal{N}(u(\tau))\, d\tau \right\|_{\mathbf{L}^p} \\
&\leq C \left\| \mathcal{N}(u) \right\|_{\mathbf{Y}} \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-\frac{\alpha-[\alpha]}{2}-\frac{n}{2}\left(\frac{1}{\widetilde{q}}-\frac{1}{2}\right)} \langle \tau \rangle^{-\eta}\, d\tau \\
&\leq C \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\min\left(1, \frac{n}{2}(\rho-1)-1\right)} \left\| u \right\|_{\mathbf{X}}^{\rho}
\end{aligned}$$

since $\widetilde{q} = \frac{2n}{n+2+2[\alpha]-2\alpha}$, $\eta = \frac{\alpha}{2} + \frac{n}{2}\rho - \frac{n}{4} - \frac{1}{2}$ for $n \geq 2$ and $\widetilde{q} = 2$, $\eta = \frac{\rho}{2} - \frac{1}{4}$, $[\alpha] = 0$ for $n = 1$. We apply Lemma 2.34 to get

$$\begin{aligned}
&\left\| \int_0^{\frac{t}{2}} \left(\mathcal{G}(t-\tau)\, \mathcal{N}(u(\tau)) - G_0(t-\tau, x) \int_{\mathbf{R}^n} \mathcal{N}(u(\tau, x))\, dx \right) d\tau \right\|_{\mathbf{L}^p} \\
&\leq C \left\| u \right\|_{\mathbf{X}}^{\rho} \int_0^{\frac{t}{2}} \left(\langle t-\tau \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-1} \langle \tau \rangle^{-\frac{n}{2}(\rho-1)} \right. \\
&\left. + \langle t-\tau \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{\delta}{2}+\frac{n}{4}} \langle \tau \rangle^{-\frac{n}{2}(\rho-1)+\frac{\delta}{2}-\frac{n}{4}} \right) d\tau \\
&\leq C \left\| u \right\|_{\mathbf{X}}^{\rho} \left(\langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-1} + \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{\delta}{2}+\frac{n}{4}} \right) \\
&\leq C \left\| u \right\|_{\mathbf{X}}^{\rho} \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\min\left(1, \frac{\delta}{2}-\frac{n}{4}\right)},
\end{aligned}$$

where $2 \leq p \leq \frac{2n}{n-2\alpha}$ if $\alpha < \frac{n}{2}$, $2 \leq p < \infty$ if $\alpha = \frac{n}{2}$, and $2 \leq p \leq \infty$ if $\alpha > \frac{n}{2}$. We also have

$$\begin{aligned}
&\left\| \int_0^{\frac{t}{2}} \left(G_0(t-\tau, x) - G_0(t, x) \right) \int_{\mathbf{R}^n} \mathcal{N}(u(\tau, x))\, dx d\tau \right\|_{\mathbf{L}^p} \\
&\leq C \left\| u \right\|_{\mathbf{X}}^{\rho} \int_0^{\frac{t}{2}} \langle t-\tau \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-1} \langle \tau \rangle^{1-\frac{n}{2}(\rho-1)}\, d\tau \\
&\leq C \left\| u \right\|_{\mathbf{X}}^{\rho} \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\min\left(\frac{n}{2}(\rho-1)-1, 1\right)}
\end{aligned}$$

and

$$\left\| G_0(t, x) \int_{\frac{t}{2}}^{\infty} \int_{\mathbf{R}^n} \mathcal{N}(u(\tau, x))\, dx d\tau \right\|_{\mathbf{L}^p} \leq C \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{n}{2}(\rho-1)+1} \left\| u \right\|_{\mathbf{X}}^{\rho}.$$

Thus by Theorem 2.4 we see that there exists a constant

$$A = \theta_1 + \theta_0 + \int_0^\infty \int_{\mathbf{R}^n} \mathcal{N}(u(\tau, x))\, dx d\tau$$

such that the following asymptotics is valid

$$\|u(t) - AG_0(t)\|_{\mathbf{L}^p} \leq C \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\min\left(1, \frac{\delta}{2}-\frac{n}{4}, \frac{n}{2}(\rho-1)-1\right)}$$

for all $t > 0$, where $2 \leq p \leq \frac{2n}{n-2\alpha}$ if $\alpha < \frac{n}{2}$, $2 \leq p < \infty$ if $\alpha = \frac{n}{2}$, and $2 \leq p \leq \infty$ if $\alpha > \frac{n}{2}$. Theorem 2.28 is proved.

2.3.1 Large initial data

We study the one dimensional nonlinear damped wave equation

$$\begin{cases} u_{tt} + u_t - u_{xx} = \lambda |u|^\sigma u, \; x \in \mathbf{R}, \; t > 0, \\ u(0, x) = u_0(x), \; u_t(0, x) = u_1(x), \; x \in \mathbf{R} \end{cases} \tag{2.59}$$

in the supercritical case $\sigma > 2$, where $\lambda < 0$.

In this section we will prove the large time asymptotic formulas for the solutions of the Cauchy problem (2.59) without any restriction on the size of the initial data $u_0(x)$, $u_1(x)$. We study the one dimensional case for simplicity. Higher dimensions can also be considered by our method.

Theorem 2.35. *Let $\lambda < 0$, $\sigma > 2$. Suppose that the initial data $u_0 \in \mathbf{W}_1^2(\mathbf{R}) \cap \mathbf{H}^2(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})$, $u_1 \in \mathbf{W}_1^1(\mathbf{R}) \cap \mathbf{H}^1(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})$, $a \in (0, 1]$. Then there exists a unique solution $u \in \mathbf{C}([0, \infty); \mathbf{W}_1^2(\mathbf{R}) \cap \mathbf{H}^2(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})$ to the Cauchy problem (2.59). Moreover the asymptotics is true*

$$u(t, x) = A(4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} + O\left(t^{-\frac{1}{2}-\gamma}\right)$$

for large time $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $0 < \gamma < \frac{1}{2}\min(a, \sigma - 2)$ and

$$A = \int_{\mathbf{R}} (u_0(x) + u_1(x))\, dx + \lambda \int_0^\infty \int_{\mathbf{R}} |u|^\sigma u(\tau, x)\, dx d\tau$$

In Subsection 2.3.1 we obtain some preliminary estimates of the Green operator solving the linearized Cauchy problem corresponding to (2.59). Subsection 2.3.1 is devoted to the proof of Theorem 2.35.

Preliminaries

First we collect some preliminary estimates for the Green operator

$$\mathcal{G}(t) = \mathcal{F}^{-1} L(t, \xi) \mathcal{F}$$

with a symbol

$$L(t,\xi) = e^{-\frac{t}{2}} \frac{\sin\left(t\sqrt{|\xi|^2 - \frac{1}{4}}\right)}{\sqrt{|\xi|^2 - \frac{1}{4}}}.$$

in the weighted Lebesgue norms $\|\phi\|_{\mathbf{L}^p}$ and $\|\phi\|_{\mathbf{L}^{1,a}}$, for $a \in (0,1)$, $1 \le p \le \infty$. Also we show that the operator $\mathcal{G}(t)$ behaves asymptotically as a Green operator $\mathcal{G}_0(t)$ for the heat equation

$$\mathcal{G}_0(t)\,\psi = t^{-\frac{1}{2}} \int_{\mathbf{R}} G_0\left((x-y)\,t^{-\frac{1}{2}}\right)\psi(y)\,dy$$

with a heat kernel

$$G_0(x) = (4\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{4}}.$$

Denote

$$\vartheta = \int_{\mathbf{R}} \phi(x)\,dx.$$

Lemma 2.36. *The estimates are fulfilled*

$$\left\|\partial_t^k \mathcal{G}(t)\,\phi\right\|_{\mathbf{L}^p} \le C\langle t\rangle^{-k} \|\phi\|_{\mathbf{W}_p^k}$$

and

$$\left\|\partial_t^2 \mathcal{G}(t)\,\phi + e^{-t}\phi\right\|_{\mathbf{L}^p} \le C\langle t\rangle^{-1} \left\|\partial_x^2\phi\right\|_{\mathbf{L}^p}$$

for all $t > 0$, *where* $k \ge 0$, $1 \le p \le \infty$. *Also the estimates are valid*

$$\left\|\mathcal{G}(t)\,\phi - \vartheta t^{-\frac{1}{2}} G_0\left((\cdot)\,t^{-\frac{1}{2}}\right)\right\|_{\mathbf{L}^\infty} \le C\langle t\rangle^{-\frac{1}{2}-\frac{a}{2}} \|\phi\|_{\mathbf{L}^{1,a}}$$

and

$$\left\|\mathcal{G}(t)\,\phi - \vartheta t^{-\frac{1}{2}} G_0\left((\cdot)\,t^{-\frac{1}{2}}\right)\right\|_{\mathbf{L}^{1,a}} \le C\|\phi\|_{\mathbf{L}^{1,a}}$$

for all $t > 0$, *where* $a \in [0,1]$, *provided that the right-hand sides are finite.*

Proof. We have

$$\mathcal{G}(t)\,\psi = \frac{1}{2}e^{-\frac{t}{2}} \int_{|y|\le t} I_0\left(\frac{1}{2}\sqrt{t^2-y^2}\right)\psi(x-y)\,dy,$$

where $I_0(x)$ is the modified Bessel function of order 0 (see Watson [1944]). Note that the function $I_0(x)$ has the following asymptotics

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}} \sum_{m=0}^{\infty} \frac{((2m-1)!!)^2}{(2x)^m\, 2^{2m} m!}$$

for $x \to +\infty$ (see Fedoryuk [1987], Olver [1997]). Thus we have the estimates

$$\left|\frac{d^k}{dt^k}\left(e^{-\frac{t}{2}} I_0\left(\frac{1}{2}\sqrt{t^2-y^2}\right)\right)\right| \le Ct^{-\frac{1}{2}-k} e^{-C\frac{y^2}{t}}$$

for all $|y| \leq \frac{t}{2}$, $t > 0$. And in the domain $\frac{t}{2} \leq |y| \leq t$, $t > 0$ we apply the estimates

$$\sup_{x \geq 0} \left| e^{-x} \frac{d^k}{dx^k} I_0 (x) \right| \leq C.$$

We rewrite the Green operator in the form

$$\mathcal{G}(t)\psi = \frac{1}{2} \int_{|y| \leq \frac{t}{2}} e^{-\frac{t}{2}} I_0 \left(\frac{1}{2} \sqrt{t^2 - y^2} \right) \psi (x - y) \, dy$$
$$+ \frac{1}{2} \int_{\frac{1}{2} < |z| \leq 1} t e^{-\frac{t}{2}} I_0 \left(\frac{t}{2} \sqrt{1 - z^2} \right) \psi (x - zt) \, dz,$$

hence

$$\left\| \partial_t^k \mathcal{G}(t)\psi \right\|_{\mathbf{L}^p} \leq C t^{-\frac{1}{2} - k} \left\| \int_{|y| \leq \frac{t}{2}} e^{-C\frac{y^2}{t}} \psi (x - y) \, dy \right\|_{\mathbf{L}^p}$$
$$+ C e^{-Ct} \|\psi\|_{\mathbf{W}_p^k} \leq C t^{-k} \|\psi\|_{\mathbf{W}_p^k}$$

for all $t > 0$, $k \geq 0$. Thus the first estimate is true.

To prove the second estimate we represent the symbol

$$\frac{1}{\xi^2} \left(\partial_t^2 L (t, \xi) + e^{-t} \right)$$
$$= \frac{1}{\xi^2} \left(\frac{e^{-\frac{t}{2}} \left(1 - 2\xi^2\right) \sin \left(t\sqrt{\xi^2 - \frac{1}{4}} \right)}{2\sqrt{\xi^2 - \frac{1}{4}}} - e^{-\frac{t}{2}} \cos \left(t\sqrt{\xi^2 - \frac{1}{4}} \right) + e^{-t} \right)$$
$$= -e^{-\frac{t}{2}} \frac{\sin t\xi}{\xi} + \widehat{R}(t, \xi),$$

where

$$\widehat{R}(t, \xi) = \frac{e^{-\frac{t}{2}}}{\xi^2} \left(\frac{\sinh \left(\frac{t}{2} \sqrt{1 - 4\xi^2} \right)}{\sqrt{1 - 4\xi^2}} - \sinh \left(\frac{t}{2} \right) \right.$$
$$\left. - \cosh \left(\frac{t}{2} \sqrt{1 - 4\xi^2} \right) + \cosh \left(\frac{t}{2} \right) \right)$$
$$- \frac{e^{-\frac{t}{2}} \sin \left(t\sqrt{\xi^2 - \frac{1}{4}} \right)}{\sqrt{\xi^2 - \frac{1}{4}}} + e^{-\frac{t}{2}} \frac{\sin t\xi}{\xi}.$$

Note that $\widehat{R}(t) \in \mathbf{C}^\infty (\mathbf{R})$. As $\xi \to 0$ we have

$$\widehat{R}(t,\xi) = \frac{\left(1-2\xi^2-\sqrt{1-4\xi^2}\right)\exp\left(\frac{t}{2}\sqrt{1-4\xi^2}-\frac{t}{2}\right)}{\xi^2\sqrt{1-4\xi^2}}$$
$$-\frac{e^{-\frac{t}{2}}}{2\xi^2}\left(\left(1+\left(1-4\xi^2\right)^{-\frac{1}{2}}\right)\exp\left(-\frac{t}{2}\sqrt{1-4\xi^2}\right)-2e^{-\frac{t}{2}}\right)$$
$$+e^{-\frac{t}{2}}\frac{\exp\left(-\frac{t}{2}\sqrt{1-4\xi^2}\right)}{\sqrt{1-4\xi^2}} = O\left(\xi^2 e^{-Ct\xi^2}\right)+O\left(e^{-\frac{t}{2}}\right).$$

Moreover $\widehat{R}(t,\xi)$ decays at infinity along with all derivatives with respect to ξ, so that the estimate is true

$$\left|\partial_\xi^m \widehat{R}(t,\xi)\right| \le C\langle t\rangle^{\frac{m}{2}-1} e^{-Ct\xi^2} + Ce^{-\frac{t}{2}}\langle\xi\rangle^{-2}$$

for all $t>0$, $\xi\in\mathbf{R}$, $m=0,1,2$. Therefore there exists an inverse Fourier transform $R(t,x)=\mathcal{F}_{\xi\to x}\widehat{R}(t,\xi)$, which satisfy the estimate

$$\sup_{x\in\mathbf{R}}\left\langle x\langle t\rangle^{-\frac{1}{2}}\right\rangle^2 |R(t,x)| \le C\langle t\rangle^{-\frac{3}{2}}.$$

Hence applying the Young inequality we obtain

$$\begin{aligned}
&\left\|\partial_t^2\mathcal{G}(t)\phi+e^{-t}\phi\right\|_{\mathbf{L}^p} \\
&\le C\left\|e^{-\frac{t}{2}}\int_{|x-y|<t}\phi_{yy}(y)\,dy\right\|_{\mathbf{L}^p} + C\left\|\int_{\mathbf{R}}R(t,x-y)\phi_{yy}(y)\,dy\right\|_{\mathbf{L}^p} \\
&\le Ct^{-1}\|\phi_{xx}\|_{\mathbf{L}^p}
\end{aligned}$$

for all $t>0$, where $1\le p\le\infty$.

The last two estimates of the lemma follow from Lemma 1.33. Lemma 2.36 is proved.

By using a standard contraction mapping principle we have the following result.

Proposition 2.37. *Let $\sigma>2$, $\lambda\in\mathbf{R}$. Let $u_0\in\mathbf{W}_1^2(\mathbf{R})\cap\mathbf{H}^2(\mathbf{R})\cap\mathbf{L}^{1,a}(\mathbf{R})$, $u_1\in\mathbf{W}_1^1(\mathbf{R})\cap\mathbf{H}^1(\mathbf{R})\cap\mathbf{L}^{1,a}(\mathbf{R})$, $a\in(0,1]$. Then there exists a positive time T and a unique solution $u\in\mathbf{C}\left([0,T];\mathbf{W}_1^2(\mathbf{R})\cap\mathbf{H}^2(\mathbf{R})\cap\mathbf{L}^{1,a}(\mathbf{R})\right.$ to the Cauchy problem (2.59).*

We now prove a global existence of solutions to the Cauchy problem (2.59) in the super critical case $\sigma>2$.

Proposition 2.38. *Let $\sigma>2$, $\lambda\in\mathbf{R}$. Let $u_0\in\mathbf{W}_1^2(\mathbf{R})\cap\mathbf{H}^2(\mathbf{R})\cap\mathbf{L}^{1,a}(\mathbf{R})$, $u_1\in\mathbf{W}_1^1(\mathbf{R})\cap\mathbf{H}^1(\mathbf{R})\cap\mathbf{L}^{1,a}(\mathbf{R})$, $a\in(0,1]$. Suppose that the local solutions u constructed in Proposition 2.37 satisfy the optimal time decay estimate*

$$\sup_{t\in[0,T]}\langle t\rangle^{\frac{1}{2}}\|u(t)\|_{\mathbf{L}^\infty}\le C,$$

where $C > 0$ does not depend on the existence time T. Then there exists a unique global solution $u \in \mathbf{C}\left([0,\infty);\mathbf{W}_1^2(\mathbf{R}) \cap \mathbf{H}^2(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})\right)$ of the Cauchy problem (2.59). Moreover the asymptotics is valid

$$u(t,x) = A(4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} + O\left(t^{-\frac{1}{2}-\gamma}\right) \tag{2.60}$$

for large time $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $0 < \gamma < \min\left(\frac{a}{2}, \frac{\sigma}{2} - 1\right)$ and a constant A is defined in Theorem 2.35.

Proof of Proposition 2.38. We write problem (2.59) as integral equation (2.50). Let us prove a priori estimate

$$\sup_{t\in[0,T]} \left(\langle t\rangle^{-\frac{a}{2}} \|u(t)\|_{\mathbf{L}^{1,a}} + \langle t\rangle^{\frac{1}{2}} \|u(t)\|_{\mathbf{L}^\infty} + \|u(t)\|_{\mathbf{W}_1^2} + \langle t\rangle^{\frac{1}{4}} \|u(t)\|_{\mathbf{H}^2}\right) \le C, \tag{2.61}$$

where $C > 0$ does not depend on the existence time T. Applying Lemma 2.36 we obtain

$$\begin{aligned}
&\|u\|_{\mathbf{L}^{1,a}} \le C\langle t\rangle^{\frac{a}{2}} \left(\|u_0\|_{\mathbf{L}^{1,a}} + \|u_1\|_{\mathbf{L}^{1,a}}\right) \\
&+ C\int_0^t \||u|^\sigma u(\tau)\|_{\mathbf{L}^{1,a}} d\tau + C\int_0^t \||u|^\sigma u(\tau)\|_{\mathbf{L}^1} \langle t-\tau\rangle^{\frac{a}{2}} d\tau \\
&\le C\langle t\rangle^{\frac{a}{2}} + C\int_0^t \langle\tau\rangle^{-\frac{\sigma}{2}} \|u(\tau)\|_{\mathbf{L}^{1,a}} d\tau + C\int_0^t \langle\tau\rangle^{-\frac{\sigma}{2}} \langle t-\tau\rangle^{\frac{a}{2}} \|u(\tau)\|_{\mathbf{L}^1} d\tau,
\end{aligned}$$

$$\begin{aligned}
&\|u\|_{\mathbf{W}_1^2} \le C\left(\|u_0\|_{\mathbf{W}_1^2} + \|u_1\|_{\mathbf{W}_1^1}\right) + C\int_0^t \||u|^\sigma u(\tau)\|_{\mathbf{W}_1^1} d\tau \\
&\le C + C\int_0^t \langle\tau\rangle^{-\frac{\sigma}{2}} \|u(\tau)\|_{\mathbf{W}_1^1} d\tau,
\end{aligned}$$

and

$$\begin{aligned}
&\|u\|_{\mathbf{H}^2} \le C\langle t\rangle^{-\frac{1}{4}} \left(\|u_0\|_{\mathbf{W}_1^2} + \|u_0\|_{\mathbf{H}^2} + \|u_1\|_{\mathbf{W}_1^1} + \|u_1\|_{\mathbf{H}^1}\right) \\
&+ C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{1}{4}} \||u|^\sigma u(\tau)\|_{\mathbf{W}_1^1} d\tau + C\int_{\frac{t}{2}}^t \||u|^\sigma u(\tau)\|_{\mathbf{H}^1} d\tau \\
&\le C + C\langle t\rangle^{-\frac{1}{4}} \int_0^{\frac{t}{2}} \langle\tau\rangle^{-\frac{\sigma}{2}} \|u(\tau)\|_{\mathbf{W}_1^1} d\tau + C\int_{\frac{t}{2}}^t \langle\tau\rangle^{-\frac{\sigma}{2}} \|u(\tau)\|_{\mathbf{H}^1} d\tau.
\end{aligned}$$

Therefore the Gronwall inequality gives the desired estimate (2.61). Now the global existence of solutions to the Cauchy problem (2.59) follows by Proposition 2.37 via a standard continuation argument.

We now prove the asymptotics (2.60). We use the integral representation (2.50) with $f = \lambda|u|^\sigma u$. By virtue of the third estimate of Lemma 2.36 we have the following asymptotic representation for the Green operator

$$(\partial_t + 1)\,\mathcal{G}(t)\,u_0 + \mathcal{G}(t)\,u_1$$
$$= \theta t^{-\frac{1}{2}} G_0\left(xt^{-\frac{1}{2}}\right) + O\left(t^{-\frac{1+a}{2}}\left(\|u_0\|_{\mathbf{L}^{1,a}} + \|u_1\|_{\mathbf{L}^{1,a}}\right)\right) \tag{2.62}$$

for large time $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $G_0(x) = (4\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{4}}$,

$$\theta = \int_{\mathbf{R}} (u_0(x) + u_1(x))\,dx,$$

and $a \in (0, 1]$. Denote

$$\vartheta(\tau) = \lambda \int_{\mathbf{R}} |u|^{\sigma} u(\tau)\,dy,$$

and consider the difference

$$\lambda \int_0^t \mathcal{G}(t-\tau)\,|u|^{\sigma} u(\tau)\,d\tau - t^{-\frac{1}{2}} G_0\left(xt^{-\frac{1}{2}}\right) \int_0^{\infty} \vartheta(\tau)\,d\tau$$
$$= \lambda \int_{\frac{t}{2}}^t \mathcal{G}(t-\tau)\,|u|^{\sigma} u(\tau)\,d\tau$$
$$+ \lambda \int_0^{\frac{t}{2}} \left(\mathcal{G}(t-\tau)\,|u|^{\sigma} u(\tau) - \vartheta(\tau)(t-\tau)^{-\frac{1}{2}} G_0\left(x(t-\tau)^{-\frac{1}{2}}\right)\right) d\tau$$
$$+ \int_0^{\frac{t}{2}} \left((t-\tau)^{-\frac{1}{2}} G_0\left(x(t-\tau)^{-\frac{1}{2}}\right) - t^{-\frac{1}{2}} G_0\left(xt^{-\frac{1}{2}}\right)\right) \vartheta(\tau)\,d\tau$$
$$+ t^{-\frac{1}{2}} G_0\left(xt^{-\frac{1}{2}}\right) \int_{\frac{t}{2}}^{\infty} \vartheta(\tau)\,d\tau. \tag{2.63}$$

In the domain $0 < \tau < \frac{t}{2}$ we apply the third estimate of Lemma 2.36 to get

$$\left\|\lambda \mathcal{G}(t-\tau)\,|u|^{\sigma} u(\tau) - \vartheta(\tau)(t-\tau)^{-\frac{1}{2}} G_0\left(x(t-\tau)^{-\frac{1}{2}}\right)\right\|_{\mathbf{L}^{\infty}}$$
$$\leq C(t-\tau)^{-\frac{1}{2}-\frac{a}{2}} \left\||u|^{\sigma} u(\tau)\right\|_{\mathbf{L}^{1,a}}$$

and in the domain $\frac{t}{2} \leq \tau < t$ we use the estimate

$$\left\|\lambda \mathcal{G}(t-\tau)\,|u|^{\sigma} u(\tau)\right\|_{\mathbf{L}^{\infty}} \leq C \left\||u|^{\sigma} u(\tau)\right\|_{\mathbf{L}^{\infty}}.$$

Therefore we have by virtue of estimates (2.61)

$$\left\|\lambda \int_{\frac{t}{2}}^t \mathcal{G}(t-\tau)\,|u|^{\sigma} u(\tau)\,d\tau\right\|_{\mathbf{L}^{\infty}}$$
$$\leq C\,\|u\|_{\mathbf{X}}^{\sigma+1} \int_{\frac{t}{2}}^t \langle\tau\rangle^{-\frac{1}{2}(\sigma+1)}\,d\tau \leq C\,\langle t\rangle^{-\frac{1}{2}-\gamma},$$

and

$$\left\| \int_0^{\frac{t}{2}} \left(\lambda \mathcal{G}(t-\tau) |u|^{\sigma} u(\tau) - \vartheta(\tau)(t-\tau)^{-\frac{1}{2}} G_0\left(x(t-\tau)^{-\frac{1}{2}}\right)\right) d\tau \right\|_{\mathbf{L}^\infty}$$
$$\leq C \|u\|_{\mathbf{X}}^{\sigma+1} \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{2}-\frac{a}{2}} \langle\tau\rangle^{-\frac{\sigma}{2}} d\tau \leq C\langle t\rangle^{-\frac{1}{2}-\gamma},$$

where $0<\gamma<\min\left(\frac{a}{2},\frac{\sigma}{2}-1\right)$. Now we estimate the third summand in (2.63)

$$\left\| \int_0^{\frac{t}{2}} \left(G_0(t-\tau,x) - t^{-\frac{1}{2}} G_0\left(xt^{-\frac{1}{2}}\right)\right) \vartheta(\tau)\, d\tau \right\|_{\mathbf{L}^\infty}$$
$$\leq C\int_0^{\frac{t}{2}} \left\|(t-\tau)^{-\frac{1}{2}} G_0\left(x(t-\tau)^{-\frac{1}{2}}\right) - t^{-\frac{1}{2}} G_0\left(xt^{-\frac{1}{2}}\right)\right\|_{\mathbf{L}^\infty} \langle\tau\rangle^{-\frac{\sigma}{2}} d\tau$$
$$\leq C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{1}{2}-\gamma} \langle\tau\rangle^{\gamma-\frac{\sigma}{2}} d\tau \leq C\langle t\rangle^{-\frac{1}{2}-\gamma}.$$

For the last summand in (2.63) we have

$$\left\| t^{-\frac{1}{2}} G_0\left(xt^{-\frac{1}{2}}\right) \int_{\frac{t}{2}}^{\infty} \vartheta(\tau)\, d\tau \right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{2}} \int_{\frac{t}{2}}^{\infty} \langle\tau\rangle^{-\frac{\sigma}{2}} d\tau \leq C\langle t\rangle^{-\frac{1}{2}-\gamma}.$$

Thus in view of (2.62) we see from the integral equation (2.50) that there exists a constant

$$A = \theta + \int_0^\infty \vartheta(\tau)\, d\tau \tag{2.64}$$

such that asymptotics (2.60) is valid. Proposition 2.38 is proved.

Proof of Theorem 2.35

Define the norms

$$\|\phi\|_{p,q} \equiv \left\| \|\phi(t,x)\|_{\mathbf{L}^q(\mathbf{R}_x)} \right\|_{\mathbf{L}^p(0,\infty)}.$$

We first prove a global existence result for large data.

Proposition 2.39. *Let $\lambda<0,\sigma>0$. Suppose that the initial data $u_0 \in \mathbf{W}_1^2(\mathbf{R}) \cap \mathbf{H}^2(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})$, $u_1 \in \mathbf{W}_1^1(\mathbf{R}) \cap \mathbf{H}^1(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})$, $a\in(0,1]$. Then there exists a unique global solution $u \in \mathbf{C}\left([0,\infty);\mathbf{W}_1^2(\mathbf{R}) \cap \mathbf{H}^2(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})\right)$ to the Cauchy problem (2.59). Moreover the a-priory estimates of a solution are valid*

$$\|u\|_{\infty,2} + \|u\|_{\infty,\sigma+2} + \|u_t\|_{\infty,2} + \|u_x\|_{\infty,2} \leq C, \tag{2.65}$$

and

$$\|u\|_{\sigma+2,\sigma+2} + \|u_x\|_{2,2} + \|u_t\|_{2,2} \leq C. \tag{2.66}$$

Proof. Let u be a solution constructed in Proposition 2.37. We now multiply equation (2.59) by $2(2u_t+u)$. Then integrating the result with respect to $x \in \mathbf{R}$ we get

$$2\int_{\mathbf{R}} \left((u_t+u)(u_{tt}+u_t)+u_t u_{tt}\right) dx$$
$$= -2\int_{\mathbf{R}} (u_t)^2\, dx + 2\int_{\mathbf{R}} (2u_t+u)\, u_{xx} dx - 2|\lambda| \int_{\mathbf{R}} (2u_t+u)\,|u|^{\sigma}\, u dx,$$

therefore

$$\frac{d}{dt}\left(\|u_t+u\|^2_{\mathbf{L}^2} + \|u_t\|^2_{\mathbf{L}^2} + 2\|u_x\|^2_{\mathbf{L}^2} + \frac{4|\lambda|}{\sigma+2}\|u\|^{\sigma+2}_{\mathbf{L}^{\sigma+2}}\right)$$
$$= -2\|u_t\|^2_{\mathbf{L}^2} - 2\|u_x\|^2_{\mathbf{L}^2} - 2|\lambda|\,\|u\|^{\sigma+2}_{\mathbf{L}^{\sigma+2}}$$

from which the a-priory estimate $\|u(t)\|^2_{\mathbf{L}^\infty} \leq C\|u_x(t)\|_{\mathbf{L}^2}\|u(t)\|_{\mathbf{L}^2} \leq C$ follows. In the same way as in the proof of Proposition 2.38, applying Lemma 2.36 we obtain

$$\|u\|_{\mathbf{L}^{1,a}} \leq C\langle t\rangle^{\frac{a}{2}}\left(\|u_0\|_{\mathbf{L}^{1,a}} + \|u_1\|_{\mathbf{L}^{1,a}}\right)$$
$$+ C\int_0^t \||u|^{\sigma} u(\tau)\|_{\mathbf{L}^{1,a}}\, d\tau + C\int_0^t \||u|^{\sigma} u(\tau)\|_{\mathbf{L}^1}\langle t-\tau\rangle^{\frac{a}{2}}\, d\tau$$
$$\leq C\langle t\rangle^{\frac{a}{2}} + C\int_0^t \|u(\tau)\|_{\mathbf{L}^{1,a}}\, d\tau + C\int_0^t \langle t-\tau\rangle^{\frac{a}{2}}\|u(\tau)\|_{\mathbf{L}^1}\, d\tau,$$

$$\|u\|_{\mathbf{W}^2_1} \leq C\left(\|u_0\|_{\mathbf{W}^2_1} + \|u_1\|_{\mathbf{W}^1_1}\right) + C\int_0^t \||u|^{\sigma} u(\tau)\|_{\mathbf{W}^1_1}\, d\tau$$
$$\leq C + C\int_0^t \|u(\tau)\|_{\mathbf{W}^1_1}\, d\tau,$$

and

$$\|u\|_{\mathbf{H}^2} \leq C\left(\|u_0\|_{\mathbf{H}^2} + \|u_1\|_{\mathbf{H}^1}\right) + C\int_0^t \||u|^{\sigma} u(\tau)\|_{\mathbf{H}^1}\, d\tau$$
$$\leq C + C\int_0^t \|u(\tau)\|_{\mathbf{H}^1}\, d\tau$$

for all $t \in [0,T]$. Then the Gronwall lemma yields the estimate

$$e^{-Ct}\left(\|u(t)\|_{\mathbf{L}^{1,a}} + \|u(t)\|_{\mathbf{W}^2_1} + \|u(t)\|_{\mathbf{H}^2}\right) \leq C$$

for all $t \in [0,T]$, where $C > 0$ does not depend on T. Therefore we can prolong the local solution to the global one. Moreover we have the desired estimates (2.65) and (2.66). Proposition 2.39 is proved.

We now prepare several lemmas.

Lemma 2.40. *Let $\sigma > 2\left(\sqrt{3}-1\right)$, $\lambda < 0$. Suppose that the initial data $u_0 \in \mathbf{W}_1^2(\mathbf{R}) \cap \mathbf{H}^2(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})$, $u_1 \in \mathbf{W}_1^1(\mathbf{R}) \cap \mathbf{H}^1(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})$, $a \in (0,1]$. Let u be a global solution constructed in Proposition 2.39. Then the estimate is true*

$$\|u_{tt}\|_{1,1} \le C.$$

Proof. Since $\|u\|_{\infty,2} + \|u\|_{\sigma+2,\sigma+2} \le C$, by the Hölder inequality we get for $2 \le p \le \sigma + 2$

$$\|u(t)\|_{\mathbf{L}^p} \le \|u(t)\|_{\mathbf{L}^2}^{1-\left(1-\frac{2}{p}\right)\left(1+\frac{2}{\sigma}\right)} \|u(t)\|_{\mathbf{L}^{2+\sigma}}^{\left(1-\frac{2}{p}\right)\left(1+\frac{2}{\sigma}\right)}$$

hence

$$\|u\|_{s,p} \le \|u\|_{\infty,2}^{1-\left(1-\frac{2}{p}\right)\left(1+\frac{2}{\sigma}\right)} \|u\|_{\sigma+2,\sigma+2}^{\left(1-\frac{2}{p}\right)\left(1+\frac{2}{\sigma}\right)} \le C \tag{2.67}$$

for $s = \frac{\sigma p}{p-2}$, $2 \le p \le \sigma + 2$. Since $\|u\|_{\infty,2} + \|u_x\|_{2,2} \le C$, by the Cauchy-Schwartz inequality we have $\|u\|_{\mathbf{L}^\infty}^2 \le 2\|u\|_{\mathbf{L}^2}\|u_x\|_{\mathbf{L}^2}$, hence

$$\|u\|_{4,\infty} \le C. \tag{2.68}$$

Combining estimates (2.67) and (2.68) we obtain

$$\|u\|_{s,p} \le C \tag{2.69}$$

for $s = \frac{4p}{p+2-\sigma}$, $2 + \sigma \le p \le \infty$.

Now we estimate the second derivative u_{xx}. By the integral representation (2.50) with $f = \lambda |u|^\sigma u$ we find

$$u_{xx} = (\partial_t + 1)\partial_x^2 \mathcal{G}(t) u_0 + \partial_x^2 \mathcal{G}(t) u_1 + \lambda \int_0^t \partial_x^2 \mathcal{G}(t-\tau) |u(\tau)|^\sigma u(\tau)\, d\tau.$$

In view of estimates (2.67), (2.68) and $\|u_x\|_{2,2} \le C$, using Lemma 2.36 and the Young inequality we obtain

$$\begin{aligned}
&\left\| \left\| \int_0^t \partial_x^2 \mathcal{G}(t-\tau) |u(\tau)|^\sigma u(\tau)\, d\tau \right\|_{\mathbf{L}_x^2} \right\|_{\mathbf{L}_t^s(0,\infty)} \\
&\le C \left\| \int_0^t \langle t-\tau \rangle^{-1} \left(\|u(\tau)\|_{\mathbf{L}^{2(\sigma+1)}}^{\sigma+1} + \|u(\tau)\|_{\mathbf{L}^2}^{\frac{\sigma}{2}} \|u_x(\tau)\|_{\mathbf{L}^2}^{1+\frac{\sigma}{2}} \right) d\tau \right\|_{\mathbf{L}_t^s(0,\infty)} \\
&\le C \left\| \langle t \rangle^{-1} \right\|_{\mathbf{L}_t^{s_1}(0,T)} \left\| \|u(t)\|_{\mathbf{L}^{2(\sigma+1)}}^{\sigma+1} \right\|_{\mathbf{L}_t^s(0,\infty)} \\
&+ C \left\| \langle t \rangle^{-1} \right\|_{\mathbf{L}_t^{s_1}(0,\infty)} \|u\|_{\infty,2}^{\frac{\sigma}{2}} \|u_x\|_{2,2}^{1+\frac{\sigma}{2}} \\
&\le C \|u\|_{(\sigma+1)s_2, 2(\sigma+1)} + C \|u\|_{\infty,2}^{\frac{\sigma}{2}} \|u_x\|_{2,2}^{1+\frac{\sigma}{2}},
\end{aligned}$$

where $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} - 1 < \frac{1}{s_2}$ and $\frac{1}{s} = \frac{1}{s_1} + \frac{2+\sigma}{4} - 1 < \frac{1}{s_2}$, since $s_1 > 1$, $s_2 = \frac{8}{\sigma+4}$. Thus we get the estimate

$$\|u_{xx}\|_{s,2} \leq C \tag{2.70}$$

for $s > \frac{8}{\sigma+4}$, if $\sigma > 1$.

Next we estimate $\partial_t^2 u$. We have by the integral representation (2.50) with $f = \lambda |u|^\sigma u$

$$\partial_t^2 u(t) = (\partial_t + 1)\, \partial_t^2 \mathcal{G}(t)\, u_0 + \partial_t^2 \mathcal{G}(t)\, u_1$$
$$+ \lambda \partial_t \mathcal{G}(t)\, |u_0|^\sigma u_0 + \lambda \int_0^t \partial_t \mathcal{G}(t-\tau)\, \partial_\tau |u(\tau)|^\sigma u(\tau)\, d\tau.$$

By virtue of estimates of Lemma 2.36 we obtain

$$\|u_{tt}(t)\|_{\mathbf{L}^1} \leq C \langle t \rangle^{-1} + C \int_0^t \langle t-\tau \rangle^{-1} \|u(\tau)\|_{\mathbf{L}^{2\sigma}}^\sigma \|u_\tau(\tau)\|_{\mathbf{L}^2}\, d\tau.$$

In view of estimates (2.67), (2.68) and $\|u_x\|_{2,2} \leq C$, using Lemma 2.36 and the Young inequality we obtain

$$\left\| \|u_{tt}(t)\|_{\mathbf{L}^1} \right\|_{\mathbf{L}_t^s(0,\infty)}$$
$$\leq C + C \left\| \left\| \int_0^t \partial_t \mathcal{G}(t-\tau)\, |u(\tau)|^\sigma u_\tau(\tau)\, d\tau \right\|_{\mathbf{L}_x^1} \right\|_{\mathbf{L}_t^s(0,\infty)}$$
$$\leq C \left\| \int_0^t \langle t-\tau \rangle^{-1} \|u(\tau)\|_{\mathbf{L}^{2\sigma}}^\sigma \|u_\tau(\tau)\|_{\mathbf{L}^2}\, d\tau \right\|_{\mathbf{L}_t^s(0,\infty)}$$
$$\leq C \left\| \langle t \rangle^{-1} \right\|_{\mathbf{L}_t^{s_1}(0,\infty)} \left\| \|u(t)\|_{\mathbf{L}^{2\sigma}}^\sigma \right\|_{\mathbf{L}_t^{s_2}(0,\infty)} \left\| \|u_t(t)\|_{\mathbf{L}^2} \right\|_{\mathbf{L}_t^2(0,\infty)}$$
$$\leq C \|u\|_{\sigma s_2, 2\sigma} \|u_t\|_{2,2},$$

where $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} - \frac{1}{2} < \frac{1}{s_2} + \frac{1}{2}$, since $s_1 > 1$, $s_2 = \frac{8}{\sigma+2}$ for $\sigma \geq 2$, and $s_2 = \frac{2\sigma}{2\sigma-2}$ for $1 < \sigma \leq 2$. Thus we get the estimate

$$\|u_{tt}\|_{s,1} \leq C \tag{2.71}$$

for $s > \frac{8}{\sigma+6}$ for $\sigma \geq 2$, and $s > \frac{2\sigma}{3\sigma-2}$ for $1 < \sigma \leq 2$.

To prove the estimate of the lemma we write

$$\partial_t^2 u(t) = (\partial_t + 1)\, \partial_t^2 \mathcal{G}(t)\, u_0 + \partial_t^2 \mathcal{G}(t)\, u_1$$
$$+ \lambda |u(t)|^\sigma u(t) + \lambda \int_0^{\frac{t}{2}} \partial_t^2 \mathcal{G}(t-\tau)\, |u(\tau)|^\sigma u(\tau)\, d\tau$$
$$+ \lambda \int_{\frac{t}{2}}^t \left(\partial_t^2 \mathcal{G}(t-\tau) + e^{-(t-\tau)} \right) |u(\tau)|^\sigma u(\tau)\, d\tau$$
$$- \lambda \int_{\frac{t}{2}}^t e^{-(t-\tau)} |u(\tau)|^\sigma u(\tau)\, d\tau.$$

Integration by parts in view of equation (2.59) yields

$$
\begin{aligned}
&-\lambda \int_{\frac{t}{2}}^{t} e^{-(t-\tau)} |u(\tau)|^{\sigma} u(\tau)\, d\tau \\
&= -\lambda |u(t)|^{\sigma} u(t) + \lambda(\sigma+1) |u(t)|^{\sigma} u_t(t) \\
&+ \lambda e^{-\frac{t}{2}} \left| u\left(\frac{t}{2}\right)\right|^{\sigma} \left(u\left(\frac{t}{2}\right) - (\sigma+1) u\left(\frac{t}{2}\right)\right) \\
&- \lambda \int_{\frac{t}{2}}^{t} e^{-(t-\tau)} \partial_{\tau}^{2} \left(|u(\tau)|^{\sigma} u(\tau)\right) d\tau \\
&= -\lambda |u(t)|^{\sigma} u(t) + \lambda(\sigma+1) |u(t)|^{\sigma} \left(u_{xx}(t) - u_{tt}(t)\right) \\
&+ \lambda^{2} (\sigma+1) |u(t)|^{2\sigma} u(t) \\
&+ \lambda e^{-\frac{t}{2}} \left| u\left(\frac{t}{2}\right)\right|^{\sigma} \left(u\left(\frac{t}{2}\right) - (\sigma+1) u\left(\frac{t}{2}\right)\right) \\
&- \lambda \int_{\frac{t}{2}}^{t} e^{-(t-\tau)} \partial_{\tau}^{2} \left(|u(\tau)|^{\sigma} u(\tau)\right) d\tau.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\partial_t^2 u(t) = (\partial_t + 1)\, \partial_t^2 \mathcal{G}(t)\, u_0 + \partial_t^2 \mathcal{G}(t)\, u_1 \\
&+ \lambda e^{-\frac{t}{2}} \left| u\left(\frac{t}{2}\right)\right|^{\sigma} \left(u\left(\frac{t}{2}\right) - (\sigma+1) u\left(\frac{t}{2}\right)\right) \\
&+ \lambda(\sigma+1) |u(t)|^{\sigma} \left(u_{xx}(t) - u_{tt}(t)\right) + \lambda^{2}(\sigma+1) |u(t)|^{2\sigma} u(t) \\
&+ \lambda \int_0^{\frac{t}{2}} \partial_t^2 \mathcal{G}(t-\tau)\, |u(\tau)|^{\sigma} u(\tau)\, d\tau \\
&- \lambda \int_{\frac{t}{2}}^{t} e^{-(t-\tau)} \partial_{\tau}^{2} \left(|u(\tau)|^{\sigma} u(\tau)\right) d\tau \\
&+ \lambda \int_{\frac{t}{2}}^{t} \left(\partial_t^2 \mathcal{G}(t-\tau) + e^{-(t-\tau)}\right) |u(\tau)|^{\sigma} u(\tau)\, d\tau.
\end{aligned}
$$

Hence we have

$$\begin{aligned}
\|u_{tt}(t)\|_{\mathbf{L}^1} &\le \left\|(\partial_t+1)\,\partial_t^2\mathcal{G}(t)\,u_0+\partial_t^2\mathcal{G}(t)\,u_1\right\|_{\mathbf{L}^1} \\
&+Ce^{-\frac{t}{2}}\left\|\left|u\left(\frac{t}{2}\right)\right|^{\sigma}\left(u\left(\frac{t}{2}\right)-(\sigma+1)\,u\left(\frac{t}{2}\right)\right)\right\|_{\mathbf{L}^1} \\
&+C\,\||u(t)|^{\sigma}\,u_{xx}(t)\|_{\mathbf{L}^1}+C\,\||u(t)|^{\sigma}\,u_{tt}(t)\|_{\mathbf{L}^1}+C\,\|u(t)\|_{\mathbf{L}^{1+2\sigma}}^{1+2\sigma} \\
&+C\left\|\int_{\frac{t}{2}}^{t}e^{-(t-\tau)}\partial_\tau^2\left(|u(\tau)|^{\sigma}\,u(\tau)\right)d\tau\right\|_{\mathbf{L}^1} \\
&+C\left\|\int_0^{\frac{t}{2}}\partial_t^2\mathcal{G}(t-\tau)\,|u(\tau)|^{\sigma}\,u(\tau)\,d\tau\right\|_{\mathbf{L}^1} \\
&+C\left\|\int_{\frac{t}{2}}^{t}\left(\partial_t^2\mathcal{G}(t-\tau)+e^{-(t-\tau)}\right)|u(\tau)|^{\sigma}\,u(\tau)\,d\tau\right\|_{\mathbf{L}^1} \qquad (2.72)
\end{aligned}$$

By Lemma 2.36 we have

$$\begin{aligned}
&\left\|(\partial_t+1)\,\partial_t^2\mathcal{G}(t)\,u_0+\partial_t^2\mathcal{G}(t)\,u_1\right\|_{\mathbf{L}^1} \\
&\le Ce^{-\frac{t}{2}}\left\|\left|u\left(\frac{t}{2}\right)\right|^{\sigma}\left(u\left(\frac{t}{2}\right)-(\sigma+1)\,u\left(\frac{t}{2}\right)\right)\right\|_{\mathbf{L}^1}\le C\,\langle t\rangle^{-2}.
\end{aligned}$$

In view of estimates (2.69) and (2.70) we get

$$\left\|\||u(t)|^{\sigma}\,u_{xx}(t)\|_{\mathbf{L}^1_x}\right\|_{\mathbf{L}^1_t(0,T)}\le\|u_{xx}\|_{s_1,2}\,\|u\|_{\sigma s_2,2\sigma}^{\sigma}$$

where $s_1>\frac{8}{\sigma+4}$ and $s_2=\frac{8}{\sigma+2}$ for $\sigma\ge 2$, and $s_2=\frac{\sigma}{\sigma-1}$ for $1<\sigma\le 2$; so that $\frac{1}{s_1}+\frac{1}{s_2}>1$, when $\sigma>2\left(\sqrt{3}-1\right)$. In the same manner by virtue of (2.69) and (2.71) we find

$$\left\|\||u(t)|^{\sigma}\,u_{tt}(t)\|_{\mathbf{L}^1_x}\right\|_{\mathbf{L}^1_t(0,\infty)}\le\|u_{tt}\|_{s,1}\,\|u\|_{4,\infty}^{\sigma}$$

where $s>\frac{8}{\sigma+6}$ for $\sigma\ge 2$, and $s>\frac{2\sigma}{3\sigma-2}$ for $1<\sigma\le 2$; so that $\frac{1}{s}+\frac{\sigma}{4}>1$, when $\sigma>\sqrt{5}-1$. By (2.69) we get

$$\left\|\|u(t)\|_{\mathbf{L}^{1+2\sigma}}^{1+2\sigma}\right\|_{\mathbf{L}^1_t(0,\infty)}\le C\,\|u\|_{1+2\sigma,1+2\sigma}^{1+2\sigma}\le C,$$

for $\sigma>1$. Similarly we have

$$\begin{aligned}
&\left\|\left\|\int_{\frac{t}{2}}^{t}e^{-(t-\tau)}\partial_\tau^2\left(|u(\tau)|^{\sigma}\,u(\tau)\right)d\tau\right\|_{\mathbf{L}^1_x}\right\|_{\mathbf{L}^1_t(0,\infty)} \\
&\le C\left\|\left\||u(t)|^{\sigma-1}\,u_t^2(t)\right\|_{\mathbf{L}^1_x}\right\|_{\mathbf{L}^1_t(0,\infty)}+C\left\|\||u(t)|^{\sigma}\,u_{tt}(t)\|_{\mathbf{L}^1_x}\right\|_{\mathbf{L}^1_t(0,\infty)} \\
&\le C\,\|u\|_{\infty,\infty}^{\sigma-1}\,\|u_t\|_{2,2}^2+C\,\|u_{tt}\|_{s,1}\,\|u\|_{4,\infty}^{\sigma}
\end{aligned}$$

for $\sigma > \sqrt{5} - 1$.

Using Lemma 2.36, we obtain

$$\left\| \int_0^{\frac{t}{2}} \partial_t^2 \mathcal{G}(t-\tau) |u(\tau)|^\sigma u(\tau) \, d\tau \right\|_{\mathbf{L}^1} \le C \langle t \rangle^{-2} \int_0^{\frac{t}{2}} \left(\| |u(\tau)|^\sigma u(\tau) \|_{\mathbf{L}^1} + \| |u(\tau)|^\sigma u_x(\tau) \|_{\mathbf{L}^1} \right) d\tau.$$

In view of estimates (2.67) - (2.70) we have

$$\int_0^{\frac{t}{2}} \left(\| |u(\tau)|^\sigma u(\tau) \|_{\mathbf{L}^1} + \| |u(\tau)|^\sigma u_x(\tau) \|_{\mathbf{L}^1} \right) d\tau \le \int_0^{\frac{t}{2}} \left(\|u(\tau)\|_{\mathbf{L}^{\sigma+1}}^{\sigma+1} + \|u(\tau)\|_{\mathbf{L}^{2\sigma}}^{\sigma} \|u_x(\tau)\|_{\mathbf{L}^2} \right) d\tau \le C t^{\frac{1}{\sigma}},$$

therefore

$$\left\| \int_0^{\frac{t}{2}} \partial_t^2 \mathcal{G}(t-\tau) |u(\tau)|^\sigma u(\tau) \, d\tau \right\|_{1,1} \le \left\| \langle t \rangle^{-2} \int_0^{\frac{t}{2}} \left(\| |u(\tau)|^\sigma u(\tau) \|_{\mathbf{L}^1} + \| |u(\tau)|^\sigma u_x(\tau) \|_{\mathbf{L}^1} \right) d\tau \right\|_{\mathbf{L}_t^1(0,\infty)} \le C \int_0^\infty \langle t \rangle^{-2} t^{\frac{1}{\sigma}} dt \le C,$$

when $\sigma > 1$.

For the last summand in (2.72) by Lemma 2.36, we get

$$\begin{aligned} &\left\| \int_{\frac{t}{2}}^t \left(\partial_t^2 \mathcal{G}(t-\tau) + e^{-(t-\tau)} \right) |u(\tau)|^\sigma u(\tau) \, d\tau \right\|_{\mathbf{L}^1} \\ &\le C \int_{\frac{t}{2}}^t \langle t-\tau \rangle^{-1-\gamma} \left(\left\| |u(\tau)|^{\sigma-1} u_x^2(\tau) \right\|_{\mathbf{L}^1}^{1-\gamma} + \| |u(\tau)|^\sigma u_{xx}(\tau) \|_{\mathbf{L}^1}^{1-\gamma} \right) \\ &\times \left(\| |u(\tau)|^\sigma u(\tau) \|_{\mathbf{L}^1}^\gamma + \| |u(\tau)|^\sigma u_x(\tau) \|_{\mathbf{L}^1}^\gamma \right) d\tau \\ &\le C \int_{\frac{t}{2}}^t \langle t-\tau \rangle^{-1-\gamma} \left(\left\| |u(\tau)|^{\sigma-1} u_x^2(\tau) \right\|_{\mathbf{L}^1}^{1-\gamma} + \| |u(\tau)|^\sigma u_{xx}(\tau) \|_{\mathbf{L}^1}^{1-\gamma} \right) d\tau \end{aligned}$$

with some small $\gamma > 0$. When $\sigma > 2\left(\sqrt{3} - 1\right)$, and if $\gamma > 0$ is sufficiently small, by the Hölder inequality we obtain

$$\left\| \|u(\tau)\|_{\mathbf{L}^\infty}^{(\sigma-1)(1-\gamma)} \|u_x(t)\|_{\mathbf{L}^2}^{2-2\gamma} \right\|_{\mathbf{L}_t^1(0,\infty)} \le \|u\|_{(\sigma-1)(1-\gamma)s_1,\infty}^{(\sigma-1)(1-\gamma)} \|u_x\|_{(2-2\gamma)s_2,2}^{2-2\gamma} \le C$$

with $(\sigma-1)(1-\gamma)s_1 \geq 4$, $(2-2\gamma)s_2 \geq 2$; so that $\frac{1}{s_1}+\frac{1}{s_2}=1$ and

$$\left\| \|u_{xx}(\tau)\|_{\mathbf{L}^2}^{1-\gamma} \|u(\tau)\|_{\mathbf{L}^{2\sigma}}^{(1-\gamma)\sigma} \right\|_{\mathbf{L}_t^1(0,\infty)}$$
$$\leq \|u_{xx}\|_{(1-\gamma)s_1,2}^{1-\gamma} \|u\|_{(1-\gamma)\sigma s_2,2\sigma}^{(1-\gamma)\sigma} \leq C$$

with $(1-\gamma)s_1 > \frac{8}{\sigma+4}$, $(1-\gamma)s_2 = \frac{8\sigma}{\sigma+2}$ for $\sigma \geq 2$, and $(1-\gamma)s_2 = \frac{\sigma}{\sigma-1}$ for $1 < \sigma \leq 2$; so that $\frac{1}{s_1}+\frac{1}{s_2}=1$. Therefore

$$\left\| \int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-1-\gamma} \left(\left\| |u(\tau)|^{\sigma-1} u_x^2(\tau) \right\|_{\mathbf{L}^1}^{1-\gamma} + \||u(\tau)|^{\sigma} u_{xx}(\tau)\|_{\mathbf{L}^1}^{1-\gamma} \right) d\tau \right\|_{\mathbf{L}_t^1(0,\infty)}$$
$$\leq C \left\| \|u(\tau)\|_{\mathbf{L}^\infty}^{(\sigma-1)(1-\gamma)} \|u_x(t)\|_{\mathbf{L}^2}^{2-2\gamma} \right\|_{\mathbf{L}_t^1(0,\infty)}$$
$$+C \left\| \|u_{xx}(\tau)\|_{\mathbf{L}^2}^{1-\gamma} \|u(\tau)\|_{\mathbf{L}^{2\sigma}}^{(1-\gamma)\sigma} \right\|_{\mathbf{L}_t^1(0,\infty)} \leq C\,.$$

Thus from (2.72) we have the result of the lemma. Lemma 2.40 is proved.

Now we estimate the decay rate of the $\mathbf{L}^p$ - norms of the solutions.

Lemma 2.41. *Let* $\lambda < 0$. *Suppose that* $u_0 \in \mathbf{W}_1^2(\mathbf{R}) \cap \mathbf{H}^2(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})$, $u_1 \in \mathbf{W}_1^1(\mathbf{R}) \cap \mathbf{H}^1(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})$, $a \in (0,1]$. *Let the global solution constructed in Proposition 2.39 satisfy*

$$\|u_{tt}\|_{1,1} \leq C.$$

Then the estimates

$$\|u(t)\|_{\mathbf{L}^1} \leq C,\ \ \|u(t)\|_{\mathbf{L}^2} \leq C\langle t\rangle^{-\frac{1}{4}},$$

and

$$\|u_x(t)\|_{\mathbf{L}^2} \leq C\langle t\rangle^{-\frac{3}{4}}$$

are valid for all $t > 0$.

Proof. We estimate the $\mathbf{L}^1$ - norm. Denote $S(x) = 1$ for all $x > 0$ and $S(x) = -1$ for all $x < 0$; $S(0) = 0$. We multiply equation (2.59) by $S(u(t,x))$ and integrate with respect to x over $\mathbf{R}$ to get

$$\int_{\mathbf{R}} S(u(t,x))\,u_t(t,x)\,dx = \int_{\mathbf{R}} S(u(t,x))\,u_{xx}dx$$
$$+\int_{\mathbf{R}} S(u(t,x))\left(\lambda |u(t,x)|^{\sigma} u(t,x) - u_{tt}(t,x)\right)dx.$$

We have

$$\int_{\mathbf{R}} S(u(t,x))\,u_t(t,x)\,dx = \frac{d}{dt}\|u(t)\|_{\mathbf{L}^1},$$

$$\int_{\mathbf{R}} S(u(t,x))\, u_{xx}(t,x)\, dx \le 0$$

and

$$\int_{\mathbf{R}} S(u(t,x))\, \lambda |u(t,x)|^{\sigma} u(t,x)\, dx = \lambda \|u(t)\|_{\mathbf{L}^{\sigma+1}}^{\sigma+1} \le 0.$$

Therefore we find

$$\frac{d}{dt} \|u(t)\|_{\mathbf{L}^1} \le \|u_{tt}(t)\|_{\mathbf{L}^1}. \tag{2.73}$$

Integration of inequality (2.73) in view of estimate (2.75) yields the first estimate of the lemma.

In particular, we find

$$\sup_{\xi \in \mathbf{R}} |\widehat{u}(t,\xi)| \le (2\pi)^{-\frac{1}{2}} \|u(t)\|_{\mathbf{L}^1} \le C. \tag{2.74}$$

We now multiply equation (2.59) by $2(2u_t + u)$. Then integrating the result with respect to $x \in \mathbf{R}$ we get

$$\begin{aligned}
&2\int_{\mathbf{R}} ((u_t+u)(u_{tt}+u_t) + u_t u_{tt})\, dx \\
&= -2\int_{\mathbf{R}} (u_t)^2\, dx + 2\int_{\mathbf{R}} (2u_t+u)\, u_{xx} dx - 2|\lambda| \int_{\mathbf{R}} (2u_t+u)\, |u|^{\sigma} u dx,
\end{aligned}$$

therefore

$$\begin{aligned}
&\frac{d}{dt}\left(\|u_t+u\|_{\mathbf{L}^2}^2 + \|u_t\|_{\mathbf{L}^2}^2 + 2\|u_x\|_{\mathbf{L}^2}^2 + \frac{4|\lambda|}{\sigma+2} \|u\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2} \right) \\
&= -2\|u_t\|_{\mathbf{L}^2}^2 - 2\|u_x\|_{\mathbf{L}^2}^2 - 2|\lambda| \|u\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2}.
\end{aligned} \tag{2.75}$$

By the Plansherel theorem using the Fourier splitting method due to Schonbek [1995], we have

$$\begin{aligned}
\|u_x\|_{\mathbf{L}^2}^2 &= \int_{|\xi| \le \chi} |\xi \widehat{u}(t,\xi)|^2\, d\xi + \int_{|\xi| \ge \chi} |\xi \widehat{u}(t,\xi)|^2\, d\xi \\
&\ge \chi^2 \|u\|_{\mathbf{L}^2}^2 - C\chi^3,
\end{aligned}$$

where $\chi > 0$. Thus from (2.75) we have the inequality

$$\begin{aligned}
&\frac{d}{dt}\left(\|u_t+u\|_{\mathbf{L}^2}^2 + \|u_t\|_{\mathbf{L}^2}^2 + 2\|u_x\|_{\mathbf{L}^2}^2 + \frac{4|\lambda|}{2+\sigma} \|u\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2} \right) \\
&\le -\chi^2 \|u\|_{\mathbf{L}^2}^2 - \|u_x\|_{\mathbf{L}^2}^2 - 2\|u_t\|_{\mathbf{L}^2}^2 - 2|\lambda| \|u\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2} + C\chi^3 \\
&\le -\chi^2 \left(\|u_t+u\|_{\mathbf{L}^2}^2 + \|u_t\|_{\mathbf{L}^2}^2 + \|u_x\|_{\mathbf{L}^2}^2 + \frac{4|\lambda|}{2+\sigma} \|u\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2} \right) + C\chi^3.
\end{aligned} \tag{2.76}$$

We choose $\chi^2 = 2(t_0+t)^{-1}$, $t_0 \ge 2$ and change

$$\|u_t + u\|_{\mathbf{L}^2}^2 + \|u_t\|_{\mathbf{L}^2}^2 + \|u_x\|_{\mathbf{L}^2}^2 + \frac{4|\lambda|}{2+\sigma} \|u\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2} = (t_0+t)^{-2} W(t).$$

Then we get from (2.76)

$$\frac{d}{dt} W(t) \le C (t_0+t)^{\frac{1}{2}}. \tag{2.77}$$

Integration of (2.77) with respect to time yields

$$W(t) \le C (t_0+t)^{\frac{3}{2}}.$$

Therefore we obtain the second estimate of the lemma.

We now differentiate equation (2.59) with respect to x and multiply the result by $2(2u_{xt}+u_x)$. Then integrating with respect to $x \in \mathbf{R}$ we get

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbf{R}} \left((u_{xt}+u_x)^2 + (u_{xt})^2 \right) dx = 2\int_{\mathbf{R}} (2u_{xt}+u_x) u_{xxx} dx \\ &- 2\int_{\mathbf{R}} (u_{xt})^2 dx - 2|\lambda| (\sigma+1) \int_{\mathbf{R}} (2u_{xt}+u_x) |u|^{\sigma} u_x dx, \end{aligned}$$

hence

$$\begin{aligned} &\frac{d}{dt}\left(\|u_{xt}+u_x\|_{\mathbf{L}^2}^2 + \|u_{xt}\|_{\mathbf{L}^2}^2 + 2\|u_{xx}\|_{\mathbf{L}^2}^2 + 2|\lambda|(\sigma+1) \left\| |u|^{\sigma} u_x^2 \right\|_{\mathbf{L}^1} \right) \\ &= -2\|u_{xx}\|_{\mathbf{L}^2}^2 - 2\|u_{xt}\|_{\mathbf{L}^2}^2 - 2|\lambda|(\sigma+1) \left\| |u|^{\sigma} u_x^2 \right\|_{\mathbf{L}^1} \\ &+ 2|\lambda| \sigma (\sigma+1) \int_{\mathbf{R}} u_x^2 |u|^{\sigma-2} u u_t dx. \end{aligned} \tag{2.78}$$

Then using equation (2.59) we get

$$\begin{aligned} &\int_{\mathbf{R}} u_x^2 |u|^{\sigma-2} u u_t dx = \int_{\mathbf{R}} u_x^2 |u|^{\sigma-2} u u_{xx} dx + \int_{\mathbf{R}} u_x^2 |u|^{\sigma-2} u \left(\lambda |u|^{\sigma} u - u_{tt}\right) dx \\ &= -\frac{\sigma-1}{3} \left\| |u|^{\sigma-2} u_x^4 \right\|_{\mathbf{L}^1} + \lambda \int_{\mathbf{R}} u_x^2 |u|^{2\sigma} dx - \int_{\mathbf{R}} u_x^2 |u|^{\sigma-2} u u_{tt} dx \\ &\le \|u_x\|_{\mathbf{L}^2}^2 \|u\|_{\mathbf{L}^\infty}^{\sigma-1} \|u_{tt}\|_{\mathbf{L}^1} \le \frac{\mu(t)}{2|\lambda| \sigma (\sigma+1)} \|u_x\|_{\mathbf{L}^2}^2, \end{aligned}$$

where we denote

$$\mu(t) = 2|\lambda| \sigma (\sigma+1) \|u\|_{\mathbf{L}^\infty}^{\sigma-1} \|u_{tt}\|_{\mathbf{L}^1}.$$

By Lemma 2.40 we see that $\int_0^\infty \mu(t)\, dt \le C$. As above by the Plansherel theorem, we have

$$\begin{aligned} \|u_{xx}\|_{\mathbf{L}^2}^2 &= \int_{|\xi| \le \chi} \left| \xi^2 \widehat{u}(t,\xi) \right|^2 d\xi + \int_{|\xi| \ge \chi} \left| \xi^2 \widehat{u}(t,\xi) \right|^2 d\xi \\ &\ge \chi^2 \|u_x(t)\|_{\mathbf{L}^2}^2 - C\chi^5, \end{aligned}$$

where $\chi > 0$. Thus from (2.78) we have the inequality

$$\frac{d}{dt}\left(\|u_{xt}+u_x\|_{\mathbf{L}^2}^2+\|u_{xt}\|_{\mathbf{L}^2}^2+2\|u_{xx}\|_{\mathbf{L}^2}^2+2|\lambda|(\sigma+1)\left\||u|^{\sigma}u_x^2\right\|_{\mathbf{L}^1}\right)$$
$$\leq\left(\mu-\chi^2\right)\|u_x\|_{\mathbf{L}^2}^2-2\|u_{xt}\|_{\mathbf{L}^2}^2-\|u_{xx}\|_{\mathbf{L}^2}^2$$
$$-2|\lambda|(\sigma+1)\left\||u|^{\sigma}u_x^2\right\|_{\mathbf{L}^1}+C\chi^5$$
$$\leq\left(\mu-\chi^2\right)\left(\|u_{xt}+u_x\|_{\mathbf{L}^2}^2+\|u_{xt}\|_{\mathbf{L}^2}^2+2\|u_{xx}\|_{\mathbf{L}^2}^2+2|\lambda|(\sigma+1)\left\||u|^{\sigma}u_x^2\right\|_{\mathbf{L}^1}\right)$$
$$+C\chi^5. \tag{2.79}$$

We choose $\chi^2=3(t_0+t)^{-1}$, $t_0\geq 6$ and change

$$\|u_{xt}+u_x\|_{\mathbf{L}^2}^2+\|u_{xt}\|_{\mathbf{L}^2}^2+2\|u_{xx}\|_{\mathbf{L}^2}^2+2|\lambda|(\sigma+1)\left\||u|^{\sigma}u_x^2\right\|_{\mathbf{L}^1}$$
$$=(t_0+t)^{-3}e^{\int_0^t\mu(\tau)d\tau}W_1(t).$$

Then we get from (2.79)

$$\frac{d}{dt}W_1(t)\leq C(t_0+t)^{\frac{1}{2}}. \tag{2.80}$$

Integration of (2.80) with respect to time yields

$$W_1(t)\leq C(t_0+t)^{\frac{3}{2}}.$$

Therefore we obtain last estimate of the lemma. Furthermore we have the optimal time decay estimate

$$\|u(t)\|_{\mathbf{L}^p}\leq C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \tag{2.81}$$

for all $t>0$, $1\leq p\leq\infty$ since

$$\|u(t)\|_{\mathbf{L}^\infty}\leq C\|u(t)\|_{\mathbf{L}^2}^{\frac{1}{2}}\|u_x(t)\|_{\mathbf{L}^2}^{\frac{1}{2}}\leq C\langle t\rangle^{-\frac{1}{2}}$$

and

$$\|u(t)\|_{\mathbf{L}^1}\leq C.$$

Lemma 2.41 is proved.

Proof of Theorem 2.35. In view of (2.65) and (2.66) we can apply Lemma 2.40 to obtain $\|u_{tt}\|_{1,1}\leq C$. Then from Lemma 2.41 we have the estimate (2.81) for all $t>0$. Thus we have the desired result by Proposition 2.38. Theorem 2.35 is proved.

2.4 Sobolev type equations

This section is devoted to the study of the Cauchy problem for the Sobolev type equation in the supercritical case

$$\begin{cases} \partial_t (u - \Delta u) - \alpha \Delta u = \lambda |u|^\sigma u, \ x \in \mathbf{R}^n, \ t > 0, \\ u(0, x) = u_0(x), \ x \in \mathbf{R}^n, \end{cases} \tag{2.82}$$

where $\alpha > 0$, $\sigma > \frac{2}{n}$, $\lambda \in \mathbf{R}$. Equation (2.82) can be viewed as a multidimensional generalization of the Boussinesq equation; it arises in liquid filtration problems for a porous medium with cracks Kozhanov [1994], Kozhanov [1999] and in the theory of crystalline semiconductors Korpusov and Sveshnikov [2003].

Using the Duhamel principle we rewrite (2.82) in the form of the integral equation

$$u(t) = \mathcal{G}(t) u_0 + \lambda \int_0^t \mathcal{G}(t - \tau) \mathcal{B} \ |u|^\sigma u(\tau) \, d\tau, \tag{2.83}$$

where the Green operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t) \phi = e^{-\alpha t} \overline{\mathcal{F}}_{\xi \to x} e^{\frac{\alpha t}{1 + |\xi|^2}} \hat{\phi}(\xi)$$

and

$$\mathcal{B}\phi = \int_{\mathbf{R}^n} B(x - y) \phi(y) \, dy$$

with a kernel (see Titchmarsh [1986])

$$B(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{i\xi x} \left(1 + |\xi|^2\right)^{-1} d\xi = |x|^{1 - \frac{n}{2}} K_{\frac{n}{2} - 1}(|x|),$$

where

$$K_\nu(|x|) = K_{-\nu}(|x|) = 2^{-\nu - 1} |x|^\nu \int_0^\infty \xi^{-\nu - 1} e^{-\xi - \frac{|x|^2}{4\xi}} d\xi$$

is the Macdonald function (or the modified Bessel function) of order $\nu \in \mathbf{R}$ (see Watson [1944]). If we use the Taylor expansion

$$e^{\frac{\alpha t}{1 + |\xi|^2}} = \sum_{k=0}^\infty \frac{\alpha^k t^k}{k!} \left(1 + |\xi|^2\right)^{-k}$$

then we can represent

$$\mathcal{G}(t) \phi = e^{-\alpha t} \sum_{k=0}^\infty \frac{\alpha^k t^k}{k!} \mathcal{B}^k \phi,$$

where $\mathcal{B}^0 = 1$ and

$$\mathcal{B}^k \phi = \int_{\mathbf{R}^n} B_k(x - y) \phi(y) \, dy$$

with

$$B_k(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{i\xi x} \left(1+|\xi|^2\right)^{-k} d\xi$$
$$= \frac{2^{1-k}}{(k-1)!} |x|^{k-\frac{n}{2}} K_{\frac{n}{2}-k}(|x|)$$

for $k \geq 1$. Thus we can easily see that the operators $\mathcal{G}(t), \mathcal{G}(t)\mathcal{B}$

$$\mathcal{G}(t)\phi = \phi + \int_{\mathbf{R}^n} G(t, x-y)\phi(y)\,dy$$

$$\mathcal{G}(t)\mathcal{B}\phi = \int_{\mathbf{R}^n} H(t, x-y)\phi(y)\,dy$$

have positive kernels

$$G(t,x) = e^{-\alpha t} \sum_{k=1}^{\infty} \frac{\alpha^k t^k}{k!} B_k(|x|) \geq 0,$$

$$H(t,x) = e^{-\alpha t} \sum_{k=0}^{\infty} \frac{\alpha^k t^k}{k!} B_{k+1}(|x|) \geq 0 \tag{2.84}$$

for all $t \geq 0$, $x \in \mathbf{R}^n$. By the estimates of the Macdonald function (see Watson [1944]) we see that for any $k \geq 1$

$$|B_k(x)| \leq \begin{cases} C|x|^{k-\frac{n+1}{2}} e^{-|x|}, \text{ for } |x| \geq 1, \\ C\int_{|x|}^1 y^{2k-n-1} dy, \text{ for } |x| < 1 \, . \end{cases}$$

Hence for any $k \geq 0$

$$\left\|\mathcal{B}^k\phi\right\|_{\mathbf{L}^p} \leq C \left\| \int_{\mathbf{R}^n} B_k(t, x-y)\phi(y)\,dy \right\|_{\mathbf{L}^p} \leq \|B_k(t)\|_{\mathbf{L}^1} \|\phi\|_{\mathbf{L}^p} \tag{2.85}$$
$$\leq C\|\phi\|_{\mathbf{L}^p} \left(\int_{|x|<1} dx \int_{|x|}^1 y^{2k-n-1} dy + \int_{|x|\geq 1} |x|^{k-\frac{n+1}{2}} e^{-|x|} dx \right)$$
$$\leq C\|\phi\|_{\mathbf{L}^p}$$

for all $1 \leq p \leq \infty$ and

$$\left\|\mathcal{B}^k\phi\right\|_{\mathbf{L}^{1,a}} \leq C \left\| \langle x\rangle^a \int_{\mathbf{R}^n} B_k(t, x-y)\phi(y)\,dy \right\|_{\mathbf{L}^1}$$
$$\leq C\|B_k(t)\|_{\mathbf{L}^{1,a}} \|\phi\|_{\mathbf{L}^{1,a}}$$
$$\leq C\|\phi\|_{\mathbf{L}^{1,a}} \left(\int_{|x|<1} dx \int_{|x|}^1 y^{2k-n-1} dy + \int_{|x|\geq 1} |x|^{a+k-\frac{n+1}{2}} e^{-|x|} dx \right)$$
$$\leq C\|\phi\|_{\mathbf{L}^{1,a}} \tag{2.86}$$

for any $a \geq 0$.

2.4.1 Local existence

We now prove the existence of local solutions to the Cauchy problem (2.82).

Theorem 2.42. *Let $u_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n)$, $a \geq 0$. Then for some $T > 0$ there exists a unique solution $u \in \mathbf{C}\left([0,T]; \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n)\right)$ to the Cauchy problem (2.82).*

Proof. To apply Theorem 1.9, we choose functional spaces

$$\mathbf{Z} = \left\{\phi \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n) : \ ||u||_{\mathbf{Z}} < \infty\right\},$$

$$\mathbf{X_T} = \left\{u \in \mathbf{C}\left([0,T]; \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n)\right) : \ ||u||_{\mathbf{X}_T} < \infty\right\},$$

with the following norms

$$\|\phi\|_{\mathbf{Z}} = \|\phi\|_{\mathbf{L}^{1,a}} + \|\phi\|_{\mathbf{L}^\infty},$$

$$\|u\|_{\mathbf{X}_T} = \sup_{t \in [0,T]} \left(\|u(t)\|_{\mathbf{L}^{1,a}} + \|u(t)\|_{\mathbf{L}^\infty}\right).$$

Applying the Young inequality for convolutions and Lemma 1.31 we have

$$\|\mathcal{G}(t) u_0\|_{\mathbf{L}^\infty} \leq C \langle t \rangle^{-\frac{n}{2}} \left(\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^1}\right) \tag{2.87}$$

and

$$\left\| |\lambda| \int_0^t \mathcal{G}(t-\tau) \mathcal{B}\left(|v_1|^\sigma v_1(\tau) - |v_2|^\sigma v_2(\tau)\right) d\tau \right\|_{\mathbf{L}^\infty}$$
$$\leq C \int_0^t \|\mathcal{B}\left(|v_1|^\sigma v_1(\tau) - |v_2|^\sigma v_2(\tau)\right)\|_{\mathbf{L}^\infty} d\tau. \tag{2.88}$$

Similarly we obtain the estimates

$$\|\mathcal{G}(t) u_0\|_{\mathbf{L}^{1,a}} \leq C \langle t \rangle^{\frac{a}{2}} \|u_0\|_{\mathbf{L}^{1,a}} \tag{2.89}$$

and

$$\left\| |\lambda| \int_0^t \|\mathcal{G}(t-\tau) \mathcal{B}\left(|v_1|^\sigma v_1(\tau) - |v_2|^\sigma v_2(\tau)\right)\|_{\mathbf{L}^{1,a}} d\tau \right\|_{\mathbf{L}^{1,a}}$$
$$\leq C \int_0^t \langle t-\tau \rangle^{\frac{a}{2}} \|\mathcal{B}\left(|v_1|^\sigma v_1(\tau) - |v_2|^\sigma v_2(\tau)\right)\|_{\mathbf{L}^{1,a}} d\tau. \tag{2.90}$$

Using the estimate

$$||\mathcal{B}\left(|v_1|^\sigma v_1(\tau) - |v_2|^\sigma v_2(\tau)\right)||_{\mathbf{X}_T} \leq C\, ||v_1 - v_2||_{\mathbf{X}_T} \left(||v_1||^\sigma_{\mathbf{X}_T} + ||v_2||^\sigma_{\mathbf{X}_T}\right),$$

we get from (2.87) to (2.90)

$$\|\mathcal{G}\phi\|_{\mathbf{X}_T} \leq C \|\phi\|_{\mathbf{Z}}.$$

Also we have

$$\left\| \int_0^t \mathcal{G}(t-\tau)\left(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau))\right) d\tau \right\|_{\mathbf{X}_T}$$
$$\leq CT \|w-v\|_{\mathbf{X}_T} \left(\|w\|_{\mathbf{X}_T}^{\sigma} + \|v\|_{\mathbf{X}_T}^{\sigma} \right).$$

Therefore if we choose sufficiently small $T > 0$, due to Theorem 1.9 there exists a unique solution $u(t,x) \in \mathbf{C}\left([0,T]; \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^{\infty}(\mathbf{R}^n)\right)$ to the problem (2.82). Theorem 2.42 is proved.

2.4.2 Small data

In order to prove global existence in the case of the arbitrary sign of λ of the nonlinearity we have to assume a smallness condition for the initial data, since in the case of $\lambda > 0$ there could be a blow up phenomena (see Samarskii et al. [1995], Glassey [1973b], Mitidieri and Pokhozhaev [2001], Tsutsumi [1984]).

Theorem 2.43. *Let* $\sigma > \frac{2}{n}$. *Let* $u_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^{\infty}(\mathbf{R}^n)$, $a \in (0,1]$. *Suppose that the norm* $\|u_0\|_{\mathbf{L}^{1,a}} + \|u_0\|_{\mathbf{L}^{\infty}}$ *is sufficiently small. Then there exists a unique solution* $u \in \mathbf{C}\left([0,\infty); \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^{\infty}(\mathbf{R}^n)\right)$ *to the Cauchy problem (2.82). Moreover the asymptotics*

$$u(t,x) = At^{-\frac{n}{2}} e^{-\frac{x^2}{4t}} + O\left(t^{-\frac{n}{2}-\gamma}\right) \tag{2.91}$$

for large time $t \to \infty$ *is true uniformly with respect to* $x \in \mathbf{R}^n$, *where* A *is a constant,* $0 < \gamma < \min\left(\frac{a}{2}, \frac{n}{2}\sigma - 1\right)$.

Proof. We apply Theorem 1.17, as well as introduce the space

$$\mathbf{X} = \left\{ v \in \mathbf{C}\left([0,\infty); \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^{\infty}(\mathbf{R}^n)\right) : \ \|v\|_{\mathbf{X}} < \infty \right\},$$

where the norm

$$\|v\|_{\mathbf{X}} = \sup_{t>0} \left(\langle t \rangle^{-\frac{a}{2}} \|v(t)\|_{\mathbf{L}^{1,a}} + \langle t \rangle^{\frac{n}{2}} \|v(t)\|_{\mathbf{L}^{\infty}} \right).$$

Note that

$$\sup_{t>0} \|\phi(t)\|_{\mathbf{L}^1} \leq \|\phi\|_{\mathbf{X}}.$$

Also we define

$$\mathbf{Z} = \left\{ \phi \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^{\infty}(\mathbf{R}^n) : \ ||\phi||_{\mathbf{Z}} < \infty \right\},$$

with the following norms

$$\|\phi\|_{\mathbf{Z}} = \|\phi\|_{\mathbf{L}^{1,a}} + \|\phi\|_{\mathbf{L}^{\infty}}.$$

Applying the Young inequality for convolutions and Lemma 1.31 we obtain

$$\|\mathcal{G}\phi\|_{\mathbf{L}^\infty} \le C \langle t \rangle^{-\frac{n}{2}} \left(\|\phi\|_{\mathbf{L}^\infty} + \|\phi\|_{\mathbf{L}^1} \right)$$

and

$$\left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{B}\left(|w|^\sigma w(\tau) - |v|^\sigma v(\tau) \right) d\tau \right\|_{\mathbf{L}^\infty}$$
$$\le C \int_0^{\frac{t}{2}} \langle t-\tau \rangle^{-\frac{n}{2}} \left(\|\mathcal{B}\left(|w|^\sigma w(\tau) - |v|^\sigma v(\tau) \right)\|_{\mathbf{L}^\infty} \right.$$
$$\left. + \|\mathcal{B}\left(|w|^\sigma w(\tau) - |v|^\sigma v(\tau) \right)\|_{\mathbf{L}^1} d\tau \right)$$
$$+ C \int_{\frac{t}{2}}^t \|\mathcal{B}\left(|w|^\sigma w(\tau) - |v|^\sigma v(\tau) \right)\|_{\mathbf{L}^\infty} d\tau.$$

Using the estimates

$$\|\mathcal{B}\left(|w|^\sigma w(\tau) - |v|^\sigma v(\tau) \right)\|_{\mathbf{L}^\infty} \le \langle \tau \rangle^{-\frac{n}{2}(\sigma+1)} ||w-v||_{\mathbf{X}} \left(||w||_{\mathbf{X}}^\sigma + ||v||_{\mathbf{X}}^\sigma \right)$$

and

$$\|\mathcal{B}\left(|w|^\sigma w(\tau) - |v|^\sigma v(\tau) \right)\|_{\mathbf{L}^1} \le \langle \tau \rangle^{-\frac{n}{2}\sigma} ||w-v||_{\mathbf{X}} \left(||w||_{\mathbf{X}}^\sigma + ||v||_{\mathbf{X}}^\sigma \right)$$

we get

$$\left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{B}\left(|w|^\sigma w(\tau) - |v|^\sigma v(\tau) \right) d\tau \right\|_{\mathbf{L}^\infty}$$
$$\le C ||w-v||_{\mathbf{X}} \left(||w||_{\mathbf{X}}^\sigma + ||v||_{\mathbf{X}}^\sigma \right) \int_0^{\frac{t}{2}} \langle \tau \rangle^{-\frac{n}{2}\sigma} \langle t-\tau \rangle^{-\frac{n}{2}} d\tau$$
$$+ C ||w-v||_{\mathbf{X}} \left(||w||_{\mathbf{X}}^\sigma + ||v||_{\mathbf{X}}^\sigma \right) \int_{\frac{t}{2}}^t \langle \tau \rangle^{-\frac{n}{2}(\sigma+1)} d\tau$$
$$\le \langle t \rangle^{-\frac{n}{2}} ||w-v||_{\mathbf{X}} \left(||w||_{\mathbf{X}}^\sigma + ||v||_{\mathbf{X}}^\sigma \right)$$

since $\frac{n}{2}\sigma > 1$. Similarly we estimate the norm

$$\|\mathcal{G}\phi\|_{\mathbf{L}^{1,a}} \le C \langle t \rangle^{\frac{a}{2}} \left(\|\phi\|_{\mathbf{L}^\infty} + \|\phi\|_{\mathbf{L}^1} \right)$$

and

$$\left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{B}\left(|w|^\sigma w(\tau) - |v|^\sigma v(\tau) \right) d\tau \right\|_{\mathbf{L}^{1,a}}$$
$$\le C \int_0^t \|\mathcal{B}\left(|w|^\sigma w(\tau) - |v|^\sigma v(\tau) \right)\|_{\mathbf{L}^{1,a}} d\tau$$
$$C \int_0^t \|\mathcal{B}\left(|w|^\sigma w(\tau) - |v|^\sigma v(\tau) \right)\|_{\mathbf{L}^1} \langle t-\tau \rangle^{\frac{a}{2}} d\tau$$
$$\le C ||w-v||_{\mathbf{X}} \left(||w||_{\mathbf{X}}^\sigma + ||v||_{\mathbf{X}}^\sigma \right) \left(\int_0^t \langle \tau \rangle^{-\frac{n}{2}\sigma + \frac{a}{2}} d\tau \right.$$
$$\left. + \int_0^t \langle \tau \rangle^{-\frac{n}{2}\sigma} \langle t-\tau \rangle^{\frac{a}{2}} d\tau \right) \le \langle t \rangle^{\frac{a}{2}} ||w-v||_{\mathbf{X}} \left(||w||_{\mathbf{X}}^\sigma + ||v||_{\mathbf{X}}^\sigma \right),$$

where we have used the inequality

$$\|\mathcal{B}(|w|^{\sigma} w(\tau) - |v|^{\sigma} v(\tau))\|_{\mathbf{L}^{1,a}} \leq ||w - v||_{\mathbf{X}} (||w||_{\mathbf{X}}^{\sigma} + ||v||_{\mathbf{X}}^{\sigma}) \langle\tau\rangle^{-\frac{n}{2}\sigma+\frac{a}{2}}.$$

Thus we get

$$\|\mathcal{G}\phi\|_{\mathbf{X}} \leq C \|\phi\|_{\mathbf{Z}}.$$

Also the estimate is true

$$\left\|\int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)))\,d\tau\right\|_{\mathbf{X}} \leq C \|w - v\|_{\mathbf{X}} (\|w\|_{\mathbf{X}} + \|v\|_{\mathbf{X}})^{\sigma}.$$

Therefore due to Theorem 1.17 there exists a unique solution

$$u(t,x) \in \mathbf{C}\left([0,\infty); \mathbf{L}^{1,a} \cap \mathbf{L}^{\infty}\right)$$

to the problem (2.82) such that

$$\|u(t)\|_{\mathbf{L}^{\infty}} \leq C \langle t\rangle^{-\frac{n}{2}}, \text{ and } \|u(t)\|_{\mathbf{L}^{1,a}} \leq C \langle t\rangle^{\frac{a}{2}}. \tag{2.92}$$

We now prove the asymptotics (2.91) by employing Theorem 2.4. By virtue of the estimate of Lemma 1.31 we have the following asymptotic representation for the Green operator

$$\mathcal{G}(t)u_0 = \theta G_0(t,x) + O\left(t^{-\frac{n+a}{2}} \|u_0\|_{\mathbf{L}^{1,a}}\right) \tag{2.93}$$

for large time $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$, where

$$G_0(t,x) = (4\pi\alpha t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4\alpha t}},$$

$$\theta = \int_{\mathbf{R}^n} u_0(x)\,dx,$$

and $a \in (0,1]$. Now consider the difference

$$\begin{aligned}
&\lambda \int_0^t \mathcal{G}(t-\tau)\mathcal{B}|u|^{\sigma} u(\tau)\,d\tau - G_0(t,x)\int_0^{\infty} \vartheta(\tau)\,d\tau \\
&= \lambda \int_0^t (\mathcal{G}(t-\tau)\mathcal{B}|u|^{\sigma} u(\tau) - \vartheta(\tau) G_0(t-\tau,x))\,d\tau \\
&+ \int_0^t (G_0(t-\tau,x) - G_0(t,x))\vartheta(\tau)\,d\tau + G_0(t,x)\int_t^{\infty} \vartheta(\tau)\,d\tau,
\end{aligned} \tag{2.94}$$

where

$$\vartheta(\tau) = \lambda \int_{\mathbf{R}^n} \mathcal{B}|u|^{\sigma} u(\tau)\,dy = \lambda \int_{\mathbf{R}^n} |u|^{\sigma} u(\tau)\,dy$$

since $\mathcal{F}_{x\to\xi}\mathcal{B}\phi = \left(1+|\xi|^2\right)^{-1} \widehat{\phi}(\xi)$. In the domain $0 < \tau < \frac{t}{2}$ we write by the estimate of Lemma 1.31

$$\|\lambda\mathcal{G}(t-\tau)\mathcal{B}|u|^{\sigma}u(\tau)-\vartheta(\tau)G_0(t-\tau,x)\|_{\mathbf{L}^\infty}$$
$$\le C(t-\tau)^{-\frac{n}{2}-\frac{a}{2}}\left(\|\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^\infty}+\|\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^{1,a}}\right)$$

for all $0<\tau<\frac{t}{2}$. In the domain $\frac{t}{2}\le\tau<t$ we use the estimate

$$\|\lambda\mathcal{G}(t-\tau)\mathcal{B}|u|^{\sigma}u(\tau)-\vartheta(\tau)G_0(t-\tau,x)\|_{\mathbf{L}^\infty}$$
$$\le C\|\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^\infty}.$$

Therefore we have by virtue of estimate (2.92)

$$\left\|\int_0^t\left(\lambda\mathcal{G}(t-\tau)\mathcal{B}|u|^{\sigma}u(\tau)-\vartheta(\tau)G_0(t-\tau,x)\right)d\tau\right\|_{\mathbf{L}^\infty}$$
$$\le C\|u\|_{\mathbf{X}}^{\sigma+1}\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{n}{2}-\frac{a}{2}}\langle\tau\rangle^{-\frac{n}{2}\sigma}d\tau+C\|u\|_{\mathbf{X}}^{\sigma+1}\int_{\frac{t}{2}}^t\langle\tau\rangle^{-\frac{n}{2}(\sigma+1)}d\tau$$
$$\le C\|u\|_{\mathbf{X}}^{\sigma+1}\langle t\rangle^{-\frac{n}{2}-\gamma},$$

where $0<\gamma<\min\left(\frac{a}{2},\frac{n}{2}\sigma-1\right)$. Now we estimate the difference

$$\left\|\int_0^t\left(G_0(t-\tau,x)-G_0(t,x)\right)\vartheta(\tau)d\tau\right\|_{\mathbf{L}^\infty}$$
$$\le C\int_0^t\|G_0(t-\tau)-G_0(t)\|_{\mathbf{L}^\infty}\langle\tau\rangle^{-\frac{n}{2}\sigma}d\tau$$
$$\le C\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-\frac{n}{2}-\gamma}\langle\tau\rangle^{\gamma-\frac{n}{2}\sigma}d\tau+C\int_{\frac{t}{2}}^t\langle t-\tau\rangle^{-\frac{n}{2}}\langle\tau\rangle^{-\frac{n}{2}\sigma}d\tau$$
$$\le C\langle t\rangle^{-\frac{n}{2}-\gamma}\|u\|_{\mathbf{X}}^{\sigma+1}.$$

In view of the estimate $|\vartheta(t)|\le C\|u(t)\|_{\mathbf{L}^{\sigma+1}}^{\sigma+1}\le C\langle t\rangle^{-\frac{n}{2}\sigma}\|u\|_{\mathbf{X}}^{\sigma+1}\le C\langle t\rangle^{-\frac{n}{2}\sigma}$ we have for the last summand in (2.94)

$$\left\|G_0(t)\int_t^\infty\vartheta(\tau)d\tau\right\|_{\mathbf{L}^\infty}\le Ct^{-\frac{n}{2}}\int_t^\infty\langle\tau\rangle^{-\frac{n}{2}\sigma}d\tau\le C\langle t\rangle^{-\frac{n}{2}-\gamma}.$$

Thus all conditions of Theorem 2.4 are fulfilled, and, by equation (2.83) we see that there exists a constant

$$A=\int_{\mathbf{R}^n}u_0(x)dx+\lambda\int_0^\infty dt\int_{\mathbf{R}^n}\mathcal{B}|u(t)|^{\sigma}u(t)dx$$
$$=\lim_{t\to+\infty}\int_{\mathbf{R}^n}u(t,x)dx.$$

such that asymptotics (2.91) is valid. Theorem 2.43 is proved.

2.4.3 Large data

Now we consider the case of $\lambda < 0$, then we can remove the smallness condition on the initial data $u_0(x)$.

Theorem 2.44. *Let $\sigma > \frac{2}{n}$, $\lambda < 0$, $n = 1, 2$. Suppose that the initial data $u_0 \in \mathbf{W}^2_\infty(\mathbf{R}^n) \cap \mathbf{W}^2_1(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, $a \in (0, 1]$. Then there exists a unique solution $u \in \mathbf{C}\left([0,\infty); \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n)\right)$ to the Cauchy problem (2.82). Moreover the asymptotics (2.91) takes place.*

Before proving Theorem 2.44 we prepare several lemmas. Define the norms

$$\|\phi\|_{p,q} \equiv \left\| \|\phi(t,x)\|_{\mathbf{L}^q(\mathbf{R}^n_x)} \right\|_{\mathbf{L}^p\left(\mathbf{R}^+_t\right)}.$$

Lemma 2.45. *Let $n = 1, 2$. Suppose that the initial data $u_0 \in \mathbf{W}^2_\infty(\mathbf{R}^n) \cap \mathbf{W}^2_1(\mathbf{R}^n)$. Let the norms of the solution be bounded*

$$\|u\|_{\infty,2} + \|u\|_{\sigma+2,\sigma+2} \le C.$$

Then the estimate is true

$$\int_0^t \left\| |u(\tau)|^\sigma \Delta u(\tau) \right\|_{\mathbf{L}^1} d\tau \le C \langle t \rangle^\beta$$

for all $t > 0$, where $\beta = 0$ for $\sigma > 1$, and $\beta > \frac{1}{\sigma} - 1$ for $\frac{1}{2} < \sigma \le 1$.

Proof. By the Hölder inequality we get for $2 \le p \le \sigma + 2$

$$\|u(t)\|_{\mathbf{L}^p} \le \|u(t)\|_{\mathbf{L}^2}^{1-\left(1-\frac{2}{p}\right)\left(1+\frac{2}{\sigma}\right)} \|u(t)\|_{\mathbf{L}^{2+\sigma}}^{\left(1-\frac{2}{p}\right)\left(1+\frac{2}{\sigma}\right)};$$

hence

$$\|u\|_{s,p} \le \|u\|_{\infty,2}^{1-\left(1-\frac{2}{p}\right)\left(1+\frac{2}{\sigma}\right)} \|u\|_{\sigma+2,\sigma+2}^{\left(1-\frac{2}{p}\right)\left(1+\frac{2}{\sigma}\right)} \le C$$

for $s = \frac{\sigma p}{p-2}$. Next we estimate the $\mathbf{L}^p$ norm for $\sigma + 2 \le p \le \infty$

$$\|u(t)\|_{\mathbf{L}^p} \le \|\mathcal{G}(t) u_0\|_{\mathbf{L}^p} + |\lambda| \left\| \int_0^t \mathcal{G}(t-\tau) \mathcal{B} |u|^\sigma u(\tau) d\tau \right\|_{\mathbf{L}^p},$$

by Lemma 1.31 we also possess the estimate

$$\begin{aligned} \|\mathcal{G}(t-\tau) \mathcal{B} |u|^\sigma u(\tau)\|_{\mathbf{L}^p} &\le e^{-a(t-\tau)} \|\mathcal{B} |u|^\sigma u(\tau)\|_{\mathbf{L}^p} \\ &+ C \langle t-\tau \rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} \|\mathcal{B} |u|^\sigma u(\tau)\|_{\mathbf{L}^r} \end{aligned}$$

where $1 \le r \le p$. Applying the Sobolev Imbedding Theorem 1.4 we have with $\max\left(1, \frac{p}{1+\frac{2}{n}p}\right) \le r \le p$

$$\|\mathcal{B} |u|^\sigma u(\tau)\|_{\mathbf{L}^p} \le C \left\| |u|^\sigma u(\tau) \right\|_{\mathbf{L}^r} \le C \|u(\tau)\|_{\mathbf{L}^{(\sigma+1)r}}^{\sigma+1}.$$

Thus we obtain

$$\|\mathcal{G}(t-\tau)\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^p} \leq C\langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|u(\tau)\|_{\mathbf{L}^{(\sigma+1)r}}^{\sigma+1}.$$

By the Young inequality and Lemma 1.31 we then get

$$\begin{aligned}
&\left\|\int_0^t \|\mathcal{G}(t-\tau)\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^p}\,d\tau\right\|_{\mathbf{L}_t^s(0,\infty)} \\
&\leq C\left\|\int_0^t \langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|u(\tau)\|_{\mathbf{L}^{(\sigma+1)r}}^{\sigma+1}\,d\tau\right\|_{\mathbf{L}_t^s(0,\infty)} \\
&\leq C\left\|\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\right\|_{\mathbf{L}_t^{q'}(0,\infty)}\|u\|_{(\sigma+1)q,(\sigma+1)r}^{\sigma+1} \\
&\leq C\|u\|_{(\sigma+1)q,(\sigma+1)r}^{\sigma+1}
\end{aligned}$$

if $\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right) > \frac{1}{q'} = \frac{1}{s}-\frac{1}{q}+1$. We apply this inequality taking $q = r = \frac{\sigma+2}{\sigma+1}$, $n = 1, 2$; consequently $s > \max\left(1, \left(\frac{\sigma+1}{\sigma+2}\left(1+\frac{n}{2}\right)-1-\frac{n}{2p}\right)^{-1}\right)$. Taking $p = \infty$ we thus obtain

$$\|u\|_{s,\infty} \leq C \tag{2.95}$$

for $s > \frac{2(\sigma+2)}{n(\sigma+1)-2}$.

Next we estimate the second derivative Δu. Note that by the smoothing properties of operator $\mathcal{B}$ we can see that solution $u(t) \in \mathbf{W}_{\infty}^2(\mathbf{R}^n) \cap \mathbf{W}_1^2(\mathbf{R}^n)$ for all $t \geq 0$, when initial data $u_0 \in \mathbf{W}_{\infty}^2(\mathbf{R}^n) \cap \mathbf{W}_1^2(\mathbf{R}^n)$. By the estimate of Lemma 1.31 we have

$$\|\Delta\mathcal{B}\mathcal{G}(t-\tau)|u|^{\sigma}u(\tau)\|_{\mathbf{L}^p} \leq C\langle t-\tau\rangle^{-1}\|u(\tau)\|_{\mathbf{L}^{(\sigma+1)p}}^{\sigma+1},$$

where $p \geq 1$. Hence

$$\begin{aligned}
&\left\|\int_0^t \|\Delta\mathcal{G}(t-\tau)\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^p}\,d\tau\right\|_{\mathbf{L}_t^s(0,\infty)} \\
&\leq C\left\|\int_0^t \langle t-\tau\rangle^{-1}\|u(\tau)\|_{\mathbf{L}^{(\sigma+1)p}}^{\sigma+1}\,d\tau\right\|_{\mathbf{L}_t^s(0,\infty)} \\
&\leq C\left\|\langle t\rangle^{-1}\right\|_{\mathbf{L}_t^{q'}(0,\infty)}\|u\|_{(\sigma+1)q,(\sigma+1)p}^{\sigma+1} \\
&\leq C\|u\|_{(\sigma+1)q,(\sigma+1)p}^{\sigma+1},
\end{aligned}$$

where $\frac{1}{s} = \frac{1}{q}+\frac{1}{q'}-1$, $q' > 1$. Thus

$$\|\Delta u\|_{s,p} \leq C\|u\|_{(\sigma+1)q,(\sigma+1)p}^{\sigma+1} \tag{2.96}$$

for $s > q$. Applying the estimate

$$\|u(\tau)\|^{\sigma}_{\mathbf{L}^{\sigma(\sigma+2)}} \le \|u(\tau)\|^{\sigma-1}_{\mathbf{L}^{\infty}} \|u(\tau)\|_{\mathbf{L}^{\sigma+2}}$$

by the Hölder inequality we have for $\sigma > 1$

$$\int_0^t \left\| |u(\tau)|^{\sigma} \Delta u(\tau) \right\|_{\mathbf{L}^1} d\tau \le \int_0^t \|u(\tau)\|^{\sigma}_{\mathbf{L}^{\sigma(\sigma+2)}} \|\Delta u(\tau)\|_{\mathbf{L}^{\frac{\sigma+2}{\sigma+1}}} d\tau$$
$$\le t^{1-\frac{1}{s_1}-\frac{1}{s_2}-\frac{1}{\sigma+2}} \|u(\tau)\|^{\sigma-1}_{s_1(\sigma-1),\infty} \|u(\tau)\|_{\sigma+2,\sigma+2} \|u\|^{\sigma+1}_{(\sigma+1)s_3,\sigma+2} \le C,$$

since we can choose $s_1(\sigma-1) > \frac{2(\sigma+2)}{n(\sigma+1)-2}$, $s_2 > s_3 = \frac{\sigma+2}{\sigma+1}$ so that $\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{\sigma+2} > 1$. Consider $\sqrt{3}-1 < \sigma \le 1$. Then

$$\int_0^t \left\| |u(\tau)|^{\sigma} \Delta u(\tau) \right\|_{\mathbf{L}^1} d\tau \le \int_0^t \|u(\tau)\|^{\sigma}_{\mathbf{L}^{\sigma(\sigma+2)}} \|\Delta u(\tau)\|_{\mathbf{L}^{\frac{\sigma+2}{\sigma+1}}} d\tau$$
$$\le C \langle t \rangle^{1-\frac{1}{s_1}-\frac{1}{s_2}} \|u(\tau)\|^{\sigma}_{s_1\sigma,\sigma(\sigma+2)} \|u\|^{\sigma+1}_{(\sigma+1)s_3,\sigma+2} \le C \langle t \rangle^{\beta},$$

since we can choose $s_1\sigma > \frac{\sigma^2(\sigma+2)}{\sigma(\sigma+2)-2}$, $s_2 > s_3 = \frac{\sigma+2}{\sigma+1}$ so that $\beta = 1 - \frac{1}{s_1} - \frac{1}{s_2} > \frac{1}{\sigma} - 1$. Furthermore for $\frac{1}{2} < \sigma \le \sqrt{3}-1$

$$\int_0^t \left\| |u(\tau)|^{\sigma} \Delta u(\tau) \right\|_{\mathbf{L}^1} d\tau \le \int_0^t \|u(\tau)\|^{\sigma}_{\mathbf{L}^2} \|\Delta u(\tau)\|_{\mathbf{L}^{\frac{2}{2-\sigma}}} d\tau$$
$$\le C \langle t \rangle^{1-\frac{1}{s_1}} \|u(\tau)\|^{\sigma}_{\infty,2} \|u\|^{\sigma+1}_{(\sigma+1)s_2,(\sigma+1)\frac{2}{2-\sigma}} \le C \langle t \rangle^{\beta},$$

since we can choose $s_1 > s_2 = \frac{\sigma}{2\sigma-1}$ so that $\beta = 1 - \frac{1}{s_1} > \frac{1}{\sigma} - 1$. Thus Lemma 2.45 is proved.

The next lemma will be used to improve the decay estimate of Lemma 2.45 for the case of $\sigma \le 1$.

Lemma 2.46. *Let* $n = 2$, $\sigma \in \left(\frac{3}{4}, 1\right]$. *Suppose that the initial data* $u_0 \in \mathbf{W}^2_{\infty}(\mathbf{R}^2) \cap \mathbf{W}^2_1(\mathbf{R}^2)$. *Let the estimates for the solution be valid*

$$\|u(t)\|_{\mathbf{L}^p} \le C \langle t \rangle^{\alpha-1+\frac{1}{p}}$$

for all $t > 0$, *where* $1 \le p \le \sigma+2$, $\alpha \in \left[0, \frac{1}{2}\right)$. *Then the estimate is true*

$$\int_0^t \left\| |u(\tau)|^{\sigma} \Delta u(\tau) \right\|_{\mathbf{L}^1} d\tau \le C \langle t \rangle^{\beta}$$

for all $t > 0$, *where* $\beta = 0$ *if* $\max\left(\sigma\alpha + \frac{1}{\sigma+2} - \sigma, \alpha(1+2\sigma) + 1 - 2\sigma\right) < 0$ *and* $\beta > \max\left(\sigma\alpha + \frac{1}{\sigma+2} - \sigma, \alpha(1+2\sigma) + 1 - 2\sigma\right)$ *otherwise.*

Proof. By estimate (2.96) we have

$$\begin{aligned}
\|\Delta u(t)\|_{\mathbf{L}^{\frac{\sigma+2}{\sigma+1}}} &\le C + \int_0^t \|\Delta \mathcal{G}(t-\tau)\mathcal{B}|u|^{\sigma} u(\tau)\|_{\mathbf{L}^{\frac{\sigma+2}{\sigma+1}}} d\tau \\
&\le C\int_0^t \langle t-\tau\rangle^{-1} \|u(\tau)\|_{\mathbf{L}^{\sigma+2}}^{\sigma+1} d\tau \\
&\le C\int_0^t \langle t-\tau\rangle^{-1} \langle \tau\rangle^{(\sigma+1)\left(\alpha-\frac{\sigma+1}{\sigma+2}\right)} d\tau \le C\langle t\rangle^{\delta}
\end{aligned}$$

for all $t>0$, where $\delta > \max\left(-1, (\sigma+1)\left(\alpha - \frac{\sigma+1}{\sigma+2}\right)\right)$. We then have

$$\begin{aligned}
\int_0^t \||u(\tau)|^{\sigma}\Delta u(\tau)\|_{\mathbf{L}^1} d\tau &\le \int_0^t \|u(\tau)\|_{\mathbf{L}^{\sigma(\sigma+2)}}^{\sigma} \|\Delta u(\tau)\|_{\mathbf{L}^{\frac{\sigma+2}{\sigma+1}}} d\tau \\
&\le \int_0^t \langle\tau\rangle^{\sigma(\alpha-1)+\frac{1}{\sigma+2}} \langle\tau\rangle^{\delta} d\tau \le C\langle t\rangle^{\beta}
\end{aligned}$$

where

$$\beta = 0 \text{ if } \max\left(\sigma\alpha + \frac{1}{\sigma+2} - \sigma, \alpha(1+2\sigma)+1-2\sigma\right) < 0$$

and

$$\beta > \max\left(\sigma\alpha + \frac{1}{\sigma+2} - \sigma, \alpha(1+2\sigma)+1-2\sigma\right)$$

otherwise. Lemma 2.46 is proved.

Now we estimate the decay rate of the $\mathbf{L}^p$ norms of the solutions.

Lemma 2.47. *Let $u_0 \in \mathbf{H}^2(\mathbf{R}^n) \cap \mathbf{L}^1(\mathbf{R}^n)$. Assume that*

$$\int_0^t d\tau \||u(\tau)|^{\sigma}\Delta u(\tau)\|_{\mathbf{L}^1} \le C\langle t\rangle^{\beta} \tag{2.97}$$

for all $t>0$, where $\beta \in \left[0, \frac{n}{4}\right)$. Then the estimate

$$\|u(t)\|_{\mathbf{L}^p} \le C\langle t\rangle^{\beta - \frac{n}{2}\left(1-\frac{1}{p}\right)}$$

is valid for all $t>0$, where $1 \le p \le 2+\sigma$.

Proof. We change $w = (1-\Delta)u$, then we get from (2.82)

$$\partial_t w = \alpha(\mathcal{B}-1)w + \lambda|\mathcal{B}w|^{\sigma} w + \lambda|\mathcal{B}w|^{\sigma}\Delta\mathcal{B}w, \tag{2.98}$$

where $\mathcal{B} = (1-\Delta)^{-1}$. We estimate the $\mathbf{L}^1$ norm. We multiply equation (2.98) by $S(t,x) = \text{sign}(w(t,x))$ and integrate with respect to x over $\mathbf{R}^n$ to get

$$\int_{\mathbf{R}^n} \partial_t w(t,x) S(t,x) dx = \alpha \int_{\mathbf{R}^n} S(t,x) (\mathcal{B}-1) w dx$$
$$+ \lambda \int_{\mathbf{R}^n} |\mathcal{B} w(t,x)|^{\sigma} |w(t,x)| dx + \lambda \int_{\mathbf{R}^n} S(t,x) |\mathcal{B} w(t,x)|^{\sigma} \Delta \mathcal{B} w(t,x) dx.$$

We have

$$\int_{\mathbf{R}^n} w_t(t,x) S(t,x) dx = \int_{\mathbf{R}^n} \frac{\partial}{\partial t} |w(t,x)| dx = \frac{d}{dt} \|w(t)\|_{\mathbf{L}^1},$$
$$\lambda \int_{\mathbf{R}^n} |\mathcal{B} w(t,x)|^{\sigma} |w(t,x)| dx \leq 0,$$
$$\int_{\mathbf{R}^n} S(t,x) \mathcal{B} w dx \leq \int_{\mathbf{R}^n} \mathcal{B} |w| dx = \|w(t)\|_{\mathbf{L}^1}.$$

Therefore, we find

$$\frac{d}{dt} \|w(t)\|_{\mathbf{L}^1} \leq |\lambda| \left\| |u(t)|^{\sigma} \Delta u(t) \right\|_{\mathbf{L}^1}. \tag{2.99}$$

Integration of inequality (2.99) in view of estimate (2.97) yields

$$\|w(t)\|_{\mathbf{L}^1} \leq \|w_0\|_{\mathbf{L}^1} + C \langle t \rangle^{\beta}.$$

In particular, we find

$$\sup_{\xi \in \mathbf{R}} |\widehat{u}(t,\xi)| \leq (2\pi)^{-\frac{n}{2}} \|\mathcal{B} w(t)\|_{\mathbf{L}^1} \leq C \|w(t)\|_{\mathbf{L}^1} \leq C \langle t \rangle^{\beta}. \tag{2.100}$$

Thus the estimate of the lemma with $p = 1$ is fulfilled.

We now multiply equation (2.82) by $2u$, then integrating with respect to $x \in \mathbf{R}^n$ we get

$$\frac{d}{dt} \left(\|u(t)\|_{\mathbf{L}^2}^2 + \|\nabla u(t)\|_{\mathbf{L}^2}^2 \right) = -2\alpha \|\nabla u(t)\|_{\mathbf{L}^2}^2 + 2\lambda \|u(t)\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2}. \tag{2.101}$$

By the Plancherel Theorem using the Fourier splitting method from Schonbek [1991], we have

$$\|\nabla u(t)\|_{\mathbf{L}^2}^2 = \left\| |\xi| \widehat{u}(t) \right\|_{\mathbf{L}^2}^2 = \int_{|\xi| \leq \delta} |\widehat{u}(t,\xi)|^2 |\xi|^2 d\xi + \int_{|\xi| \geq \delta} |\widehat{u}(t,\xi)|^2 |\xi|^2 d\xi$$
$$\geq \delta^2 \|u(t)\|_{\mathbf{L}^2}^2 - 2\delta^{2+n} \sup_{|\xi| \leq \delta} |\widehat{u}(t,\xi)|^2,$$

where $\delta > 0$. Thus from (2.101) we have the inequality

$$\frac{d}{dt} \|u(t)\|_{\mathbf{H}^1}^2 \leq -\alpha\delta^2 \|u(t)\|_{\mathbf{H}^1}^2 + 4\alpha\delta^{2+n} \sup_{|\xi| \leq \delta} |\widehat{u}(t,\xi)|^2. \tag{2.102}$$

We choose $\alpha\delta^2 = (1+n)(t_0+t)^{-1}, t_0 = \sqrt{\frac{1+n}{\alpha}}$ and change

$$\|u(t)\|_{\mathbf{H}^1}^2 = (t_0 + t)^{-1-n} W(t).$$

Then via (2.100) we get from (2.102)

$$\frac{d}{dt} W(t) \le C (t_0 + t)^{2\beta + \frac{n}{2}}. \tag{2.103}$$

Integration of (2.103) with respect to time yields

$$W(t) \le \|u_0\|_{\mathbf{H}^1}^2 + C\left((t_0 + t)^{\frac{n}{2}+1+2\beta} - 1\right).$$

Therefore we obtain a time decay estimate of the $\mathbf{L}^2$ norm

$$\|u(t)\|_{\mathbf{L}^2} \le C (1 + t)^{\beta - \frac{n}{4}} \tag{2.104}$$

for all $t > 0$. Now we differentiate equation (2.82) and multiply the result by $2\nabla u$, then by the same considerations as above we obtain the optimal time decay estimate (see also the proof of Lemma 2.41 for details)

$$\|\nabla u(t)\|_{\mathbf{L}^2} \le C (1 + t)^{\beta - \frac{n}{4} - \frac{1}{2}}$$

for all $t > 0$. Then by the Sobolev imbedding theorem and by the Hölder inequality we arrive at the optimal time decay estimate of the lemma. Lemma 2.47 is proved.

Proposition 2.48. *Let $\sigma > 1$ for $n = 1$ and $\sigma > \frac{3}{4}$ for $n = 2$. Suppose that the initial data $u_0 \in \mathbf{W}_\infty^2(\mathbf{R}^n) \cap \mathbf{W}_1^2(\mathbf{R}^n)$. Then the estimates for the solution are valid*

$$\|u(t)\|_{\mathbf{L}^p} \le C \langle t \rangle^{-\frac{n}{2}\left(1 - \frac{1}{p}\right)}$$

for all $t > 0$, where $1 \le p \le \sigma + 2$.

Proof. Multiplying equation (2.82) by $2u$ and integrating with respect to $x \in \mathbf{R}^n$ we get

$$\frac{d}{dt}\left(\|u(t)\|_{\mathbf{L}^2}^2 + \|\nabla u(t)\|_{\mathbf{L}^2}^2\right) + 2\alpha \|\nabla u(t)\|_{\mathbf{L}^2}^2 = 2\lambda \|u(t)\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2};$$

hence by integrating we see that

$$\begin{aligned} &\|u(t)\|_{\mathbf{L}^2}^2 + \|\nabla u(t)\|_{\mathbf{L}^2}^2 + 2\alpha \int_0^t \|\nabla u(\tau)\|_{\mathbf{L}^2}^2 d\tau - 2\lambda \int_0^t \|u(\tau)\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2} d\tau \\ &\le \|u_0\|_{\mathbf{L}^2}^2 + \|\nabla u_0\|_{\mathbf{L}^2}^2 = \|u_0\|_{\mathbf{H}^1}^2 \end{aligned}$$

for all $t \ge 0$. In particular we have

$$\|u\|_{\infty,2} \equiv \sup_{t \ge 0} \|u(t)\|_{\mathbf{L}^2} \le \|u_0\|_{\mathbf{H}^1}, \tag{2.105}$$

$$\|u\|_{\sigma+2,\sigma+2} \equiv \left\| \|u(t,x)\|_{\mathbf{L}_x^{\sigma+2}} \right\|_{\mathbf{L}_t^{\sigma+2}(0,\infty)} \le C \|u_0\|_{\mathbf{H}^1}. \tag{2.106}$$

Now applying estimates (2.105) and (2.106) by Lemma 2.45 we get

$$\int_0^t d\tau \, \||u(\tau)|^{\sigma} \Delta u(\tau)\|_{\mathbf{L}^1} \le C$$

for all $t > 0$, if $\sigma > 1$. Then from Lemma 2.47 the result of the lemma follows for the case $\sigma > 1$. Now we consider the case of $\sigma \in \left(\frac{3}{4}, 1\right]$ and $n = 2$. By Lemma 2.45 we have

$$\int_0^t d\tau \, \||u(\tau)|^{\sigma} \Delta u(\tau)\|_{\mathbf{L}^1} \le C \langle t \rangle^{\beta_1} \tag{2.107}$$

for all $t > 0$, where $\beta_1 > \frac{1}{\sigma} - 1$. By Lemma 2.47 we then find the time decay

$$\|u(t)\|_{\mathbf{L}^p} \le C \langle t \rangle^{\beta_1 - \frac{n}{2}\left(1 - \frac{1}{p}\right)}$$

for all $t > 0$, where $1 \le p \le \sigma + 2$. Now by Lemma 2.46 we obtain

$$\int_0^t \||u(\tau)|^{\sigma} \Delta u(\tau)\|_{\mathbf{L}^1} d\tau \le C \langle t \rangle^{\beta_2}$$

for all $t > 0$, where $\beta_2 = 0$ if $\sigma \in \left(\frac{1}{4}\left(1 + \sqrt{5}\right), 1\right]$, and $\beta_2 > \frac{1}{\sigma} - 4\sigma + 2$ otherwise.

We again apply Lemma 2.47 to get

$$\|u(t)\|_{\mathbf{L}^p} \le C \langle t \rangle^{\beta_2 - \frac{n}{2}\left(1 - \frac{1}{p}\right)}$$

for all $t > 0$, where $1 \le p \le \sigma + 2$. Now by Lemma 2.46 we obtain

$$\int_0^t \||u(\tau)|^{\sigma} \Delta u(\tau)\|_{\mathbf{L}^1} d\tau \le C \langle t \rangle^{\beta_3}$$

for all $t > 0$, where $\beta_3 = 0$ if $\sigma > 0.775$ and $\beta_3 > \beta_2(1 + 2\sigma) + 1 - 2\sigma$ otherwise. We repeat these considerations to get

$$\int_0^t \||u(\tau)|^{\sigma} \Delta u(\tau)\|_{\mathbf{L}^1} d\tau \le C$$

for all $t > 0$, if $\sigma > \frac{3}{4}$. Therefore by virtue of Lemma 2.47 we obtain time decay estimates

$$\|u(t)\|_{\mathbf{L}^p} \le C \langle t \rangle^{-\frac{n}{2}\left(1 - \frac{1}{p}\right)}$$

for all $t > 0$, where $1 \le p \le \sigma + 2$, for $\sigma \in \left(\frac{3}{4}, 1\right]$, $n = 2$. Proposition 2.48 is proved.

Proof of Theorem 2.44

Using the result of Proposition 2.48 we can prove the following optimal time decay estimates

$$\|u(t)\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)} \tag{2.108}$$

for all $t>0$, where $1\le p\le\infty$. Let us prove (2.108) for $p=\infty$. We consider case $t\ge T>0$, since for $t\in[0,T]$ we have the estimate (2.108) from Proposition 2.48. Due to Lemma 1.31 with $p=\infty$ and $r=\frac{\sigma+2}{\sigma+1}$ and by using (2.85) and (2.95), we have

$$\begin{aligned}
&\|\mathcal{G}(t-\tau)\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^\infty}\\
&\le Ce^{-at}\|\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^\infty}+C(t-\tau)^{-\frac{n(\sigma+1)}{2(\sigma+2)}}\|\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^{\frac{\sigma+2}{\sigma+1}}}\\
&\le C(t-\tau)^{-\frac{n(\sigma+1)}{2(\sigma+2)}}\|u(\tau)\|_{\mathbf{L}^{\sigma+2}}^{\sigma+1}
\end{aligned}$$

for $\frac{t}{2}\le\tau\le t$. In interval $0\le\tau\le\frac{t}{2}$ from Lemma 1.31 with $p=\infty$ and $r=1$ we get

$$\begin{aligned}
&\|\mathcal{G}(t-\tau)\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^\infty}\\
&\le Ce^{-at}\|\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^\infty}+C(t-\tau)^{-\frac{n}{2}}\|\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^1}\\
&\le C(t-\tau)^{-\frac{n}{2}}\left(\|u(\tau)\|_{\mathbf{L}^{\sigma+2}}^{\sigma+1}+\|u(\tau)\|_{\mathbf{L}^{\sigma+1}}^{\sigma+1}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|u(t)\|_{\mathbf{L}^\infty}\le\|\mathcal{G}(t)u_0\|_{\mathbf{L}^\infty}+C\int_0^t\|\mathcal{G}(t-\tau)\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^\infty}d\tau\\
&\le\|\mathcal{G}(t)u_0\|_{\mathbf{L}^\infty}+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{n(\sigma+1)}{2(\sigma+2)}}\|u(\tau)\|_{\mathbf{L}^{\sigma+2}}^{\sigma+1}d\tau\\
&+C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{n}{2}}\left(\|u(\tau)\|_{\mathbf{L}^{\sigma+2}}^{\sigma+1}+\|u(\tau)\|_{\mathbf{L}^{\sigma+1}}^{\sigma+1}\right)d\tau\\
&\le Ct^{-\frac{n}{2}}+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{n(\sigma+1)}{2(\sigma+2)}}\langle\tau\rangle^{-\frac{n(\sigma+1)^2}{2(\sigma+2)}}d\tau\\
&+C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{n}{2}}\langle\tau\rangle^{-\frac{n}{2}\sigma}d\tau\le Ct^{-\frac{n}{2}}
\end{aligned}$$

for all $t>0$, since $\frac{n}{2}\sigma>1$. Now estimate (2.108) for all $1\le p\le\infty$ follows via the Hölder inequality.

We now estimate the $\mathbf{L}^{1,a}$ norm of the solution. By Lemma 1.31 and by using (2.85) and (2.95), we find

$$\begin{aligned}
&\|\mathcal{G}(t-\tau)\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^{1,a}}\\
&\le C\langle t-\tau\rangle^{\frac{a}{2}}\|\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^1}+C\|\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^{1,a}}\\
&\le C\langle t-\tau\rangle^{\frac{a}{2}}\|u(\tau)\|_{\mathbf{L}^\infty}^{\sigma}\|u(\tau)\|_{\mathbf{L}^1}+C\|u(\tau)\|_{\mathbf{L}^\infty}^{\sigma}\|u(\tau)\|_{\mathbf{L}^{1,a}};
\end{aligned}$$

therefore, we have

$$\begin{aligned}
&\|u(t)\|_{\mathbf{L}^{1,a}} \le \|\mathcal{G}(t)\,u_0\|_{\mathbf{L}^{1,a}} \\
&+ C\int_0^t \langle t-\tau\rangle^{\frac{a}{2}} \langle\tau\rangle^{-\frac{n}{2}\sigma}\,d\tau + C\int_0^t \langle\tau\rangle^{-\frac{n}{2}\sigma}\|u(\tau)\|_{\mathbf{L}^{1,a}}\,d\tau \\
&\le C\langle t\rangle^{\frac{a}{2}} + C\int_0^t \langle\tau\rangle^{-\frac{n}{2}\sigma}\|u(\tau)\|_{\mathbf{L}^{1,a}}\,d\tau.
\end{aligned}$$

Hence by the Gronwall's lemma we obtain

$$\|u(t)\|_{\mathbf{L}^{1,a}} \le C\langle t\rangle^{\frac{a}{2}}$$

for all $t > 0$. Therefore again we arrive at the optimal time decay estimates (2.92). Using estimates (2.92) via Theorem 1.20, we see that there exists a unique solution $u \in \mathbf{C}\left([0,\infty);\mathbf{L}^{1,a}(\mathbf{R}^n)\cap\mathbf{L}^{\infty}(\mathbf{R}^n)\right)$ to the Cauchy problem (2.82). Thus from Theorem 2.4 we obtain the asymptotics (2.91). Theorem 2.44 is proved.

2.5 Whitham type equation

2.5.1 A model equation

In this section we study the Cauchy problem for the nonlinear nonlocal evolution equations

$$\begin{cases} u_t + \mathcal{N}(u) + \mathcal{L}u = 0, \; x\in\mathbf{R}, t>0, \\ u(0,x) = u_0(x), \; x\in\mathbf{R}, \end{cases} \tag{2.109}$$

where the linear part $\mathcal{L}u = \overline{\mathcal{F}}_{\xi\to x}L(\xi)\,\hat{u}(t,\xi)$ and the nonlinearity $\mathcal{N}(u)$ are pseudodifferential operators defined by the Fourier transformations as follows

$$\begin{aligned}
\mathcal{N}(u) &= \overline{\mathcal{F}}_{\xi\to x}\int_{\mathbf{R}} a_1(t,\xi,y_1)\,\widehat{u}(t,\xi-y_1)\,\widehat{u}(t,y_1)\,dy_1 \\
&+ \overline{\mathcal{F}}_{\xi\to x}\int_{\mathbf{R}^2} a_2(t,\xi,y_1,y_2)\,\widehat{u}(t,\xi-y_1-y_2)\,\widehat{u}(t,y_1)\,\widehat{u}(t,y_2)\,dy_1dy_2.
\end{aligned}$$

We suppose that the symbols $a_1(t,\xi,y)$ and $a_2(t,\xi,y_1,y_2)$ are continuous functions with respect to time $t>0$ and the operators $\mathcal{N}$ and $\mathcal{L}$ have a finite order, that is the symbols a_1, a_2 and L grow no faster than a power

$$|a_1(t,\xi,y_1)| + |a_2(t,\xi,y_1,y_2)| + |L(\xi)| \le C\langle\xi\rangle^{\kappa} + C\langle y_1\rangle^{\kappa} + C\langle y_2\rangle^{\kappa}$$

with $C > 0$.

Model equation (2.109) combines many well-known equations of modern mathematical physics and describes various wave processes in different media.

For example, if the solution $u(t,x)$ is a real-valued function and the nonlinearity has the form $\mathcal{N}(u) = auu_x$, that is $a_1 = \frac{i}{2}a\xi$, $a_2 = 0$, $a \in \mathbf{R}$, then we obtain the Whitham equation Whitham [1999]

$$u_t + auu_x + \mathcal{L}u = 0, \tag{2.110}$$

which contains a number of famous nonlinear equations in the theory of water waves. If we take the nonlinearity $\mathcal{N}(u) = a|u|^2 u_x + c\left(|u|^2 u\right)_x$ in equation (2.109), that is we can also define the nonlinearity taking the complex conjugation

$$\mathcal{N}(u) = \overline{\mathcal{F}}_{\xi \to x} \int_{\mathbf{R}^2} a_2(t,\xi,y_1,y_2)\, \widehat{u}(t,\xi - y_1 - y_2)\, \widehat{u}(t,y_1)\, \overline{\widehat{u}(t,-y_2)} dy_1 dy_2$$

with $a_2(t,\xi,y_1,y_2) = iay_1 + ic\xi$, $a, c \in \mathbf{C}$, then we obtain the derivative nonlinear Schrödinger equation with dissipation

$$u_t + a|u|^2 u_x + c\left(|u|^2 u\right)_x - \mu u_{xx} - iu_{xx} = 0, \tag{2.111}$$

where $\mu > 0$. Finally note that under the condition $\int_{\mathbf{R}} u(t,y)\, dy = 0$ for all $t \geq 0$ we can introduce a potential $\varphi(t,x) = \int_{-\infty}^{x} u(t,y)\, dy$, which is also a decaying function with respect to the space variable x. Then we get the potential Whitham equation

$$\varphi_t + \frac{a}{2}(\varphi_x)^2 + \mathcal{L}\varphi = 0,$$

which also follows from (2.109) if we take $a_1 = -\frac{a}{2}y_0 y_1$, $a_2 = 0$, $y_0 = \xi - y_1$ with $a \in \mathbf{R}$. Some other nonlinear nonlocal equations appearing in the theory of waves can be found in book Naumkin and Shishmarev [1994b].

2.5.2 Local existence and smoothing effect

We suppose the linear operator $\mathcal{L}$ is strongly dissipative, that is

$$\operatorname{Re} L(\xi) \geq \mu |\xi|^{\nu}, \quad |L'(\xi)| \leq C|\xi|^{\nu} \tag{2.112}$$

for all $|\xi| \geq 1$, where $\mu > 0$, $\nu \geq 0$. Suppose that the symbols of the nonlinearity $\mathcal{N}$ are such that

$$\sum_{l=0}^{1} \left|\partial_{\xi}^{l} a_n(t,\xi,\mathbf{y})\right| \leq C\langle\xi\rangle^{\sigma} \tag{2.113}$$

for all $\xi \in \mathbf{R}$, $\mathbf{y} \in \mathbf{R}^n$, $n = 1, 2$, $t > 0$, where $\sigma \in [0,\nu]$. We denote $\vartheta = 2$ if the nonlinear operator $\mathcal{N}$ is quadratic: $a_2(t,\xi,\mathbf{y}) \equiv 0$ and $\vartheta = 3$ otherwise. Denote

$$S = \frac{1}{2} - \min\left(\frac{\nu - \sigma}{\vartheta - 1}, \frac{2\nu + 1}{2\vartheta}\right).$$

Theorem 2.49. *Let the linear operator $\mathcal{L}$ satisfy the dissipation condition (2.112) with $\nu \geq 0$ and the nonlinear operator $\mathcal{N}$ satisfy estimates (2.113) with $\sigma \in [0, \nu]$. Suppose that the initial data $u_0 \in \mathbf{H}^s(\mathbf{R}) \cap \mathbf{H}^{\lambda,\omega}(\mathbf{R})$, where $s > S$, $\omega \in [0,1)$, $\lambda = \min(s, -\sigma)$. If $\sigma = \nu > 0$ we assume additionally that the norm $\|u_0\|_{\mathbf{H}^{s,0}}$ is small. Then for some time $T > 0$ there exists a unique solution $u(t,x)$ to the Cauchy problem (2.109) such that $u(t,x) \in \mathbf{C}^0\left([0,T]; \mathbf{H}^s(\mathbf{R}) \cap \mathbf{H}^{\lambda,\omega}(\mathbf{R})\right)$. In the case of $\nu > 0$ we also have $u(t,x) \in \mathbf{C}^1((0,T]; \mathbf{H}^\infty(\mathbf{R}))$.*

For example, let us apply Theorem 2.49 to the Ott - Sudan - Ostrovsky equation Ostrovsky [1976], Ott and Sudan [1969]

$$u_t + auu_x + \mu \mathcal{H} u_x + u_{xxx} = 0;$$

here the symbol $L(\xi) = \mu|\xi| - i\xi^3$, $\mu > 0$, and $\mathcal{H}u = \frac{1}{\pi}\mathrm{PV}\int_{\mathbf{R}} \frac{u(y)}{x-y} dy$ is the Hilbert transformation. We have $\sigma = 1 = \nu$. Therefore if $u_0 \in \mathbf{H}^s(\mathbf{R})$, $s > \frac{1}{2}$, then for some time $T > 0$ there exists a unique solution $u(t,x) \in \mathbf{C}^\infty((0,T]; \mathbf{H}^\infty(\mathbf{R}))$.

Before proving Theorem 2.49 we consider some estimates.

Preliminary lemmas

We define the Green operator $\mathcal{G}$ for the linear Cauchy problem

$$\begin{cases} u_t + \mathcal{L}u = f, \ t > 0, x \in \mathbf{R}, \\ u(0,x) = u_0, \ x \in \mathbf{R}. \end{cases} \tag{2.114}$$

Using the Fourier transformation we can formally represent the Green operator as

$$\mathcal{G}(t)\psi = \overline{\mathcal{F}}_{\xi \to x}\left(\hat{\psi}(\xi) e^{-L(\xi)t}\right) = \int_{\mathbf{R}} G(t, x-y)\psi(y)\, dy,$$

where the kernel $G(t,x) = \overline{\mathcal{F}}_{\xi \to x} e^{-L(\xi)t}$. Therefore the solution of problem (2.114) can be written by the Duhamel's integral

$$u(t) = \mathcal{G}(t) u_0 + \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau.$$

In the following lemma we prove the smoothing property for the Green operator $\mathcal{G}(t)$. Let us denote the fractional derivative of order $\omega \in (0,1)$ as follows

$$|\partial_x|^\omega \phi(x) \equiv \int_{\mathbf{R}} |\phi(x-y) - \phi(x)| \, |y|^{-1-\omega}\, dy.$$

To prove the local existence we define two norms

$$\|\varphi\|_{\mathbf{X}^{s,\omega}} = \sup_{1\le p\le\infty} \left\| (1+|\partial_\xi|^\omega) \langle\xi\rangle^{s+\frac{\nu}{p}} E(t,\xi) \widehat{\varphi}(\xi) \right\|_{\mathbf{L}^2_\xi \mathbf{L}^p_t},$$
$$\|\varphi\|_{\mathbf{Y}^{s,\omega}} = \|(1+|\partial_\xi|^\omega) \langle\xi\rangle^{s} E(t,\xi) \widehat{\varphi}(\xi)\|_{\mathbf{L}^2_\xi \mathbf{L}^1_t},$$

where $E(t,\xi) = e^{\frac{\mu}{2}t\langle\xi\rangle^{\nu_1}}$, $\nu_1 = \min(1,\nu)$, $\omega \in [0,1)$, and $s \in \mathbf{R}$. We denote $\mathbf{L}^q_x \mathbf{L}^p_t \equiv \mathbf{L}^q(\mathbf{R}; \mathbf{L}^p(0,T))$, where $T > 0$, $1 \le p, q \le \infty$.

Lemma 2.50. *Let the linear operator $\mathcal{L}$ satisfy dissipation condition (2.112) with $\nu \ge 0$. Then the following estimates*

$$\|\mathcal{G}(t)\phi\|_{\mathbf{X}^{s,\omega}} \le C\|\phi\|_{\mathbf{H}^{s,\omega}} \quad \text{and} \quad \left\| \int_0^t \mathcal{G}(t-\tau)\phi(\tau)\,d\tau \right\|_{\mathbf{X}^{s,\omega}} \le C\|\phi\|_{\mathbf{Y}^{s,\omega}}$$

are valid for any $\omega \in [0,1)$, $s \in \mathbf{R}$, provided that the right-hand sides are bounded.

Proof. By virtue of dissipation condition (2.112) we have

$$\sup_{1\le p\le\infty} \left\| \partial_\xi^l \langle\xi\rangle^{\frac{\nu}{p}} e^{-L(\xi)t} E(t,\xi) \right\|_{\mathbf{L}^\infty \mathbf{L}^p} \le C \sup_{1\le p\le\infty} \left\| \langle\xi\rangle^{\frac{\nu}{p}} e^{-\frac{\mu}{2}t|\xi|^\nu} \right\|_{\mathbf{L}^\infty \mathbf{L}^p} \le C, \tag{2.115}$$

where $l = 0, 1$. We have the analogue of the Leibnitz rule

$$|\partial_x|^\omega (\phi\psi) \le \psi |\partial_x|^\omega \phi + [|\partial_x|^\omega, \psi]\phi,$$

where we define the commutator

$$[|\partial_x|^\omega, \psi]\phi = \int |\psi(x-y) - \psi(x)| |\phi(x-y)| |y|^{-1-\omega} dy.$$

Since

$$\|\varphi(\cdot - y) - \varphi(\cdot)\|_{\mathbf{L}^q} \le C|y| \|\partial_x\varphi\|_{\mathbf{L}^q}$$

we have estimate of the Besov norm

$$\begin{aligned} \|\varphi\|_{\mathbf{B}^{\omega,q}} &\le \int_{|y|\le 1} |y|^{-\omega} \|\partial_x\varphi\|_{\mathbf{L}^q} dy + 2\|\varphi\|_{\mathbf{L}^q} \int_{|y|>1} |y|^{-1-\omega} dy \\ &\le C\|\varphi\|_{\mathbf{L}^q} + C\|\partial_x\varphi\|_{\mathbf{L}^q}. \end{aligned}$$

By the Hölder inequality we get

$$\begin{aligned} &\|[|\partial_\xi|^\omega, \psi(t,\xi)]\phi(t,\xi)\|_{\mathbf{L}^2_\xi \mathbf{L}^p_t} \\ &\le \left\| \int d\eta |\eta|^{-1-\omega} |\phi(t,\xi-\eta)| |\psi(t,\xi-\eta) - \psi(t,\xi)| \right\|_{\mathbf{L}^2_\xi \mathbf{L}^p_t} \\ &\le \|\psi\|_{\mathbf{B}^{\omega,\infty}_\xi \mathbf{L}^p_t} \|\phi\|_{\mathbf{L}^2_\xi \mathbf{L}^\infty_t} \le C\left(\|\psi\|_{\mathbf{L}^\infty_\xi \mathbf{L}^p_t} + \|\partial_x\psi\|_{\mathbf{L}^\infty_\xi \mathbf{L}^p_t} \right) \|\phi\|_{\mathbf{L}^2_\xi \mathbf{L}^\infty_t}. \end{aligned}$$

Therefore via the Leibnitz rule in view of (2.115) we find

$$\begin{aligned}
&\|\mathcal{G}(t)\phi\|_{\mathbf{X}^{s,\omega}} \le C \sup_{1\le p\le\infty} \left\| \langle\xi\rangle^{\frac{\nu}{p}} e^{-L(\xi)t} E(t,\xi)\left(1+|\partial_\xi|^\omega\right)\langle\xi\rangle^s \hat{\phi}(\xi) \right\|_{\mathbf{L}^2_\xi \mathbf{L}^p_t} \\
&+ C \sup_{1\le p\le\infty} \left\| \left[|\partial_\xi|^\omega, \langle\xi\rangle^{\frac{\nu}{p}} e^{-L(\xi)t} E(t,\xi) \right] \langle\xi\rangle^s \hat{\phi}(\xi) \right\|_{\mathbf{L}^2_\xi \mathbf{L}^p_t} \\
&\le C \sup_{1\le p\le\infty} \left\| \langle\xi\rangle^{\frac{\nu}{p}} e^{-L(\xi)t} E(t,\xi) \right\|_{\mathbf{L}^\infty_\xi \mathbf{L}^p_t} \|\phi\|_{\mathbf{H}^{s,\omega}} \\
&+ C \sum_{l=0}^{1} \sup_{1\le p\le\infty} \left\| \partial_\xi^l \langle\xi\rangle^{\frac{\nu}{p}} e^{-L(\xi)t} E(t,\xi) \right\|_{\mathbf{L}^\infty_\xi \mathbf{L}^p_t} \|\phi\|_{\mathbf{H}^{s,0}} \le C \|\phi\|_{\mathbf{H}^{s,\omega}}.
\end{aligned}$$

The first estimate of the lemma is proved. Using the identity

$$E(t,\xi) = E(t-\tau,\xi)\,E(\tau,\xi),$$

and denoting $\phi(\tau,\xi) = \langle\xi\rangle^s E(\tau,\xi)\hat{\psi}(\tau,\xi)$ we obtain

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau \right\|_{\mathbf{X}^{s,\omega}} \\
&\le C \sup_{1\le p\le\infty} \left\| \int_0^t d\tau \langle\xi\rangle^{\frac{\nu}{p}} E(t-\tau,\xi) e^{-L(\xi)(t-\tau)} \left(1+|\partial_\xi|^\omega\right)\phi(\tau,\xi) \right\|_{\mathbf{L}^2_\xi \mathbf{L}^p_t} \\
&+ C \sup_{1\le p\le\infty} \left\| \int_0^t d\tau \left[|\partial_\xi|^\omega, \langle\xi\rangle^{\frac{\nu}{p}} E(t-\tau,\xi) e^{-L(\xi)(t-\tau)} \right] \phi(\tau,\xi) \right\|_{\mathbf{L}^2_\xi \mathbf{L}^p_t}.
\end{aligned}$$

Via the Young inequality we obtain

$$\begin{aligned}
&\left\| \int_0^t d\tau \left[|\partial_\xi|^\omega, \psi(t-\tau,\xi) \right] \phi(\tau,\xi) \right\|_{\mathbf{L}^2_\xi \mathbf{L}^p_t} \\
&\le \left\| \int_0^t d\tau \int \frac{d\eta}{|\eta|^{1+\omega}} \left| \psi(t-\tau,\xi-\eta) - \psi(t-\tau,\xi) \right| \left| \phi(\tau,\xi-\eta) \right| \right\|_{\mathbf{L}^2_\xi \mathbf{L}^p_t} \\
&\le C \|\psi(t,\xi)\|_{\mathbf{B}^{\omega,\infty}_\xi \mathbf{L}^p_t} \|\phi(t,\xi)\|_{\mathbf{L}^2_\xi \mathbf{L}^1_t} \le C \sum_{l=0}^{1} \left\| \partial_\xi^l \psi(t,\xi) \right\|_{\mathbf{L}^\infty_\xi \mathbf{L}^p_t} \|\phi\|_{\mathbf{L}^2_\xi \mathbf{L}^1_t};
\end{aligned}$$

hence via (2.115), denoting $\phi(t,\xi) = \langle\xi\rangle^s E(t,\xi)\hat{\psi}(t,\xi)$ we find

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau \right\|_{\mathbf{X}^{s,\omega}} \le C \sup_{1\le p\le\infty} \left\| \langle\xi\rangle^{\frac{\nu}{p}} E(t,\xi) e^{-L(\xi)t} \right\|_{\mathbf{L}^\infty_\xi \mathbf{L}^p_t} \\
&\times \left\| \left(1+|\partial_\xi|^\omega\right)\phi(t,\xi) \right\|_{\mathbf{L}^2_\xi \mathbf{L}^1_t} \\
&+ C \sum_{l=0}^{1} \sup_{1\le p\le\infty} \left\| \partial_\xi^l \langle\xi\rangle^{\frac{\nu}{p}} E(t,\xi) e^{-L(\xi)t} \right\|_{\mathbf{L}^\infty_\xi \mathbf{L}^p_t} \|\phi(t,\xi)\|_{\mathbf{L}^2_\xi \mathbf{L}^1_t} \le C \|\psi\|_{\mathbf{Y}^{s,\omega}}.
\end{aligned}$$

Thus the second estimate of the lemma is true, and Lemma 2.50 is proved.

Now we estimate the nonlinearity in the norms $\mathbf{Y}^{s,\omega}$. Denote

$$\mathcal{N}(\phi_0,\phi_1,\phi_2)=\overline{\mathcal{F}}_{\xi\to x}\int_{\mathbf{R}} a_1(t,\xi,y_1)\widehat{\phi_0}(t,\xi-y_1)\widehat{\phi_1}(t,y_1)\,dy_1$$
$$+\overline{\mathcal{F}}_{\xi\to x}\int_{\mathbf{R}^2} a_2(t,\xi,y_1,y_2)\widehat{\phi_0}(t,\xi-y_1-y_2)\widehat{\phi_1}(t,y_1)\widehat{\phi_2}(t,y_2)\,dy_1dy_2.$$

As above we denote $\vartheta = 2$ if the nonlinear operator $\mathcal{N}$ is quadratic: $a_2(t,\xi,\mathbf{y})\equiv 0$ and $\vartheta=3$ otherwise.

Lemma 2.51. *Let the symbols a_n of the nonlinear operator $\mathcal{N}(\phi_0,\phi_1,\phi_2)$ satisfy condition (2.113) with $\sigma\in[0,\nu]$, $\nu\geq 0$. Then the inequalities*

$$\|\mathcal{N}(\phi_0,\phi_1,\phi_2)\|_{\mathbf{Y}^{s,0}}\leq CT^{\gamma}\prod_{k=0}^{\vartheta-1}\|\phi_k\|_{\mathbf{X}^{s,0}},$$
$$\|\mathcal{N}(\phi_0,\phi_1,\phi_2)\|_{\mathbf{Y}^{\lambda,\omega}}\leq CT^{\gamma}\sum_{k=0}^{\vartheta-1}\|\phi_k\|_{\mathbf{X}^{\lambda,\omega}}\prod_{j=0,j\neq k}^{\vartheta-1}\|\phi_j\|_{\mathbf{X}^{s,0}}$$

are valid for any functions $\phi_k\in\mathbf{X}^{s,0}\cap\mathbf{X}^{\lambda,\omega}$ with $s>S$, $\lambda=\min(-\sigma,s)$, where $\gamma=1$ if $\nu=0$ and $\gamma=\min\left(1-\frac{\sigma}{\nu},\frac{s-S}{2\nu}\right)$ if $\nu>0$.

Proof. First we consider the case of $\nu=0$ in which we have $\sigma=0$ and $s>S=\frac{1}{2}$; by the Young inequality we then obtain

$$\|\mathcal{N}(\phi_0,\phi_1,\phi_2)\|_{\mathbf{Y}^{s,0}}\leq C\sum_{n=1}^{\vartheta-1}\left\|\langle\xi\rangle^{s}\int_{\mathbf{R}^n}d\mathbf{y}a_n\prod_{k=0}^{n}\widehat{\phi}_k(t,y_k)\right\|_{\mathbf{L}^2_{\xi}\mathbf{L}^1_t}$$
$$\leq CT\sum_{n=1}^{\vartheta-1}\left\|\int_{\mathbf{R}^n}d\mathbf{y}\sum_{j=0}^{n}\langle y_j\rangle^{s}\prod_{k=0}^{n}\widehat{\phi}_k(t,y_k)\right\|_{\mathbf{L}^2_{\xi}\mathbf{L}^{\infty}_t}$$
$$\leq CT\|\langle\xi\rangle^{s}\phi_j(t,\xi)\|_{\mathbf{L}^2_{\xi}\mathbf{L}^{\infty}_t}\prod_{k=0,k\neq j}^{\vartheta-1}\|\phi_k\|_{\mathbf{L}^1_{\xi}\mathbf{L}^{\infty}_t}\leq CT\prod_{k=0}^{\vartheta-1}\|\phi_k\|_{\mathbf{X}^{s,0}}.$$

In addition,

$$\|\mathcal{N}(\phi_0,\phi_1,\phi_2)\|_{\mathbf{Y}^{0,\omega}} \le C\sum_{n=1}^{\vartheta-1}\left\|\int_{\mathbf{R}^n} d\mathbf{y} a_n \langle\partial_\xi\rangle^\omega \prod_{k=0}^{n}\widehat{\phi}_k(t,y_k)\right\|_{\mathbf{L}^2_\xi\mathbf{L}^1_t}$$
$$+C\sum_{n=1}^{\vartheta-1}\left\|\int_{\mathbf{R}^n} d\mathbf{y}\left[|\partial_\xi|^\omega, a_n(t,\xi,\mathbf{y})\right]\prod_{k=0}^{n}\widehat{\phi}_k(t,y_k)\right\|_{\mathbf{L}^2_\xi\mathbf{L}^1_t}$$
$$\le CT\|\langle\partial_\xi\rangle^\omega\phi_0\|_{\mathbf{L}^2_\xi\mathbf{L}^\infty_t}\prod_{j=1}^{\vartheta-1}\|\phi_j\|_{\mathbf{L}^1_\xi\mathbf{L}^\infty_t}+CT\|\phi_0\|_{\mathbf{L}^2_\xi\mathbf{L}^\infty_t}\prod_{j=1}^{\vartheta-1}\|\phi_j\|_{\mathbf{L}^1_\xi\mathbf{L}^\infty_t}$$
$$\le CT\|\phi_0\|_{\mathbf{X}^{0,\omega}}\prod_{j=1}^{\vartheta-1}\|\phi_j\|_{\mathbf{X}^{s,0}},$$

since

$$\left\|\int_{\mathbf{R}^n} d\mathbf{y}\left[|\partial_\xi|^\omega, a_n(t,\xi,\mathbf{y})\right]\prod_{k=0}^{n}\phi_k(t,y_k)\right\|_{\mathbf{L}^2_\xi\mathbf{L}^\infty_t}$$
$$\le C\|a_n(t,\xi,\mathbf{y})\|_{\mathbf{B}^{\omega,\infty}_\xi\mathbf{L}^\infty_{\mathbf{y}}\mathbf{L}^\infty_t}\|\phi_0(t,\xi)\|_{\mathbf{L}^2_\xi\mathbf{L}^\infty_t}\prod_{k=1}^{n}\|\phi_k(t,y_k)\|_{\mathbf{L}^1_{y_k}\mathbf{L}^\infty_t}$$
$$\le C\sum_{l=0}^{1}\left\|\partial_\xi^l a_n(t,\xi,\mathbf{y})\right\|_{\mathbf{L}^\infty_\xi\mathbf{L}^\infty_{\mathbf{y}}\mathbf{L}^\infty_t}\prod_{k=0}^{n}\|\phi_k\|_{\mathbf{X}^{s,0}}.$$

Thus we get the results of the lemma in the case of $\nu=0$.

If $\nu>0$ we denote $\gamma=\min\left(1-\frac{\sigma}{\nu},\frac{s-S}{4\nu}\right)\ge 0$, $\theta=\max(0,s+\sigma)$. By the Hölder inequality we have

$$\|\psi\|_{\mathbf{L}^2\mathbf{L}^1}\le CT^\gamma\|\psi\|_{\mathbf{L}^2\mathbf{L}^p}$$

with $p=\frac{1}{1-\gamma}$. Applying the Young inequality we obtain

$$\|\mathcal{N}(\phi_0,\phi_1,\phi_2)\|_{\mathbf{Y}^{s,0}}\le C\sum_{n=1}^{2}\left\|\langle\xi\rangle^s E\int_{\mathbf{R}^n} d\mathbf{y} a_n\prod_{k=0}^{n}\widehat{\phi}_k(t,y_k)\right\|_{\mathbf{L}^2_\xi\mathbf{L}^1_t}$$
$$\le CT^\gamma\sum_{n=1}^{\vartheta-1}\left\|\int_{\mathbf{R}^n} d\mathbf{y}\sum_{j=0}^{n}\langle y_j\rangle^\theta\prod_{k=0}^{n}E(t,y_k)\widehat{\phi}_k(t,y_k)\right\|_{\mathbf{L}^2_\xi\mathbf{L}^p_t}$$
$$\le CT^\gamma\left\|\langle\xi\rangle^\theta E\phi_j\right\|_{\mathbf{L}^2_\xi\mathbf{L}^q_t}\prod_{k=0,k\ne j}^{\vartheta-1}\|E\phi_k\|_{\mathbf{L}^1_\xi\mathbf{L}^r_t}$$
$$\le CT^\gamma\left\|\langle\xi\rangle^{s+\frac{\nu}{q}}E\phi_j\right\|_{\mathbf{L}^2_\xi\mathbf{L}^q_t}\prod_{k=0,k\ne j}^{\vartheta-1}\left\|\langle\xi\rangle^{s+\frac{\nu}{r}}E\phi_k\right\|_{\mathbf{L}^2_\xi\mathbf{L}^r_t}\le CT^\gamma\prod_{k=0}^{n}\|\phi_k\|_{\mathbf{X}^{s,0}},$$

since $\frac{1}{p} = \frac{1}{q} + \frac{\vartheta - 1}{r}$, $s + \frac{\nu}{q} \geq \theta = \max(0, s + \sigma)$ and $s + \frac{\nu}{r} > \frac{1}{2}$, which follows from the conditions $s > S$ and $0 \leq \gamma \leq \frac{s-S}{4\nu}$. We have

$$\nu(1-\gamma) = \frac{\nu}{q} + \frac{\vartheta - 1}{r}\nu > \theta + \frac{\vartheta - 1}{2} - \vartheta s;$$

hence $s - \frac{\theta}{\vartheta} - \left(\frac{1}{2} - \frac{1+2\nu}{2\vartheta}\right) > \frac{\gamma\nu}{\vartheta}$. Also denoting

$$\widetilde{a_n}(t, \xi, \mathbf{y}) = \langle\xi\rangle^{\lambda} a_n(t, \xi, \mathbf{y}) E(t, \xi) \prod_{k=0}^{n} E^{-1}(t, y_k)$$

we get

$$\begin{aligned}
&\|\mathcal{N}(\phi_0, \phi_1, \phi_2)\|_{\mathbf{Y}^{\lambda,\omega}} \\
&\leq CT^{\gamma} \sum_{n=1}^{\vartheta-1} \left\| \int_{\mathbf{R}^n} d\mathbf{y} \widetilde{a_n} \langle\partial_\xi\rangle^{\omega} \prod_{k=0}^{n} E(t, y_k) \widehat{\phi}_k(t, y_k) \right\|_{\mathbf{L}^2_\xi \mathbf{L}^p_t} \\
&+ CT^{\gamma} \sum_{n=1}^{\vartheta-1} \left\| \int_{\mathbf{R}^n} d\mathbf{y} \left[|\partial_\xi|^{\omega}, \widetilde{a_n}(t, \xi, \mathbf{y})\right] \prod_{k=0}^{n} E(t, y_k) \widehat{\phi}_k(t, y_k) \right\|_{\mathbf{L}^2_\xi \mathbf{L}^p_t};
\end{aligned}$$

thus by using estimate

$$\begin{aligned}
&\left\| \int_{\mathbf{R}^n} d\mathbf{y} \left[|\partial_\xi|^{\omega}, \widetilde{a_n}(t, \xi, \mathbf{y})\right] \prod_{k=0}^{n} \phi_k(t, y_k) \right\|_{\mathbf{L}^2_\xi \mathbf{L}^p_t} \\
&\leq C \|\widetilde{a_n}(t, \xi, \mathbf{y})\|_{\mathbf{B}^{\omega,\infty}_\xi \mathbf{L}^\infty_\mathbf{y} \mathbf{L}^\infty_t} \|\phi_0(t, \xi)\|_{\mathbf{L}^2_\xi \mathbf{L}^q_t} \prod_{k=1}^{n} \|\phi_k(t, y_k)\|_{\mathbf{L}^1_{y_k} \mathbf{L}^r_t} \\
&\leq C \sum_{l=0}^{1} \left\|\partial_\xi^l \widetilde{a_n}\right\|_{\mathbf{L}^\infty_\xi \mathbf{L}^\infty_\mathbf{y} \mathbf{L}^\infty_t} \left\| \langle\xi\rangle^{\lambda + \frac{\nu}{q}} \phi_0 \right\|_{\mathbf{L}^2_\xi \mathbf{L}^q_t} \prod_{k=1}^{n} \left\| \langle\xi\rangle^{s + \frac{\nu}{r}} \phi_k \right\|_{\mathbf{L}^2_\xi \mathbf{L}^r_t}
\end{aligned}$$

we get

$$\begin{aligned}
&\|\mathcal{N}(\phi_0, \phi_1, \phi_2)\|_{\mathbf{Y}^{\lambda,\omega}} \\
&\leq CT^{\gamma} \left(\|\langle\partial_\xi\rangle^{\omega} E\phi_0\|_{\mathbf{L}^2_\xi \mathbf{L}^q_t} + \|E\phi_0\|_{\mathbf{L}^2_\xi \mathbf{L}^q_t} \right) \prod_{j=1}^{\vartheta-1} \|E\phi_j\|_{\mathbf{L}^1_\xi \mathbf{L}^r_t} \\
&\leq CT^{\gamma} \left\| \langle\partial_\xi\rangle^{\omega} \langle\xi\rangle^{\lambda + \frac{\nu}{q}} E(t, \xi) \phi_0(t, \xi) \right\|_{\mathbf{L}^2_\xi \mathbf{L}^q_t} \prod_{j=1}^{\vartheta-1} \left\| \langle\xi\rangle^{s + \frac{\nu}{r}} E\phi_j \right\|_{\mathbf{L}^2_\xi \mathbf{L}^r_t} \\
&\leq CT^{\gamma} \sum_{k=0}^{\vartheta-1} \|\phi_k\|_{\mathbf{X}^{\lambda,\omega}} \prod_{j=0, j\neq k}^{\vartheta-1} \|\phi_j\|_{\mathbf{X}^{s,0}},
\end{aligned}$$

since $\frac{1}{p} = \frac{1}{q} + \frac{\vartheta-1}{r}$, $\lambda + \frac{\nu}{q} \geq 0$ and $s + \frac{\nu}{r} > \frac{1}{2}$. Lemma 2.51 is proved.

Proof of Theorem 2.49

We apply general local existence Theorems 1.9 and 1.11. Via the Green operator $\mathcal{G}(t)$ of the linear Cauchy problem (2.114) we write the nonlinear Cauchy problem (2.109) as the integral equation

$$u(t) = \mathcal{G}(t)\, u_0 - \int_0^t \mathcal{G}(t-\tau)\, \mathcal{N}(u)(\tau)\, d\tau. \tag{2.116}$$

By virtue of Lemma 2.50 and Lemma 2.51 we can estimate

$$\begin{aligned}
&\|\mathcal{G}u_0\|_{\mathbf{X}^{s,0}\cap\mathbf{X}^{\lambda,\omega}} + \left\|\int_0^t \mathcal{G}(t-\tau)\, \mathcal{N}(u)(\tau)\, d\tau\right\|_{\mathbf{X}^{s,0}\cap\mathbf{X}^{\lambda,\omega}} \\
&\le C\,\|u_0\|_{\mathbf{H}^{s,0}\cap\mathbf{H}^{\lambda,\omega}} + C\,\|\mathcal{N}(u)\|_{\mathbf{Y}^{s,0}\cap\mathbf{Y}^{\lambda,\omega}} \\
&\le C\,\|u_0\|_{\mathbf{H}^{s,0}\cap\mathbf{H}^{\lambda,\omega}} + CT^{\gamma}\,\|u\|_{\mathbf{X}^{s,0}\cap\mathbf{X}^{\lambda,\omega}} \left(\|u\|_{\mathbf{X}^{s,0}} + \|u\|_{\mathbf{X}^{s,0}}^{\vartheta-1}\right),
\end{aligned}$$

where

$$\|v\|_{\mathbf{X}^{s,0}\cap\mathbf{X}^{\lambda,\omega}} = \|v\|_{\mathbf{X}^{s,0}} + \|v\|_{\mathbf{X}^{\lambda,\omega}}\,.$$

The norms $\|v\|_{\mathbf{Y}^{s,0}\cap\mathbf{Y}^{\lambda,\omega}}$ and $\|v\|_{\mathbf{H}^{s,0}\cap\mathbf{H}^{\lambda,\omega}}$ are defined similarly. Here $s > S$, $\omega \in [0,1)$, $\gamma = 1$ if $\nu = 0$ and $\gamma = \min\left(1 - \frac{\sigma}{\nu}, \frac{s-S}{4\nu}\right)$ if $\nu > 0$. Similarly we estimate the difference

$$\begin{aligned}
&\left\|\int_0^t \mathcal{G}(t-\tau)\left(\mathcal{N}(u_1) - \mathcal{N}(u_2)\right) d\tau\right\|_{\mathbf{X}^{s,0}\cap\mathbf{X}^{\lambda,\omega}} \\
&\le C\,\|\mathcal{N}(u_1) - \mathcal{N}(u_2)\|_{\mathbf{Y}^{s,0}\cap\mathbf{Y}^{\lambda,\omega}} \\
&\le CT^{\gamma}\left(\|u_1\|_{\mathbf{X}^{s,0}} + \|u_2\|_{\mathbf{X}^{s,0}}^{\vartheta-1}\right)\|u_1 - u_2\|_{\mathbf{X}^{s,0}\cap\mathbf{X}^{\lambda,\omega}} \\
&\le \frac{1}{2}\,\|u_1 - u_2\|_{\mathbf{X}^{s,0}\cap\mathbf{X}^{\lambda,\omega}}\,.
\end{aligned}$$

Consequently by virtue of Theorems 1.9 and 1.11 there exists a unique solution $u(t,x) \in \mathbf{C}\left([0,T]; \mathbf{H}^{s,0}(\mathbf{R}) \cap \mathbf{H}^{\lambda,\omega}(\mathbf{R})\right)$ of the Cauchy problem (2.109), such that

$$\|u\|_{\mathbf{X}^{s,0}\cap\mathbf{X}^{\lambda,\omega}} \le C\,\|u_0\|_{\mathbf{H}^{s,0}\cap\mathbf{H}^{\lambda,\omega}}\,.$$

By Remark 1.10 we see that the existence time $T = T\left(\|u_0\|_{\mathbf{H}^{s,0}}\right)$ is sufficiently small if $\gamma > 0$. In the case of $\gamma = 0$ the norm $\|u_0\|_{\mathbf{H}^{s,0}}$ is sufficiently small and the existence time T can be chosen such that $T \ge 1$. By the definition of the norm $\mathbf{X}^{s,0}$ we have the smoothing property for the case $\nu > 0$

$$\begin{aligned}
&\sup_{t\in[T_0,T]} \|v(t)\|_{\mathbf{H}^{k,0}} = \sup_{t\in[T_0,T]} \left\|\langle\xi\rangle^{k}\, \hat{v}(t,\xi)\right\|_{\mathbf{L}^2_\xi} \\
&\le C(k,T_0) \left\|\sup_{t\in[T_0,T]} E(t,\xi)\,\langle\xi\rangle^{s}\,|\hat{v}(t,\xi)|\right\|_{\mathbf{L}^2_\xi} \le C\,\|v\|_{\mathbf{X}^{s,0}}\,,
\end{aligned}$$

for all $k \geq 0$, where $T_0 \in (0, T]$. The derivatives with respect to time $t > 0$ can be estimated directly from equation (2.109), since the symbols $a_n(t)$ are continuous in time. Therefore in the case of $\nu > 0$ the solution $u(t,x) \in \mathbf{C}^1\left((0,T];\mathbf{H}^{\infty,0}(\mathbf{R})\right)$. Theorem 2.49 is proved.

2.5.3 Small initial data

We now consider a rather general class of nonlinearities in equation (2.109), however, we have to assume a smallness condition on the initial data in order to be able to prove the global existence of solutions. Suppose that the symbols of the nonlinear operator $\mathcal{N}$ are such that

$$\left|\partial_\xi^l a_n(t,\xi,\mathbf{y})\right| \leq C\langle\xi\rangle^\sigma \{y_0\}^{\alpha_{0,n}-l} \prod_{j=1}^n \{y_j\}^{\alpha_{j,n}} \tag{2.117}$$

for all $\xi \in \mathbf{R}$, $\mathbf{y} \in \mathbf{R}^n$, $n = 1,2$, $t > 0$, where σ, $\alpha_{j,n} \geq 0$, $l = 0,1$, $y_0 = \xi - \sum_{j=1}^n y_j$. Note that condition (2.117) is general enough. We can see that the examples mentioned above satisfy this assumption if we take into account the fact that the symbols $a_n(t,\xi,\mathbf{y})$ are defined not uniquely. For example, in the case of nonlinearity $\mathcal{N} = auu_x$, with u real-valued, we have chosen above the symbol $a_1(t,\xi,\mathbf{y}) = \frac{i}{2}a\xi$. Also in this case we can take $a_1(t,\xi,\mathbf{y}) = iay_0$, or $a_1(t,\xi,\mathbf{y}) = i(\xi - y_0)a$ (in view of the symmetry of the integral). If we make a combination

$$a_1(t,\xi,\mathbf{y}) = \begin{cases} iay_0 \text{ if } |y_0| \leq 1 \text{ or } |\xi - y_0| \leq 1, \\ \frac{1}{2}a\xi, \text{ otherwise,} \end{cases}$$

then the symbol $a_1(t,\xi,\mathbf{y})$ satisfies condition (2.117) since $|y_0| \leq C\langle\xi\rangle\{y_0\}$ in the domain ($|y_0| \leq 1$ or $|\xi - y_0| \leq 1$) and $|\xi| \leq C\langle\xi\rangle\{y_0\}$ in the domain ($|y_0| \geq 1$ and $|\xi - y_0| \geq 1$).

Remark 2.52. In the case of nonlinearity $\mathcal{N}(u) = a|u|^2 u_x + c\left(|u|^2 u\right)_x$ we can use more general conditions than (2.117)

$$\left|\partial_\xi^l a_n(t,\xi,\mathbf{y})\right| \leq C\langle\xi\rangle^{\sigma-\zeta} \{y_0\}^{\alpha_{0,n}-l} \prod_{j=1}^n \{y_j\}^{\alpha_{j,n}} \langle y_j\rangle^\zeta$$

for $n = 1,2$. By the change of dependent variable $v = \overline{\mathcal{F}}_{\xi\to x}\langle\xi\rangle^\zeta \mathcal{F}_{x\to\xi}u$ this case is reduced to (2.117), therefore all the results are true if we replace the regularity condition $s > S$ on the initial data by the following condition $s > S + \zeta$.

Let the linear operator $\mathcal{L}$ satisfy the dissipation condition

$$\operatorname{Re} L(\xi) \geq \mu\{\xi\}^\delta \langle\xi\rangle^\nu \tag{2.118}$$

for all $\xi \in \mathbf{R}$, where $\mu > 0$, $\nu \geq \sigma \geq 0$, $\delta > 0$. To find the asymptotic formulas for the solution we assume that the symbol $L(\xi)$ has the following asymptotic representation in the origin

$$L(\xi) = \mu |\xi|^{\delta} + O\left(|\xi|^{\delta+\gamma}\right) \tag{2.119}$$

for all $|\xi| \leq 1$, where $\gamma > 0$. Also we suppose that the symbol is smooth $L(\xi) \in \mathbf{C}^1(\mathbf{R})$ and has the estimate

$$|L'(\xi)| \leq C \{\xi\}^{\delta-1} \langle\xi\rangle^{\nu} \tag{2.120}$$

for all $\xi \in \mathbf{R}$. We denote $\vartheta = 2$ if the nonlinear operator $\mathcal{N}$ is quadratic: $a_2(t,\xi,\mathbf{y}) \equiv 0$ for all $\mathbf{y} \in \mathbf{R}^3$ and let $\vartheta = 3$ otherwise. Denote

$$S = S(\sigma,\nu,\vartheta) \equiv \frac{1}{2} - \min\left(\frac{\nu-\sigma}{\vartheta-1}, \frac{2\nu+1}{2\vartheta}\right),$$
$$\delta_c = \min(1+|\alpha_1|, 2+|\alpha_2|),$$

where $|\alpha_n| = \sum_{j=0}^{n} \alpha_{j,n}$, $n = 1,2$. The constant δ_c denotes the critical order with respect to the large time behavior.

Theorem 2.53. *Let the linear operator $\mathcal{L}$ satisfy conditions (2.118) through (2.120) with $\delta < \delta_c$. Suppose that the nonlinear operator $\mathcal{N}$ satisfies estimates (2.117) with $\sigma \in [0,\nu)$ if $\nu > 0$ or $\sigma = 0 = \nu$. Let the initial data $u_0 \in \mathbf{H}^{s,0}(\mathbf{R}) \cap \mathbf{H}^{\lambda,\omega}(\mathbf{R})$ be sufficiently small, where $s > S$, $\omega > \frac{1}{2}$, $\lambda = \min(s,-\sigma)$. Then there exists a unique solution $u(t,x) \in \mathbf{C}^0\left([0,\infty); \mathbf{H}^{s,0}(\mathbf{R}) \cap \mathbf{H}^{\lambda,\omega}(\mathbf{R})\right)$ of the Cauchy problem (2.109). In the case of $\nu > 0$ we also have a smoothing property $u(t,x) \in \mathbf{C}^1\left((0,\infty); \mathbf{H}^{\infty,0}(\mathbf{R})\right)$. Moreover, there exists a unique number U, such that the solution $u(t,x)$ has the following asymptotics*

$$u(t,x) = Ut^{-\frac{1}{\delta}} G\left(xt^{-\frac{1}{\delta}}\right) + O\left(t^{-\frac{1}{\delta}-\gamma}\right) \tag{2.121}$$

for large time $t > 0$ uniformly with respect to $x \in \mathbf{R}$, where $\gamma > 0$, $G(x) = \overline{\mathcal{F}}_{\xi\to x} e^{-\mu|\xi|^{\delta}}$.

Remark 2.54. Note that if the nonlinear term has a zero total mass

$$\int_{\mathbf{R}} \mathcal{N}(u)\, dx = 0,$$

then the coefficient $U = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} u_0(x)\, dx$, that is the main term of the asymptotics is the same as for the linear case. We can guarantee that the coefficient U in the asymptotic formula (2.121) does not vanish if the initial data are small with nonzero total mass $\int_{\mathbf{R}} u_0(x)\, dx \neq 0$. If $\int_{\mathbf{R}} u_0(x)\, dx = 0$, the coefficient U can vanish; in that case the solutions decay faster than described by (2.121).

As an example we apply Theorem 2.53 to the modified Korteweg-de Vries-Burgers equation

$$u_t + u^2 u_x - \mu u_{xx} + u_{xxx} = 0.$$

We have $\sigma = 1, \nu = 2 = \delta$, $\delta_c = 3$, $\mu > 0$ and $S = 0$. Therefore if the initial data $u_0 \in \mathbf{H}^{s,\omega}(\mathbf{R})$ are sufficiently small, where $s > 0$, $\omega > \frac{1}{2}$, then there exists a unique solution $u(t,x) \in \mathbf{C}^{\infty}\left((0,\infty); \mathbf{H}^{\infty,0}(\mathbf{R})\right)$, and asymptotics

$$u(t,x) = (4\pi\mu t)^{-\frac{1}{2}} e^{-\frac{x^2}{2\mu t}} \int_{\mathbf{R}} u_0(x)\, dx + O\left(t^{-\frac{1}{2}-\gamma}\right)$$

is true for $t \to \infty$, where $\gamma > 0$. Before proving Theorem 2.53 we consider some estimates.

Preliminary lemmas

In the next lemmas we give large time decay estimates for the Green function $\mathcal{G}(t)$ in the norm

$$\|\varphi\|_{s,\rho,\omega} = \|(1 + |\partial_\xi|^\omega)\{\cdot\}^\rho \langle\cdot\rangle^s \hat{\varphi}(t,\cdot)\|_{\mathbf{L}^2_\xi},$$

where $s \in \mathbf{R}$, $\omega \geq 0$, $\rho \geq 0$. Denote $\mathbf{A}^{s,\rho,\omega} = \left\{\phi \in \mathbf{L}^2(\mathbf{R}) : \|\varphi\|_{s,\rho,\omega} < \infty\right\}$. By the result of Lemma 1.38 we obtain the following.

Lemma 2.55. *Let the linear operator $\mathcal{L}$ satisfy dissipation conditions (2.118) and (2.120). Then the estimates are valid*

$$\|\mathcal{G}(t)\psi(\tau)\|_{q\nu,\rho,0} \leq C\langle t\rangle^{-\frac{\rho}{\delta}}\{t\}^{-q}\min\left(\|\psi(\tau)\|_{0,0,0}, \langle t\rangle^{-\frac{1}{2\delta}}\|\psi(\tau)\|_{0,0,\omega}\right),$$

$$\|\mathcal{G}(t)\psi(\tau)\|_{q\nu,\rho,\omega} \leq C\langle t\rangle^{\frac{\omega-\rho}{\delta}-\frac{1}{2\delta}}\{t\}^{-q}\|\psi(\tau)\|_{0,0,\omega}$$

for all $0 \leq \tau \leq t$, where $q \geq 0$, $\rho \geq 0$, $\omega \in \left(\frac{1}{2}, 1\right)$ is such that $\omega < \frac{1}{2} + \delta$ if $\rho = 0$ and $\omega < \frac{1}{2} + \min(\rho, \delta)$ if $\rho > 0$.

By Lemma 1.38 in the next lemma we find the asymptotic formulas for the linear Cauchy problem (2.114).

Lemma 2.56. *Let the linear operator $\mathcal{L}$ satisfy dissipation conditions (2.118) and (2.120) and asymptotic representation (2.119). Then for any $\phi(x) \in \mathbf{A}^{s,0,0} \cap \mathbf{A}^{0,0,\omega}$ and $\psi(t,x) \in \mathbf{C}\left((0,\infty); \mathbf{A}^{s,0,0} \cap \mathbf{A}^{0,0,\omega}\right)$, where $s > \frac{1}{2}$, $\omega \in \left(\frac{1}{2}, \frac{1}{2} + \delta\right)$ we have the asymptotic representation as $t \geq 1$ uniformly with respect to $x \in \mathbf{R}$*

$$\mathcal{G}(t)\phi = \hat{\phi}(0)t^{-\frac{1}{\delta}}G\left(xt^{-\frac{1}{\delta}}\right) + O\left(t^{-\frac{1}{\delta}-\gamma}\left(\|\phi\|_{s,0,0} + \|\phi\|_{0,0,\omega}\right)\right),$$

and

$$\int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau = t^{-\frac{1}{\delta}} G\left(xt^{-\frac{1}{\delta}}\right)\int_0^\infty \hat{\psi}(\tau)\,d\tau$$
$$+O\left(t^{-\frac{1}{\delta}-\gamma}\sup_{\tau>0}\langle\tau\rangle^\theta\left(\|\psi(\tau)\|_{s,0,0}+\|\psi(\tau)\|_{0,0,\omega}+\langle\tau\rangle^{\frac{1}{\delta}}\|\psi(\tau)\|_{s,\beta,0}\right)\right),$$

where $G(x)=\overline{\mathcal{F}}_{\xi\to x}e^{-\mu|\xi|^\delta}$, $0<\gamma<\min\left(\frac{\omega}{\delta},\theta-1\right)$, $\theta>1$, $\beta\in\left(0,\frac{1}{2}\right)$.

Now we estimate the nonlinearity in the norms $\mathbf{A}^{s,\rho,\omega}$. Remind that $\vartheta=2$ if the nonlinear operator $\mathcal{N}$ is quadratic and let $\vartheta=3$ otherwise.

Lemma 2.57. *Let the nonlinear operator* $\mathcal{N}$ *satisfy condition (2.117). Then the inequalities*

$$\|\mathcal{N}(\phi_0,\phi_1,\phi_2)\|_{s,0,0}\le C\sum_{n=1}^{\vartheta-1}\|\phi_0\|_{s+\sigma,|\alpha_n|,0}\prod_{j=1}^n\|\phi_j\|_{r,\beta,0}$$

and

$$\|\mathcal{N}(\phi_0,\phi_1,\phi_2)\|_{s,0,\omega}\le C\sum_{n=1}^{\vartheta-1}\sum_{k=0}^n\left(\|\phi_0\|_{\sigma_{0,k},\alpha_{0,n},\omega}+\|\phi_0\|_{\sigma_{0,k},\alpha_{0,n}-\omega-\gamma,0}\right)$$
$$\times\prod_{j=1}^n\|\phi_j\|_{r+\sigma_{j,k},\beta+\alpha_{j,n},0}$$

are valid, provided that the right-hand sides are bounded, where $\omega\in\left(\frac{1}{2},1\right)$, $s\in\mathbf{R}$, $r>\frac{1}{2}$, $\beta<\frac{1}{2}$, $\omega<\frac{1}{2}+\alpha_{0,n}$ *if* $\alpha_{0,n}>0$, $n=1,2$; $\sigma_{j,k}=s+\sigma$ *if* $j=k$ *and* $\sigma_{j,k}=0$ *otherwise.*

Proof. By virtue of condition (2.117) we obtain

$$\|\mathcal{N}(\phi_0,\phi_1,\phi_2)\|_{s,0,0}\le C\sum_{n=1}^{\vartheta-1}\left\|\langle\xi\rangle^s\int_{\mathbf{R}^n}d\mathbf{y}\,a_n\prod_{k=0}^n\widehat{\phi}_k(y_k)\right\|_{\mathbf{L}^2_\xi}$$
$$\le C\sum_{n=1}^{\vartheta-1}\left\|\langle\xi\rangle^{s+\sigma}\int_{\mathbf{R}^n}d\mathbf{y}\prod_{k=0}^n\{y_l\}^{\alpha_{k,n}}\left|\widehat{\phi}_k(y_k)\right|\right\|_{\mathbf{L}^2_\xi}$$
$$\le C\sum_{n=1}^{\vartheta-1}\left\|\langle\xi\rangle^{s+\sigma}\{\xi\}^{|\alpha_n|}\widehat{\phi}_0\right\|_{\mathbf{L}^2_\xi}\prod_{k=1}^n\left\|\widehat{\phi}_k\right\|_{\mathbf{L}^1_\xi}$$
$$\le C\sum_{n=1}^{\vartheta-1}\|\phi_0\|_{s+\sigma,|\alpha_n|,0}\prod_{j=1}^n\|\phi_j\|_{r,\beta,0},$$

where $\beta<\frac{1}{2}$ and $r>\frac{1}{2}$. We now prove the second estimate. Denote

$$\widetilde{a_n}(t,\xi,\mathbf{y})=a_n(t,\xi,\mathbf{y})\langle\xi\rangle^{-\sigma}\{y_0\}^{-\alpha_{0,n}}\prod_{k=1}^n\{y_k\}^{-\alpha_{k,n}}$$

for $n = 1, 2$. Via condition (2.117) we get

$$\begin{aligned}&\left\|\left[|\partial_\xi|^\omega, \widetilde{a_n}(t,\xi,\mathbf{y})\right]\Phi(t,y_0,\mathbf{y})\right\|_{\mathbf{L}^2_\xi}\\ &\leq \left\|\int_{|y_0|\leq 2|\eta|}\frac{d\eta}{|\eta|^{1+\omega}}\left(|\widetilde{a_n}(t,\xi-\eta,\mathbf{y})| + |\widetilde{a_n}(t,\xi,\mathbf{y})|\right)\Phi(t,y_0-\eta,\mathbf{y})\right\|_{\mathbf{L}^2_\xi}\\ &+\left\|\int_{|y_0|>2|\eta|}\frac{d\eta}{|\eta|^{1+\omega}}\left|\widetilde{a_n}(t,\xi-\eta,\mathbf{y}) - \widetilde{a_n}(t,\xi,\mathbf{y})\right|\Phi(t,y_0-\eta,\mathbf{y})\right\|_{\mathbf{L}^2_\xi};\end{aligned}$$

hence

$$\begin{aligned}&\left\|\left[|\partial_\xi|^\omega, \widetilde{a_n}(t,\xi,\mathbf{y})\right]\Phi(t,y_0,\mathbf{y})\right\|_{\mathbf{L}^2_\xi}\\ &\leq C\left\|\int_{|y_0|\leq 2|\eta|}\frac{d\eta}{|\eta|^{1+\omega}}\{\eta\}^\omega\{y_0-\eta\}^{-\omega-\gamma}\Phi(t,y_0-\eta,\mathbf{y})\right\|_{\mathbf{L}^2_\xi}\\ &+C\left\|\int_{|y_0|>2|\eta|}\frac{d\eta}{|\eta|^{1+\omega}}\{y_0-\eta\}^\omega\left|\int_{y_0}^{y_0-\eta}\{\zeta\}^{-1}d\zeta\right|\right.\\ &\times\left.\{y_0-\eta\}^{-\omega}\Phi(t,y_0-\eta,\mathbf{y})\right\|_{\mathbf{L}^2_\xi}\\ &\leq C\left\|\{y_0\}^{-\omega-\gamma}\Phi(t,y_0,\mathbf{y})\right\|_{\mathbf{L}^2_\xi}\left(1+\{y_0\}^{\omega-1}\int_{|y_0|>2|\eta|}\frac{d\eta}{|\eta|^\omega}\right)\\ &\leq C\left\|\{y_0\}^{-\omega-\gamma}\Phi(t,y_0,\mathbf{y})\right\|_{\mathbf{L}^2_\xi}\end{aligned}$$

for all $\xi \in \mathbf{R}$, $\mathbf{y} \in \mathbf{R}^n$, $n = 1, 2$, $t > 0$, where $\omega \in \left(\frac{1}{2}, \frac{1}{2} + \alpha_{0,n}\right)$ if $\alpha_{0,n} > 0$, $n = 1, 2$. Therefore, by now taking

$$\Phi_n(t,y_0,\mathbf{y}) = \langle\xi\rangle^{s+\sigma}\prod_{k=0}^{n}\{y_k\}^{\alpha_{k,n}}\widehat{\phi}_k(y_k)$$

we get

$$\begin{aligned}
\|\mathcal{N}(u)\|_{s,0,\omega} &\le C\sum_{n=1}^{\vartheta-1}\left\|\int_{\mathbf{R}^n} d\mathbf{y}\widetilde{a_n}(t,\xi,\mathbf{y})\,|\partial_\xi|^{\omega}\,\Phi_n(t,y_0,\mathbf{y})\right\|_{\mathbf{L}^2_\xi}\\
&+C\sum_{n=1}^{\vartheta-1}\left\|\int_{\mathbf{R}^n} d\mathbf{y}\left[|\partial_\xi|^{\omega},\widetilde{a_n}(t,\xi,\mathbf{y})\right]\Phi_n(t,y_0,\mathbf{y})\right\|_{\mathbf{L}^2_\xi}\\
&\le C\sum_{n=1}^{\vartheta-1}\sum_{k=0}^{n}\left\||\partial_\xi|^{\omega}\{\xi\}^{\alpha_{0,n}}\langle\xi\rangle^{\sigma_{0,k}}\widehat{\phi}_0\right\|_{\mathbf{L}^2_\xi}\prod_{j=1}^{n}\left\|\{\xi\}^{\alpha_{j,n}}\langle\xi\rangle^{\sigma_{j,k}}\widehat{\phi}_j\right\|_{\mathbf{L}^1_\xi}\\
&+C\sum_{n=1}^{\vartheta-1}\sum_{k=0}^{n}\left\|\{\xi\}^{\alpha_{0,n}-\omega-\gamma}\langle\xi\rangle^{\sigma_{0,k}}\widehat{\phi}_0\right\|_{\mathbf{L}^2_\xi}\prod_{j=1}^{n}\left\|\{\xi\}^{\alpha_{j,n}}\langle\xi\rangle^{\sigma_{j,k}}\widehat{\phi}_j\right\|_{\mathbf{L}^1_\xi}\\
&\le\sum_{n=1}^{\vartheta-1}\sum_{k=0}^{n}\left(\|\phi_0\|_{\sigma_{0,k},\alpha_{0,n},\omega}+\|\phi_0\|_{\sigma_{0,k},\alpha_{0,n}-\omega-\gamma,0}\right)\prod_{j=1}^{n}\|\phi_j\|_{r+\sigma_{j,k},\beta+\alpha_{j,k},0}\,.
\end{aligned}$$

Lemma 2.57 is proved.

Proof of Theorem 2.53

By the regularizing effect described in the local existence Theorem 2.49 we can consider the Cauchy problem (2.109) for $t > t_1 > 0$ with initial data $\widetilde{u}_0 = u(t_1) \in \mathbf{H}^{r,0}(\mathbf{R}) \cap \mathbf{H}^{0,\omega}(\mathbf{R})$ with $r > \frac{1}{2}$, $\omega \in \left(\frac{1}{2}, 1\right)$. Changing $t - t_1 = \widetilde{t} \ge 0$ via the Green operator $\mathcal{G}(t)$ of the linear Cauchy problem (2.114) we write the nonlinear Cauchy problem (2.109) as the integral equation (tildes are omitted)

$$u(t) = \mathcal{G}(t)\,u_0 - \int_0^t \mathcal{G}(t-\tau)\,\mathcal{N}(u)(\tau)\,d\tau. \tag{2.122}$$

We apply Theorem 1.17. Denote $\rho_m = \max(\beta, |\alpha_1|, |\alpha_2|)$ and define

$$\begin{aligned}
\|\phi\|_{\mathbf{X}} = &\sup_{t>0}\sup_{q\in[0,1)}\sup_{\rho\in[0,\rho_m]}\{t\}^{q}\langle t\rangle^{\frac{\rho}{\delta}+\frac{1}{2\delta}}\|\phi(t)\|_{r+q\nu,\rho,0}\\
&+\sup_{t>0}\sup_{q\in[0,1)}\sup_{\rho\in[0,\rho_m]}\{t\}^{q}\langle t\rangle^{\frac{\rho-\omega}{\delta}+\frac{1}{2\delta}}\|\phi(t)\|_{q\nu,\rho,\omega}
\end{aligned}$$

with $r > \frac{1}{2}$, where the norm

$$\|\varphi\|_{s,\rho,\omega} = \left\|(1+|\partial_\xi|^{\omega})\{\xi\}^{\rho}\langle\xi\rangle^{s}\hat{\varphi}(t,\xi)\right\|_{\mathbf{L}^2_\xi}.$$

Using Lemma 2.55 we get for $q \in [0,1)$, $\rho \in [0,\rho_m]$

$$\|\mathcal{G}(t)u_0\|_{r+q\nu,\rho,0} \le C\{t\}^{-q}\|u_0\|_{r,0,\omega},$$

$$\left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau\right\|_{r+q\nu,\rho,0} \le C\int_0^t d\tau\,\{t-\tau\}^{-q}\|\mathcal{N}(u)(\tau)\|_{r,0,0}$$

for $t \in [0,1]$,

$$\begin{aligned}&\|\mathcal{G}(t)u_0\|_{r+q\nu,\rho,0} + \langle t\rangle^{-\frac{\omega}{\delta}} \|\mathcal{G}(t)u_0\|_{q\nu,\rho,\omega}\\ &\leq C\langle t\rangle^{-\frac{\rho}{\delta}-\frac{1}{2\delta}}\{t\}^{-q}\|u_0\|_{r,0,\omega}\end{aligned}$$

and

$$\begin{aligned}&\left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v)(\tau)\,d\tau\right\|_{r+q\nu,\rho,0} + \langle t\rangle^{-\frac{\omega}{\delta}}\left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v)(\tau)\,d\tau\right\|_{q\nu,\rho,\omega}\\ &\leq C\int_0^{t/2} d\tau\,\langle t-\tau\rangle^{-\frac{\rho}{\delta}-\frac{1}{2\delta}}\|\mathcal{N}(u)(\tau)\|_{0,0,\omega}\\ &+C\int_{t/2}^t d\tau\,\langle t-\tau\rangle^{-\frac{\rho}{\delta}}\{t-\tau\}^{-q}\|\mathcal{N}(u)(\tau)\|_{r,0,0}\\ &+C\langle t\rangle^{-\frac{\omega}{\delta}}\int_0^t d\tau\,\langle t-\tau\rangle^{\frac{\omega-\rho}{\delta}-\frac{1}{2\delta}}\{t-\tau\}^{-q}\|\mathcal{N}(u)(\tau)\|_{0,0,\omega}\end{aligned}$$

for $t \geq 1$. By Lemma 2.57 we have

$$\begin{aligned}\|\mathcal{N}(u)\|_{r,0,0} &\leq C\sum_{n=1}^{\vartheta-1}\|u\|_{r+\sigma,|\alpha_n|,0}\|u\|_{r,\beta,0}^n\\ &\leq C\|u\|_{\mathbf{X}}^2\sum_{n=1}^{\vartheta-1}\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{-\frac{|\alpha_n|+\beta n}{\delta}-\frac{n+1}{2\delta}}\end{aligned}$$

and

$$\begin{aligned}&\|\mathcal{N}(u)(\tau)\|_{0,0,\omega} \leq C\sum_{n=1}^{\vartheta-1}\sum_{k=0}^{n}\left(\|u\|_{\sigma_{0,k},\alpha_{0,n},\omega} + \|u\|_{\sigma_{0,k},\alpha_{0,n}-\omega-\gamma,0}\right)\\ &\times\|u\|_{r+\sigma_{j,k},\beta+\alpha_{j,n},0}^n \leq C\|u\|_{\mathbf{X}}^2\sum_{n=1}^{\vartheta-1}\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{\frac{\omega+\gamma}{\delta}-\frac{|\alpha_n|+\beta n}{\delta}-\frac{n+1}{2\delta}};\end{aligned}$$

hence

$$\begin{aligned}&\left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau\right\|_{r+q\nu,\rho,0} + \langle t\rangle^{-\frac{\omega}{\delta}}\left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau\right\|_{q\nu,\rho,\omega}\\ &\leq C\|u\|_{\mathbf{X}}^2\sum_{n=1}^{\vartheta-1}\int_{\frac{t}{2}}^t\langle t-\tau\rangle^{-\frac{\rho}{\delta}}\langle\tau\rangle^{-\frac{|\alpha_n|+\beta n}{\delta}-\frac{n+1}{2\delta}}\frac{d\tau}{\{t-\tau\}^q\{\tau\}^{\frac{\sigma}{\nu}}}\\ &+C\|u\|_{\mathbf{X}}^2\sum_{n=1}^{\vartheta-1}\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-\frac{\rho}{\delta}-\frac{1}{2\delta}}\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{\frac{\omega+\gamma}{\delta}-\frac{|\alpha_n|+\beta n}{\delta}-\frac{n+1}{2\delta}}\,d\tau\\ &+C\|u\|_{\mathbf{X}}^2\sum_{n=1}^{\vartheta-1}\langle t\rangle^{-\frac{\omega}{\delta}}\int_0^t\langle t-\tau\rangle^{\frac{\omega-\rho}{\delta}-\frac{1}{2\delta}}\langle\tau\rangle^{\frac{\omega+\gamma}{\delta}-\frac{|\alpha_n|+\beta n}{\delta}-\frac{n+1}{2\delta}}\frac{d\tau}{\{t-\tau\}^q\{\tau\}^{\frac{\sigma}{\nu}}}\\ &\leq 2\|u\|_{\mathbf{X}}^2\langle t\rangle^{-\frac{\rho}{\delta}-\frac{1}{2\delta}}\{t\}^{-q}\end{aligned}$$

since $0 < \delta < \delta_c = \min\left(1 + |\alpha_1|, 2 + |\alpha_2|\right)$. Finally, by choosing $\beta = \frac{1}{2} - \gamma$, $\omega = \frac{1}{2} + \gamma$ with sufficiently small $\gamma > 0$ we get $|\alpha_n| + \beta n + \frac{n+1}{2} - \omega > \delta$, for $n = 1, \vartheta - 1$.

In the same manner we estimate a difference

$$\left\| \int_0^t \mathcal{G}(t-\tau)\left(\mathcal{N}(w)(\tau) - \mathcal{N}(v)(\tau)\right) d\tau \right\|_{\mathbf{X}} \leq C \|w - v\|_{\mathbf{X}} (\|w\|_{\mathbf{X}} + \|v\|_{\mathbf{X}}).$$

Therefore by virtue of general global existence Theorem 1.17 there exists a unique solution $u(t,x) \in \mathbf{C}\left([t_1, \infty), \mathbf{H}^{r,0}(\mathbf{R}) \cap \mathbf{H}^{0,\omega}(\mathbf{R})\right)$. Now applying Lemma 2.56 to integral equation (2.122) we can see that the conditions of Theorem 2.4 are fulfilled. Hence we find the asymptotics (2.121) with coefficient

$$U = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} u_0(x)\, dx - \frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \int_{\mathbf{R}} \mathcal{N}(u)(\tau, x)\, dx.$$

Note that in view of the estimates for the solution $u(t,x)$, the coefficient U can be calculated approximately with any desired accuracy via the integral equation (2.122). Theorem 2.53 is proved.

2.5.4 Large initial data

Although Theorem 2.53 is general enough, however the application of Theorem 2.53 to the particular equations gives us rough results since we did not take into account a special character of the nonlinearity. We now intend to remove the smallness condition for the initial data. As we know the condition of strong dissipation (2.118) prevents the effect of blow up for solutions to the Whitham equation and the nonlinear Schrödinger equation. Therefore, the classical solutions with any large initial data exist globally in time. Because of some special symmetry of the nonlinearity of these equations which easily allows us to estimate the $\mathbf{L}^2$ norm of the solution. We express this symmetry property in the following form

$$\operatorname{Re} \int \overline{\mathcal{B}\varphi} \mathcal{B}\mathcal{N}(\varphi) dx = 0 \tag{2.123}$$

for any function $\varphi \in \mathbf{C}_0^\infty(\mathbf{R})$, where $\mathcal{B}$ is a pseudodifferential operator, such that the norm $\|\mathcal{B}u\|_{\mathbf{L}^2}$ is equivalent to the norm of the Sobolev space $\|u\|_{\mathbf{H}^{b,0}}$ for some $b \geq 0$. The symmetry property (2.123) with $\mathcal{B} \equiv 1$ and $b = 0$ is fulfilled for the Whitham equation (2.110) and the derivative nonlinear Schrödinger equation (2.111) if $\operatorname{Re} a = -\operatorname{Re} c$.

Now we can state the result analogous to Theorem 2.53 without any restriction on the size of the initial data; however the critical order with respect to the large time behavior $\delta_c = \min\left(1 + |\alpha_1|, 2 + |\alpha_2|\right)$ now must be shifted by $\frac{1}{2}$. Denote

$$S = S(\sigma, \nu, \vartheta) \equiv \frac{1}{2} - \min\left(\frac{\nu - \sigma}{\vartheta - 1}, \frac{2\nu + 1}{2\vartheta}\right),$$

where $|\alpha_n| = \sum_{j=0}^{n} \alpha_{j,n}$, $n = 1, 2$, $\vartheta = 2$ if the nonlinear operator $\mathcal{N}$ is quadratic and $\vartheta = 3$ otherwise.

Theorem 2.58. *Let the linear operator $\mathcal{L}$ satisfy conditions (2.118) through (2.120) with $\delta < \delta_c - \frac{1}{2}$. Suppose that the nonlinear operator $\mathcal{N}$ satisfies estimates (2.117) with $\sigma \in [0, \nu)$, $\nu > 0$ and the symmetry property (2.123) with $b > \frac{1-\nu}{2} - \frac{\nu-\sigma}{\vartheta-1}$. Let the initial data $u_0 \in \mathbf{H}^{s,0}(\mathbf{R}) \cap \mathbf{H}^{\lambda,\omega}(\mathbf{R})$, with $s > S$, $\omega > \frac{1}{2}$, $\lambda = \min(s, -\sigma)$. Then there exists a unique solution $u \in \mathbf{C}^0\left([0,\infty); \mathbf{H}^{s,0}(\mathbf{R}) \cap \mathbf{H}^{\lambda,\omega}(\mathbf{R})\right) \cap \mathbf{C}^1\left((0,\infty)); \mathbf{H}^{\infty,0}(\mathbf{R})\right)$ of the Cauchy problem (2.109). Moreover, the solution has asymptotics (2.121).*

For example, let us apply Theorem 2.58 to the derivative nonlinear Schrödinger equation (2.111) with $\operatorname{Re} a = -\operatorname{Re} c$. In this case we have $\nu = 2 = \delta$, $\delta_c = 3$, $\sigma = 1 = \zeta$ and condition (2.123) is fulfilled with $b = 0$, so if the initial data $u_0 \in \mathbf{H}^{s,\omega}(\mathbf{R})$, where $s > 1$, $\omega > \frac{1}{2}$, then there exists a unique solution $u \in \mathbf{C}^\infty\left((0,\infty); \mathbf{H}^{\infty,0}(\mathbf{R})\right)$ with asymptotics

$$u(t,x) = Ut^{-\frac{1}{2}}\, e^{-\frac{x^2}{2t}} + O\left(t^{-\frac{1}{2}-\gamma}\right),$$

where $\gamma > 0$.

Proof of Theorem 2.58

Since $\nu > 0$ because of the smoothing effect the solution $u(t,x)$ of the Cauchy problem (2.109) belongs to the space $\mathbf{H}^{r,\omega}(\mathbf{R})$ for all $t \geq t_1$ with $r > \frac{1}{2}$. Therefore, we consider Cauchy problem (2.109) for $t \geq t_1 > 0$. Multiplying equation (2.109) by $\overline{\mathcal{B}u}\mathcal{B}$, then by integrating with respect to $x \in \mathbf{R}$, and by using property (2.123), we get

$$\begin{aligned} \frac{d}{dt}\|\mathcal{B}u\|_{\mathbf{L}^2}^2 &= -2\operatorname{Re}\int_{\mathbf{R}} \left(\overline{\mathcal{B}u}\mathcal{B}\mathcal{N}(u) + \overline{\mathcal{B}u}\mathcal{B}\mathcal{L}u\right) dx \\ &= -2\int_{\mathbf{R}} \operatorname{Re} L(\xi)\left|\widehat{\mathcal{B}u}\right|^2 d\xi; \end{aligned}$$

hence by integrating with respect to $t \geq t_1$, we obtain

$$\sup_{t \geq t_1} \|\mathcal{B}u(t)\|_{\mathbf{L}^2}^2 + 2\mu \int_{t_1}^{\infty} \|\mathcal{B}u(t)\|_{\nu/2,\delta/2,0}^2\, dt \leq C\,\|\mathcal{B}u(t_1)\|_{\mathbf{L}^2}^2.$$

Thus the norm $\|\mathcal{B}u(t)\|_{\mathbf{L}^2}$ is bounded, and there exists a time $T \geq t_1$ such that the norms $\|\mathcal{B}u(t)\|_{\nu/2,\delta/2,0}$ are sufficiently small. Hence we can consider the Cauchy problem (2.109) for $t \geq T > 0$ with sufficiently small initial data $u(T)$ in $\mathbf{A}^{r_1,\rho_1,0}$, $r_1 = b + \frac{\nu}{2} > S$, $\rho_1 > 0$ and apply the method of proof of Theorem 2.53. Changing $t - T = \widetilde{t} \geq 0$ via the Green operator $\mathcal{G}(t)$ of the linear Cauchy problem (2.114) we write the nonlinear Cauchy problem

(2.109) as the integral equation (2.122) with initial data u_0, which is small in the norm $\mathbf{A}^{r_1,\rho_1,0}$, $r_1 = b + \frac{\nu}{2} > S$, $\rho_1 > 0$. We define a set

$$\mathbf{Y}= \left\{ \phi \in \mathbf{S}' : \sup_{t>0} \sup_{q\in[0,1)} \sup_{\rho\in[0,\rho_m]} \{t\}^q \langle t\rangle^{\frac{\rho}{\delta}} \|u(t)\|_{r_1+q\nu,\rho_1+\rho,0} \le 2\varepsilon \right.$$
$$\left. \sup_{t>0} \sup_{q\in[0,1)} \sup_{\rho\in[0,\rho_m]} \{t\}^q \langle t\rangle^{\frac{\rho-\omega}{\delta}+\frac{1}{2\delta}} \|u(t)\|_{r_1+q\nu,\rho_1+\rho,\omega} \le C \right\}.$$

Using Lemma 2.55, we get for $q \in [0,1)$, $\rho \in [0,\rho_m]$

$$\|\mathcal{G}(t) u_0\|_{r_1+q\nu,\rho_1+\rho,0} \le C \langle t\rangle^{-\frac{\rho}{\delta}} \{t\}^{-q} \|u_0\|_{r_1,\rho_1,0}$$

and

$$\left\| \int_0^t \mathcal{G}(t-\tau) \mathcal{N}(v)(\tau)\, d\tau \right\|_{r_1+q\nu,\rho_1+\rho,0}$$
$$\le C \int_0^t d\tau \langle t-\tau\rangle^{-\frac{\rho}{\delta}} \{t-\tau\}^{-q} \|\mathcal{N}(v)(\tau)\|_{r_1,\rho_1,0}.$$

By Lemma 2.57 we have

$$\|\mathcal{N}(u)\|_{r_1,\rho_1,0} \le C \sum_{n=1}^{\vartheta-1} \|u\|_{r_1+\sigma,|\alpha_n|+\rho_1,0} \|u\|^n_{r,\beta,0}$$
$$\le C\varepsilon^2 \sum_{n=1}^{\vartheta-1} \{\tau\}^{-\frac{\sigma-ns_1}{\nu}} \langle\tau\rangle^{-\frac{|\alpha_n|+\beta n}{\delta}-\frac{n-1}{2\delta}+\frac{n\rho_1}{\delta}},$$

where $s_1 = \max\left(0, \frac{1}{2} - r_1\right)$. Hence

$$\langle t\rangle^{\frac{\rho}{\delta}} \{t\}^q \left\| \int_0^t \mathcal{G}(t-\tau) \mathcal{N}(u)(\tau)\, d\tau \right\|_{r_1+q\nu,\rho_1+\rho,0}$$
$$\le \varepsilon^2 \sum_{n=1}^{\vartheta-1} \langle t\rangle^{\frac{\rho}{\delta}} \{t\}^q \int_0^t \{\tau\}^{-\frac{\sigma-ns_1}{\nu}} \langle\tau\rangle^{-\frac{|\alpha_n|+\beta n}{\delta}-\frac{n-1}{2\delta}+\frac{n\rho_1}{\delta}} \frac{d\tau}{\langle t-\tau\rangle^{\frac{\rho}{\delta}} \{t-\tau\}^q}$$
$$\le 2\varepsilon$$

since by the condition $b > \frac{1-\nu}{2} - \frac{\nu-\sigma}{\vartheta-1}$ we have $\frac{\sigma-ns_1}{\nu} < 1$. Via the condition $0 < \delta < \delta_c - \frac{1}{2}$ we get

$$|\alpha_n| + \beta n + \frac{n-1}{2} + \frac{n\rho_1}{\delta} > \delta,$$

for $n = 1, \vartheta - 1$ if $\beta = \frac{1}{2} - \gamma$, $\rho_1 = \gamma$, and $\gamma > 0$ is small. Similarly

$$\|\mathcal{G}(t) u_0\|_{r_1+q\nu,\rho,\omega} \le C \langle t\rangle^{\frac{\omega-\rho}{\delta}-\frac{1}{2\delta}} \{t\}^{-q} \|u_0\|_{r_1,0,\omega}$$

and

$$\left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau \right\|_{r_1+q\nu,\rho,\omega}$$
$$\leq C\int_0^t d\tau \,\langle t-\tau\rangle^{\frac{\omega-\rho}{\delta}-\frac{1}{2\delta}} \{t-\tau\}^{-q} \|\mathcal{N}(u)(\tau)\|_{r_1,0,\omega}\,.$$

By Lemma 2.57 we have

$$\|\mathcal{N}(u)\|_{r_1,0,\omega} \leq C\|u\|_{\mathbf{Y}}^2 \sum_{n=1}^{\vartheta-1} \{\tau\}^{-\frac{\sigma}{\nu}} \langle\tau\rangle^{\frac{\omega+\gamma}{\delta}-\frac{|\alpha_n|+\beta n}{\delta}-\frac{n}{2\delta}+\frac{\rho_1}{\delta}};$$

hence

$$\langle t\rangle^{\frac{\rho-\omega}{\delta}+\frac{1}{2\delta}}\{t\}^q \left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau \right\|_{r_1+q\nu,\rho,\omega}$$
$$\leq C\|u\|_{\mathbf{Y}}^2 \sum_{n=1}^{\vartheta-1} \langle t\rangle^{\frac{\rho-\omega}{\delta}+\frac{1}{2\delta}}\{t\}^q \int_0^t \langle t-\tau\rangle^{\frac{\omega-\rho}{\delta}-\frac{1}{2\delta}}$$
$$\times\{t-\tau\}^{-q}\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{\frac{\omega+\gamma}{\delta}-\frac{|\alpha_n|+\beta n}{\delta}-\frac{n}{2\delta}+\frac{\rho_1}{\delta}}\,d\tau \leq C$$

since $|\alpha_n| + \beta n + \frac{n}{2} - \omega > \delta$, for $n = 1, \vartheta - 1$, if we choose $\omega = \frac{1}{2} + \gamma$ with small $\gamma > 0$. Likewise we prove that

$$\left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(w)(\tau) - \mathcal{N}(v)(\tau)\,d\tau \right\|_{\mathbf{Y}} \leq C\|w-v\|_{\mathbf{Y}}\left(\|w\|_{\mathbf{Y}} + \|v\|_{\mathbf{Y}}\right).$$

Therefore by the global existence Theorem 1.17 there exists a unique solution $u(t,x) \in \mathbf{C}([T,\infty), \mathbf{H}^{r,\omega}(\mathbf{R}))$. Applying Lemma 2.56 to integral equation (2.122) we find the asymptotics (2.121) of the solution. Theorem 2.58 is proved.

2.6 Weak dissipation, strong dispersion

Consider the Cauchy problem

$$\begin{cases} u_t + \mathcal{N}(u) + \mathcal{L}u = 0,\ x \in \mathbf{R},\ t > 0, \\ u(0,x) = u_0(x),\ x \in \mathbf{R}; \end{cases} \tag{2.124}$$

here the linear part is a pseudodifferential operator defined by the Fourier transformation $\mathcal{L}u = \overline{\mathcal{F}}_{\xi\to x}(L(\xi)\,\widehat{u}(\xi))$, and the nonlinearity $\mathcal{N}(u)$ is a quadratic pseudodifferential operator

$$\mathcal{N}(u) = \overline{\mathcal{F}}_{\xi\to x}\int_{\mathbf{R}} a(\xi,y)\,\widehat{u}(t,\xi-y)\,\widehat{u}(t,y)\,dy,$$

defined by the symbol $a(\xi, y)$. We consider here the real-valued solutions $u(t,x)$. We suppose that the operators $\mathcal{N}$ and $\mathcal{L}$ have a finite order, that is the symbols $a(\xi, y)$ and $L(\xi)$ grow with respect to ξ and y no faster than a power:

$$|L(\xi)| \leq C\langle\xi\rangle^{\kappa}, |a(\xi,y)| \leq C(\langle\xi\rangle^{\kappa} + \langle y\rangle^{\kappa}),$$

where $C > 0$.

Model equation (2.124) contains, for example, the Korteweg-de Vries equation with linear dissipation

$$u_t + uu_x + u_{xxx} + \lambda u = 0, \ x \in \mathbf{R}, \ t > 0, \tag{2.125}$$

if we take $\mathcal{N}(u) = uu_x$, $\mathcal{L}u = u_{xxx} + \lambda u$, that is $a(\xi, y) = i\xi$, $L(\xi) = \lambda - i\xi^3$. Another example is the Benjamin-Ono equation with dissipation

$$u_t + uu_x + \mathcal{H}u_{xx} + \mathcal{L}u = 0, \ x \in \mathbf{R}, \ t > 0, \tag{2.126}$$

which comes from (2.124) if we assume $a(\xi,y) = i\xi$, $L(\xi) = \sqrt{|\xi|} + i\xi|\xi|$. For some other examples we refer to Naumkin and Shishmarev [1994b], Whitham [1999].

Suppose that the linear operator $\mathcal{L}$ satisfies the dissipation condition which in terms of the symbol $L(\xi)$ has the form

$$\operatorname{Re} L(\xi) \geq \mu\{\xi\}^{\delta}\langle\xi\rangle^{\nu} \tag{2.127}$$

for all $\xi \in \mathbf{R}$, where $\mu > 0$, $\nu \geq 0$, $\delta > 0$. Thus we assume that $\operatorname{Re} L(\xi)$ does not grow at all or grows slowly for $\xi \to \infty$. This behavior of the real part of the symbol $L(\xi)$ we refer by the weak dissipation.

Also we suppose that the symbol is smooth $L(\xi) \in \mathbf{C}^1(\mathbf{R})$ and has the estimate

$$\left|\partial_\xi^l L(\xi)\right| \leq C\{\xi\}^{\delta - l}\langle\xi\rangle^{\nu} \tag{2.128}$$

for all $\xi \in \mathbf{R} \setminus \{0\}$, $l = 0, 1$. We assume that the imaginary part $|\operatorname{Im} \partial_\xi L(\xi)|$ of the symbol $L(\xi)$ grows monotonically and satisfies the estimate

$$|\operatorname{Im} \partial_\xi L(\xi)| \geq C\langle\xi\rangle^{\rho} \tag{2.129}$$

for all $\xi \in \mathbf{R}$, where $\rho > \max\left(\frac{3\sigma - \nu}{2}, \sigma\right)$. This property we call by strong dispersion.

To find the asymptotic formulas for the solution we assume that the symbol $L(\xi)$ has the following asymptotic representation in the origin

$$L(\xi) = L_0(\xi) + O\left(|\xi|^{\delta + \gamma}\right) \tag{2.130}$$

for all $|\xi| \leq 1$, where $L_0(\xi) = \mu_1|\xi|^{\delta} + i\mu_2|\xi|^{\delta - 1}\xi$, $\mu_1 > 0$, $\mu_2 \in \mathbf{R}$, $\gamma > 0$.

We suppose that the symbols of the nonlinear operator $\mathcal{N}$ are such that

$$\left|\partial_\xi^l a(\xi, y)\right| \leq C\{\xi - y\}^{-l}\left(\{\xi - y\}^{\alpha}\langle\xi - y\rangle^{\sigma} + \{y\}^{\alpha}\langle y\rangle^{\sigma}\right) \tag{2.131}$$

for all $\xi, y \in \mathbf{R}$, $t > 0$, $l = 0, 1$, where $\alpha \geq 0$. Without a loss of generality we can assume the symmetry property $a(\xi, y) = a(\xi, \xi - y)$.

Denote

$$G_0(x) = \overline{\mathcal{F}}_{\xi \to x}\left(e^{-L_0(\xi)}\right)$$

and define spaces

$$\begin{aligned} \mathbf{A}^{0,\infty} &= \left\{\phi \in \mathbf{L}^2(\mathbf{R}) : \|\varphi\|_{\mathbf{A}^{0,\infty}} < \infty\right\}, \\ \mathbf{B}^{\sigma,1} &= \left\{\phi \in \mathbf{L}^2(\mathbf{R}) : \|\varphi\|_{\mathbf{B}^{\sigma,1}} < \infty\right\}, \\ \mathbf{D}^{0,\sigma} &= \left\{\phi \in \mathbf{L}^2(\mathbf{R}) : \|\varphi\|_{\mathbf{D}^{0,\sigma}} < \infty\right\} \end{aligned}$$

with the following norms

$$\|\varphi\|_{\mathbf{A}^{0,\infty}} = \|\widehat{\varphi}(\cdot)\|_{\mathbf{L}^\infty_\xi(|\xi| \leq 1)}, \ \|\varphi\|_{\mathbf{B}^{\sigma,1}} = \||\cdot|^\sigma \widehat{\varphi}(\cdot)\|_{\mathbf{L}^1_\xi(|\xi| \geq 1)}$$

and

$$\|\varphi\|_{\mathbf{D}^{0,\sigma}} = \||\partial_\xi|^\gamma |\cdot|^\sigma \widehat{\varphi}(\cdot)\|_{\mathbf{L}^\infty_\xi}$$

with some fixed $\gamma > 0$.

Theorem 2.59. *Let the linear operator $\mathcal{L}$ satisfy conditions (2.127) through (2.130) with $\delta < \delta_c = 1 + \alpha$. Suppose that the nonlinear operator $\mathcal{N}$ satisfies estimates (2.131). Let the initial data $u_0 \in \mathbf{A}^{0,\infty} \cap \mathbf{B}^{\sigma,1} \cap \mathbf{D}^{0,\sigma}$ and be sufficiently small. Then there exists a unique solution $u \in \mathbf{C}^0\left([0,\infty); \mathbf{A}^{0,\infty} \cap \mathbf{B}^{\sigma,1} \cap \mathbf{D}^{0,\sigma}\right)$ of the Cauchy problem (2.124). In the case of $\nu > 0$ we also have the smoothing property $u \in \mathbf{C}^\infty((0,\infty) \times \mathbf{R})$. Moreover, there exists a unique number A, such that the solution u has the following asymptotics*

$$u(t,x) = At^{-\frac{1}{\delta}} G_0\left(xt^{-\frac{1}{\delta}}\right) + O\left(t^{-\frac{1}{\delta} - \gamma_1}\right) \tag{2.132}$$

for large time $t > 0$ uniformly with respect to $x \in \mathbf{R}$, with some $\gamma_1 > 0$.

Remark 2.60. The conditions of the theorem on the initial data u_0 can also be expressed in terms of the usual weighted Sobolev spaces as follows

$$\|u_0\|_{\mathbf{H}^{\omega+\sigma,\omega}} \leq \varepsilon,$$

where $\omega > \frac{1}{2}$. However, the conditions on the initial data u_0 are described more precisely in the norms $\mathbf{A}^{0,\infty} \cap \mathbf{B}^{\sigma,1} \cap \mathbf{D}^{0,\sigma}$.

Remark 2.61. As an example we apply Theorem 2.59 to the Korteweg-de Vries equation with linear dissipation (2.125). We have $\sigma = \alpha = 1$, $\nu = \delta = 0$, and $\rho = 2$. Therefore if the initial data $u_0 \in \mathbf{H}^{1+\omega,\omega}(\mathbf{R})$ are sufficiently small, where $\omega > \frac{1}{2}$, then there exists a unique solution $u(t,x) \in \mathbf{C}^\infty((0,\infty) \times \mathbf{R})$, and asymptotics

$$u(t,x) = e^{-\lambda t} t^{-\frac{1}{3}} G_0\left(xt^{-\frac{1}{3}}\right) \int u_0(x)\, dx + O\left(e^{-2\lambda t}\right)$$

is true for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$.

To apply results of Section2.1 we obtain in the next subsection some preliminary estimates for the linear and nonlinear operators of equation (2.124) and give large time asymptotics formulas for the linearized Cauchy problem.

2.6.1 Preliminary Lemmas

First we collect some preliminary estimates for the Green operator $\mathcal{G}(t)\phi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-L(\xi)t}\hat{\phi}(\xi)\right)$ in the norms

$$\|\varphi(t)\|_{\mathbf{A}^{\rho,p}} = \||\cdot|^{\rho}\widehat{\varphi}(t)\|_{\mathbf{L}^p_\xi(|\xi|\le 1)}, \quad \|\varphi(t)\|_{\mathbf{B}^{s,p}} = \||\cdot|^{s}\widehat{\varphi}(t)\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)},$$
$$\|\varphi(t)\|_{\mathbf{D}^{\rho,s}} = \||\partial_\xi|^{\gamma}\{\cdot\}^{\rho}\langle\cdot\rangle^{s}\widehat{\varphi}(t)\|_{\mathbf{L}^\infty_\xi},$$

where $\rho, s \in \mathbf{R}$, $\gamma \in (0,1)$. The norm $\mathbf{A}^{\rho,p}$ is responsible for the large time asymptotic properties of solutions and the norm $\mathbf{B}^{s,p}$ describes the regularity of solutions. Applying Lemma 1.38 we obtain

Lemma 2.62. *Let the linear operator $\mathcal{L}$ satisfy dissipation conditions (2.127) and (2.128). Then the estimates are valid for all $t > 0$*

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{A}^{\rho,p}} \le C\langle t\rangle^{-\frac{1}{\delta}\left(\rho+\frac{1}{p}-\frac{1}{q}\right)}\|\varphi\|_{\mathbf{A}^{0,q}},$$

for $\rho \ge 0$, if $p = q$ and $\rho + \frac{1}{p} - \frac{1}{q} > 0$ if $1 \le p < q \le \infty$,

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{B}^{s,p}} \le Ce^{-\frac{\mu}{2}t}\{t\}^{-\frac{s}{\nu}}\|\varphi\|_{\mathbf{B}^{0,p}}$$

where $1 \le p \le \infty$, $s \ge 0$; in the case of $\nu = 0$ we take $s = 0$, and

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{D}^{\rho,s}} \le C\langle t\rangle^{-\frac{\rho}{\delta}}\{t\}^{-\frac{s}{\nu}}\|\varphi\|_{\mathbf{D}^{0,0}}$$
$$+ C\langle t\rangle^{\frac{\gamma-\rho}{\delta}}\{t\}^{-\frac{s}{\nu}}\left(\|\varphi\|_{\mathbf{A}^{0,\infty}} + \|\varphi\|_{\mathbf{B}^{0,\infty}}\right)$$

where $1 \le p \le q \le \infty$, $s \ge 0$, $\rho \ge 0$, $\gamma \in (0,1)$ is such that $\gamma < \delta$ if $\rho = 0$ and $\gamma < \min(\rho,\delta)$ if $\rho > 0$.

Define $\mathcal{G}_0(t)\phi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-L_0(\xi)t}\hat{\phi}(\xi)\right)$, where the symbol $L_0(\xi) = \mu_1|\xi|^{\delta} + i\mu_2|\xi|^{\delta-1}\xi$ is homogeneous, $\mu_1, \mu_2 \in \mathbf{R}$ and $G_0(x) = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-L_0(\xi)}\right)$. By Lemma 1.39 in the next lemma we find the asymptotic formulas for the Green operator $\mathcal{G}(t)$.

Lemma 2.63. *Let the linear operator $\mathcal{L}$ satisfy dissipation conditions (2.127) through (2.128) and asymptotic representation (2.130). Then the asymptotics are true*

$$\mathcal{G}(t)\phi = t^{-\frac{1}{\delta}}G_0\left(xt^{-\frac{1}{\delta}}\right)\hat{\phi}(0)$$
$$+ O\left(\langle t\rangle^{-\frac{1+\gamma}{\delta}}\left(\|\phi\|_{\mathbf{A}^{0,\infty}} + \|\phi\|_{\mathbf{B}^{0,1}} + \|\phi\|_{\mathbf{D}^{0,0}}\right)\right)$$

and

$$\int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau = t^{-\frac{1}{\delta}} G_0\left(xt^{-\frac{1}{\delta}}\right)\int_0^\infty \widehat{\psi}(\tau,0)\,d\tau$$
$$+O\left(\langle t\rangle^{-\frac{1+\gamma}{\delta}} \sup_{\tau\in(0,t)} \langle\tau\rangle^{\theta+\frac{1}{\delta}} \left(\|\psi(\tau)\|_{\mathbf{A}^{0,1}} + \|\psi(\tau)\|_{\mathbf{B}^{0,1}}\right)\right)$$
$$+O\left(\langle t\rangle^{-\frac{1+\gamma}{\delta}} \sup_{\tau\in(0,t)} \langle\tau\rangle^{\theta} \left(\|\psi(\tau)\|_{\mathbf{A}^{0,\infty}} + \|\psi(\tau)\|_{\mathbf{D}^{0,0}}\right)\right)$$

as $t>0$ *uniformly with respect to* $x\in\mathbf{R}$, *where* $\theta\geq 1+\frac{\gamma}{\delta}$, $\gamma>0$.

Now let us prove the following estimates.

Lemma 2.64. *Let* $s\in[0,\nu)$ *if* $\nu>0$ *and* $s=0$ *if* $\nu=0$. *The following estimates are true for all* $t>0$

$$\left\|\int_0^t \mathcal{G}(t-\tau)f(\tau)\,d\tau\right\|_{\mathbf{A}^{\rho,p}} \leq C\langle t\rangle^{-\frac{\rho}{\delta}-\frac{1}{\delta p}} \sup_{1\leq q\leq p}\sup_{\tau>0}\langle\tau\rangle^{\theta+\frac{1}{\delta q}}\|f(\tau)\|_{\mathbf{A}^{0,q}}$$

for $\theta>1$, $\rho>-1$ *if* $p=1$ *and* $\rho\geq 0$ *if* $p=\infty$,

$$\left\|\int_0^t \mathcal{G}(t-\tau)f(\tau)\,d\tau\right\|_{\mathbf{B}^{s,p}} \leq C\langle t\rangle^{-\theta-\frac{1}{\delta p}} \sup_{\tau>0}\langle\tau\rangle^{\theta+\frac{1}{\delta p}}\|f(\tau)\|_{\mathbf{B}^{0,p}}$$

for $1\leq p\leq\infty$, *and*

$$\left\|\int_0^t \mathcal{G}(t-\tau)f(\tau)\,d\tau\right\|_{\mathbf{D}^{\rho,s}} \leq C\langle t\rangle^{\frac{\gamma-\rho}{\delta}}\left(\sup_{\tau>0}\langle\tau\rangle^{\theta-\frac{\gamma}{\delta}}\|f(\tau)\|_{\mathbf{D}^{0,0}}\right.$$
$$\left.+\sup_{\tau>0}\langle\tau\rangle^{\theta}\left(\|f(\tau)\|_{\mathbf{A}^{0,\infty}}+\|f(\tau)\|_{\mathbf{B}^{0,\infty}}\right)\right)$$

for $\rho=0,\alpha$, *and* $\gamma\in(0,1)$ *such that* $\gamma<\delta$ *if* $\alpha=0$ *and* $\gamma<\min(\alpha,\delta)$ *if* $\alpha>0$.

Proof. By virtue of the first estimate of Lemma 2.62 we obtain for $\rho>-1$ if $p=1$ and $\rho\geq 0$ if $p=\infty$

$$\left\|\int_0^t \mathcal{G}(t-\tau)f(\tau)\,d\tau\right\|_{\mathbf{A}^{\rho,p}}$$
$$\leq C\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-\frac{\rho}{\delta}-\frac{1}{\delta p}}\langle\tau\rangle^{-\theta}\,d\tau\sup_{\tau>0}\langle\tau\rangle^{\theta}\|f(\tau)\|_{\mathbf{A}^{0,\infty}}$$
$$+C\int_{\frac{t}{2}}^t\langle t-\tau\rangle^{-\frac{\rho}{\delta}}\langle\tau\rangle^{-\frac{1}{\delta p}-\theta}\,d\tau\sup_{\tau>0}\langle\tau\rangle^{\theta+\frac{1}{\delta p}}\|f(\tau)\|_{\mathbf{A}^{0,p}}$$
$$\leq C\langle t\rangle^{-\frac{\rho}{\delta}-\frac{1}{\delta p}}\sup_{p\leq q\leq\infty}\sup_{\tau>0}\langle\tau\rangle^{\theta+\frac{1}{\delta q}}\|f(\tau)\|_{\mathbf{A}^{0,q}}.$$

Similarly by the second estimate of Lemma 2.62 we get

$$\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{B}^{s,p}} \leq C \int_0^t e^{-\frac{\mu}{2}(t-\tau)} \langle \tau \rangle^{-\theta - \frac{\gamma}{\delta} - \frac{1}{\delta p}} \{t-\tau\}^{-\frac{s}{\nu}}\, d\tau$$
$$\times \sup_{\tau>0} \langle \tau \rangle^{\theta + \frac{1}{\delta p}} \|f(\tau)\|_{\mathbf{B}^{0,p}} \leq C \langle t \rangle^{-\theta - \frac{1}{\delta p}} \sup_{\tau>0} \langle \tau \rangle^{\theta + \frac{1}{\delta p}} \|f(\tau)\|_{\mathbf{B}^{0,p}}.$$

for all $t > 0$, where $1 \leq p \leq \infty$, $s \in [0, \nu)$ if $\nu > 0$ and $s = 0$ if $\nu = 0$. Finally by the third estimate of Lemma 2.62 we find

$$\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{D}^{\rho,s}} \leq C \int_0^t d\tau \langle t-\tau \rangle^{-\frac{\rho}{\delta}} \{t-\tau\}^{-\frac{s}{\nu}} \|f(\tau)\|_{\mathbf{D}^{0,0}}$$
$$+ C \int_0^t d\tau \langle t-\tau \rangle^{\frac{\gamma-\rho}{\delta}} \{t-\tau\}^{-\frac{s}{\nu}} \left(\|f(\tau)\|_{\mathbf{A}^{0,\infty}} + \|f(\tau)\|_{\mathbf{B}^{0,\infty}} \right)$$
$$\leq C \int_0^t \langle t-\tau \rangle^{-\frac{\rho}{\delta}} \langle \tau \rangle^{\frac{\gamma}{\delta} - \theta} \{t-\tau\}^{-\frac{s}{\nu}}\, d\tau \sup_{\tau>0} \langle \tau \rangle^{\theta - \frac{\gamma}{\delta}} \|f(\tau)\|_{\mathbf{D}^{0,0}}$$
$$+ C \int_0^t \langle t-\tau \rangle^{\frac{\gamma-\rho}{\delta}} \langle \tau \rangle^{-\theta} \{t-\tau\}^{-\frac{s}{\nu}}\, d\tau$$
$$\times \sup_{\tau>0} \langle \tau \rangle^{\theta} \left(\|f(\tau)\|_{\mathbf{A}^{0,\infty}} + \|f(\tau)\|_{\mathbf{B}^{0,\infty}} \right)$$

for $s \in [0, \nu)$, $\rho = 0, \alpha$. Thus the third estimate of the lemma is true. Lemma 2.64 is proved.

In the next lemma we estimate the nonlinear terms

$$\mathcal{N}_1(\phi) = \overline{\mathcal{F}}_{\xi \to x} \int_{\mathbf{R}} a_1(\xi, y) \widehat{\phi}(\xi - y) \widehat{\phi}(y)\, dy$$

and

$$\mathcal{N}_2(\phi, \varphi) = \overline{\mathcal{F}}_{\xi \to x} \int_{\mathbf{R}^2} a_3(\xi, y, z) \widehat{\phi}(\xi - y) \widehat{\phi}(y - z) \widehat{\varphi}(z)\, dy dz.$$

Lemma 2.65. *Suppose that*

$$|a_1(\xi, y)| \leq C \left(\{\xi - y\}^{\alpha} \langle \xi - y \rangle^{\sigma} + \{y\}^{\alpha} \langle y \rangle^{\sigma} \right)$$

for $|\xi| \leq 1$, $y \in \mathbf{R}$ *and*

$$|a_1(\xi, y)| \leq C \left(\{\xi - y\}^{\beta} \{y\}^{\lambda} + \{\xi - y\}^{\lambda} \{y\}^{\beta} \right) \langle \xi - y \rangle^{\sigma} \langle y \rangle^{\sigma}$$

for $|\xi| \geq 1$, $y \in \mathbf{R}$. *Also assume that*

$$|a_2(\xi, y, z)| \leq C \{\xi - y\}^{\lambda} \left(\{\xi - y + z\}^{\lambda} + \{\xi - z\}^{\lambda} \right)$$
$$\times \left(\{\xi - y\}^{\beta} + \{y - z\}^{\beta} + \{z\}^{\beta} \right) \langle y - z \rangle^{\sigma_1} \langle z \rangle^{\sigma_2}$$

for $|\xi| \geq 1,\ y, z \in \mathbf{R}$, *where* $\beta \geq \alpha \geq 0,\ \lambda > -1,\ \sigma,\ \sigma_1,\ \sigma_2 \geq 0$. *Then the estimates*

$$\|\mathcal{N}_1(\varphi)\|_{\mathbf{A}^{0,p}} \leq C\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}} + \|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,p}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,p}}\right),$$
$$\|\mathcal{N}_1(\varphi)\|_{\mathbf{B}^{0,p}} \leq C\left(\|\varphi(t)\|_{\mathbf{A}^{\lambda,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\beta,p}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,p}}\right)$$

and

$$\begin{aligned}
&\|\mathcal{N}_2(\phi,\varphi)\|_{\mathbf{B}^{0,p}} \leq C\left(\|\phi\|_{\mathbf{A}^{\beta+\lambda,1}} + \|\phi\|_{\mathbf{B}^{0,1}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}} + \|\phi\|_{\mathbf{B}^{\sigma_1,p}}\right)\\
&\times\left(\|\varphi\|_{\mathbf{A}^{0,\infty}} + \|\varphi\|_{\mathbf{B}^{0,\infty}} + \|\varphi\|_{\mathbf{B}^{\sigma_2,1}}\right)\\
&+ C\left(\|\phi\|_{\mathbf{A}^{0,\infty}} + \|\phi\|_{\mathbf{B}^{0,\infty}} + \|\phi\|_{\mathbf{B}^{0,1}}\right)\left(\|\phi\|_{\mathbf{A}^{\beta,p}} + \|\phi\|_{\mathbf{B}^{\sigma_1,p}}\right)\\
&\times\left(\|\varphi\|_{\mathbf{A}^{0,\infty}} + \|\varphi\|_{\mathbf{B}^{0,\infty}} + \|\varphi\|_{\mathbf{B}^{\sigma_2,1}}\right)\\
&+ C\left(\|\phi\|_{\mathbf{A}^{0,\infty}} + \|\phi\|_{\mathbf{B}^{0,\infty}} + \|\phi\|_{\mathbf{B}^{0,1}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}} + \|\phi\|_{\mathbf{B}^{\sigma_1,p}}\right)\\
&\times\left(\|\varphi\|_{\mathbf{A}^{\beta,\infty}} + \|\varphi\|_{\mathbf{B}^{0,\infty}} + \|\varphi\|_{\mathbf{B}^{\sigma_2,1}}\right)
\end{aligned}$$

are true, where $\beta \geq \alpha \geq 0$.

Proof. By virtue of the Young inequality we obtain

$$\begin{aligned}
&\|\mathcal{N}_1(\varphi)\|_{\mathbf{A}^{0,p}} \leq \left\|\int_{\mathbf{R}} |a(\xi,y)|\,|\widehat{\varphi}(t,\xi-y)\,\widehat{\varphi}(t,y)|\,dy\right\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)}\\
&\leq C\left\|\int_{\mathbf{R}} \left(\{\xi-y\}^\alpha\langle\xi-y\rangle^\sigma + \{y\}^\alpha\langle y\rangle^\sigma\right)|\widehat{\varphi}(t,\xi-y)\,\widehat{\varphi}(t,y)|\,dy\right\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|\mathcal{N}_1(\varphi)\|_{\mathbf{A}^{0,p}} \leq C\,\|\widehat{\varphi}(t,\cdot)\|_{\mathbf{L}^1_\xi}\Big(\|\{\cdot\}^\alpha\,\widehat{\varphi}(t,\cdot)\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)}\\
&+ \|\langle\cdot\rangle^\sigma\,\widehat{\varphi}(t,\cdot)\|_{\mathbf{L}^p_\xi(|\xi|>1)}\Big)\\
&\leq C\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}} + \|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,p}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,p}}\right).
\end{aligned}$$

In the same manner we have

$$\begin{aligned}
&\|\mathcal{N}_1(\varphi)\|_{\mathbf{B}^{0,p}} \leq C\left\|\langle\cdot\rangle^\sigma\{\cdot\}^\lambda\,\widehat{\varphi}(t,\cdot)\right\|_{\mathbf{L}^1_\xi}\left\|\langle\cdot\rangle^\sigma\{\cdot\}^\beta\,\widehat{\varphi}(t,\cdot)\right\|_{\mathbf{L}^p_\xi}\\
&\leq C\left(\|\varphi(t)\|_{\mathbf{A}^{\lambda,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\beta,p}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,p}}\right).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\left\| \mathcal{N}_2(\phi,\varphi)\right\|_{\mathbf{B}^{0,p}} \\
&\le C \left\| \int_{\mathbf{R}^2} \{\xi - y\}^{\lambda} \left(\{\xi - y + z\}^{\lambda} + \{\xi - z\}^{\lambda} \right) \langle y - z\rangle^{\sigma_1} \langle z\rangle^{\sigma_2} \right. \\
&\times \left. \left(\{\xi - y\}^{\beta} + \{y - z\}^{\beta} + \{z\}^{\beta} \right) \left| \widehat{\phi}(\xi - y) \widehat{\phi}(y - z) \widehat{\varphi}(z) \right| \right\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)} \\
&\le C \left\| \{\cdot\}^{\beta+\lambda} \widehat{\phi}(\cdot) \right\|_{\mathbf{L}^1_\xi} \left\| \langle\cdot\rangle^{\sigma_1} \widehat{\phi}(\cdot) \right\|_{\mathbf{L}^p_\xi} \\
&\times \left(\left\| \langle\cdot\rangle^{\sigma_2} \widehat{\varphi}(\cdot) \right\|_{\mathbf{L}^\infty_\xi} + \left\| \langle\cdot\rangle^{\sigma_2} \widehat{\varphi}(\cdot) \right\|_{\mathbf{L}^1_\xi} \right) \\
&+ C \left(\left\| \widehat{\phi} \right\|_{\mathbf{L}^\infty_\xi} + \left\| \widehat{\phi} \right\|_{\mathbf{L}^1_\xi} \right) \left\| \{\cdot\}^{\beta} \langle\cdot\rangle^{\sigma_1} \widehat{\phi}(\cdot) \right\|_{\mathbf{L}^p_\xi} \\
&\times \left(\left\| \langle\cdot\rangle^{\sigma_2} \widehat{\varphi}(\cdot) \right\|_{\mathbf{L}^\infty_\xi} + \left\| \langle\cdot\rangle^{\sigma_2} \widehat{\varphi}(\cdot) \right\|_{\mathbf{L}^1_\xi} \right) \\
&+ C \left(\left\| \widehat{\phi} \right\|_{\mathbf{L}^\infty_\xi} + \left\| \widehat{\phi} \right\|_{\mathbf{L}^1_\xi} \right) \left\| \langle\cdot\rangle^{\sigma_1} \widehat{\phi}(\cdot) \right\|_{\mathbf{L}^p_\xi} \\
&\times \left(\left\| \{\cdot\}^{\beta} \langle\cdot\rangle^{\sigma_2} \widehat{\varphi}(\cdot) \right\|_{\mathbf{L}^\infty_\xi} + \left\| \{\cdot\}^{\beta} \langle\cdot\rangle^{\sigma_2} \widehat{\varphi}(\cdot) \right\|_{\mathbf{L}^1_\xi} \right);
\end{aligned}$$

hence, the last estimate of the lemma follows. Lemma 2.65 is proved.

Now we estimate the nonlinearity $\mathcal{N}(u)$ in the norm $\mathbf{D}^{0,0}$.

Lemma 2.66. *Let condition (2.131) be fulfilled. Then the inequality*

$$\begin{aligned}
&\left\| \mathcal{N}(\varphi)\right\|_{\mathbf{D}^{0,0}} \le C \left\| \varphi(t)\right\|_{\mathbf{D}^{\alpha,\sigma}} \left(\left\| \varphi(t)\right\|_{\mathbf{A}^{0,1}} + \left\| \varphi(t)\right\|_{\mathbf{B}^{0,1}} \right) \\
&+ C \left\| \varphi(t)\right\|_{\mathbf{D}^{0,0}} \left(\left\| \varphi(t)\right\|_{\mathbf{A}^{\alpha,1}} + \left\| \varphi(t)\right\|_{\mathbf{B}^{\sigma,1}} \right) \\
&+ C \left(\left\| \varphi(t)\right\|_{\mathbf{A}^{\alpha-\gamma,1}} + \left\| \varphi(t)\right\|_{\mathbf{B}^{\sigma,1}} \right) \left(\left\| \varphi(t)\right\|_{\mathbf{A}^{0,\infty}} + \left\| \varphi(t)\right\|_{\mathbf{B}^{0,\infty}} \right) \\
&+ C \left(\left\| \varphi(t)\right\|_{\mathbf{A}^{-\gamma,1}} + \left\| \varphi(t)\right\|_{\mathbf{B}^{0,1}} \right) \left(\left\| \varphi(t)\right\|_{\mathbf{A}^{\alpha,\infty}} + \left\| \varphi(t)\right\|_{\mathbf{B}^{\sigma,\infty}} \right)
\end{aligned}$$

is valid for $1 \le p \le \infty$, provided that the right-hand side is bounded.

Proof. Denote

$$\widetilde{a}(\xi, y) = a(\xi, y) \left(\{\xi - y\}^{\alpha} \langle \xi - y\rangle^{\sigma} + \{y\}^{\alpha} \langle y\rangle^{\sigma} \right)^{-1}$$

and

$$\Phi(t,\xi,y) = \left(\{\xi - y\}^{\alpha} \langle \xi - y\rangle^{\sigma} + \{y\}^{\alpha} \langle y\rangle^{\sigma} \right) \widehat{\varphi}(t, \xi - y) \widehat{\varphi}(t, y).$$

We have

$$\begin{aligned}
&\left\| \mathcal{N}(\varphi)\right\|_{\mathbf{D}^{0,0}} = \left\| |\partial_\xi|^{\gamma} \int_{\mathbf{R}} \widetilde{a}(\cdot, y) \Phi(t, \cdot, y)\, dy \right\|_{\mathbf{L}^\infty_\xi} \\
&\le \left\| \int_{\mathbf{R}} \int_{\mathbf{R}} \left| \Phi(t, \cdot - \eta, y) - \Phi(t, \cdot, y) \right| \widetilde{a}(\cdot - \eta, y) \frac{d\eta dy}{|\eta|^{1+\gamma}} \right\|_{\mathbf{L}^\infty_\xi} \\
&+ \left\| \int_{\mathbf{R}} \Phi(t, \cdot, y) |\partial_\xi|^{\gamma} \widetilde{a}(\cdot, y)\, dy \right\|_{\mathbf{L}^\infty_\xi}. \qquad (2.133)
\end{aligned}$$

By virtue of condition (2.131) we estimate the first summand as follows

$$\begin{aligned}
&\left\| \int_{\mathbf{R}} \int_{\mathbf{R}} |\Phi(t, \cdot - \eta, y) - \Phi(t, \cdot, y)| \, \widetilde{a}(\cdot - \eta, y) \frac{d\eta dy}{|\eta|^{1+\gamma}} \right\|_{\mathbf{L}^\infty_\xi} \\
&\leq C \left\| \int_{\mathbf{R}} \left(|\partial_\xi|^\gamma \{\cdot - y\}^\alpha \langle \cdot - y \rangle^\sigma \widehat{\varphi}(t, \cdot - y) \right) \widehat{\varphi}(t, y) \, dy \right\|_{\mathbf{L}^\infty_\xi} \\
&+ C \left\| \int_{\mathbf{R}} \left(|\partial_\xi|^\gamma \widehat{\varphi}(t, \cdot - y) \right) \{y\}^\alpha \langle y \rangle^\sigma \widehat{\varphi}(t, y) \, dy \right\|_{\mathbf{L}^\infty_\xi} \\
&\leq C \left\| |\partial_\xi|^\gamma \{\cdot\}^\alpha \langle \cdot \rangle^\sigma \widehat{\varphi}(t, \cdot) \right\|_{\mathbf{L}^\infty_\xi} \|\widehat{\varphi}(t)\|_{\mathbf{L}^1_\xi} \\
&+ C \left\| |\partial_\xi|^\gamma \widehat{\varphi}(t) \right\|_{\mathbf{L}^\infty_\xi} \|\{\cdot\}^\alpha \langle \cdot \rangle^\sigma \widehat{\varphi}(t, \cdot)\|_{\mathbf{L}^1_\xi} \\
&\leq C \|\varphi(t)\|_{\mathbf{D}^{\alpha,\sigma}} \left(\|\varphi(t)\|_{\mathbf{A}^{0,1}} + \|\varphi(t)\|_{\mathbf{B}^{0,1}} \right) \\
&+ C \|\varphi(t)\|_{\mathbf{D}^{0,0}} \left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}} \right). \qquad (2.134)
\end{aligned}$$

Via (2.131) we obtain

$$\begin{aligned}
&|\partial_\xi|^\gamma \widetilde{a}(\xi, y) = \int |\widetilde{a}(\xi - \eta, y) - \widetilde{a}(\xi, y)| \frac{d\eta}{|\eta|^{1+\gamma}} \\
&\leq C \int_{|\eta| \geq \frac{1}{2}\{\xi - y\}} \frac{d\eta}{|\eta|^{1+\gamma}} + C \{\xi - y\}^{-1} \int_{|\eta| \leq \frac{1}{2}\{\xi - y\}} \frac{d\eta}{|\eta|^\gamma} \leq C \{\xi - y\}^{-\gamma}
\end{aligned}$$

for all $\xi, y \in \mathbf{R}$. Therefore,

$$\begin{aligned}
&\left\| \int_{\mathbf{R}} \Phi(t, \cdot, y) \, |\partial_\xi|^\gamma \widetilde{a}(\cdot, y) \, dy \right\|_{\mathbf{L}^\infty_\xi} \\
&\leq C \left\| \int_{\mathbf{R}} \{\cdot - y\}^{\alpha - \gamma} \langle \cdot - y \rangle^\sigma \widehat{\varphi}(t, \cdot - y) \, \widehat{\varphi}(t, y) \, dy \right\|_{\mathbf{L}^\infty_\xi} \\
&+ C \left\| \int_{\mathbf{R}} \{\cdot - y\}^{-\gamma} \widehat{\varphi}(t, \cdot - y) \{y\}^\alpha \langle y \rangle^\sigma \widehat{\varphi}(t, y) \, dy \right\|_{\mathbf{L}^\infty_\xi} \\
&\leq C \left\| \{\cdot\}^{\alpha - \gamma} \langle \cdot \rangle^\sigma \widehat{\varphi}(t, \cdot) \right\|_{\mathbf{L}^1_\xi} \|\widehat{\varphi}(t)\|_{\mathbf{L}^\infty_\xi} \\
&+ C \left\| \{\cdot\}^{-\gamma} \widehat{\varphi}(t, \cdot) \right\|_{\mathbf{L}^1_\xi} \|\{\cdot\}^\alpha \langle \cdot \rangle^\sigma \widehat{\varphi}(t, \cdot)\|_{\mathbf{L}^\infty_\xi} \\
&\leq C \left(\|\varphi(t)\|_{\mathbf{A}^{\alpha-\gamma,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}} \right) \left(\|\varphi(t)\|_{\mathbf{A}^{0,\infty}} + \|\varphi(t)\|_{\mathbf{B}^{0,\infty}} \right) \\
&+ C \left(\|\varphi(t)\|_{\mathbf{A}^{-\gamma,1}} + \|\varphi(t)\|_{\mathbf{B}^{0,1}} \right) \left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,\infty}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,\infty}} \right). \qquad (2.135)
\end{aligned}$$

In view of (2.133) through (2.135) we get the estimate of the lemma. Lemma 2.66 is finally proved.

2.6.2 Proof of Theorem 2.59

First we prove the existence of classical solutions for the Cauchy problem (2.124) in the spaces $\mathbf{A}^{\alpha,p} \cap \mathbf{B}^{\sigma,p}$. Applying the Fourier transformation to equation (2.124) we transform equation (2.124) by the method similar to the normal forms of Shatah (see Shatah [1985])

$$\widehat{u}_t(t,\xi) + L(\xi)\,\widehat{u}(t,\xi) = -\int_{\mathbf{R}} a(\xi,y)\,\widehat{u}(t,\xi-y)\,\widehat{u}(t,y)\,dy.$$

We put $v(t,\xi) = \widehat{u}(t,\xi)\,e^{-tL(\xi)}$, then we get

$$\partial_t v(t,\xi) = \int_{\mathbf{R}} e^{tQ(\xi,y)} a(\xi,y)\,v(t,\xi-y)\,v(t,y)\,dy,$$

where $Q(\xi,y) \equiv L(\xi) - L(\xi-y) - L(y)$. Integrating by parts with respect to time, and considering the symmetry property $a(\xi,y) = a(\xi,\xi-y)$ we obtain

$$\begin{aligned} v(t,\xi) &= v(0,\xi) + \int_0^t d\tau \int_{\mathbf{R}} e^{\tau Q(\xi,y)} a_2(\xi,y)\,v(\tau,\xi-y)\,v(\tau,y)\,dy \\ &+ \int_{\mathbf{R}} e^{\tau Q(\xi,y)} a_1(\xi,y)\,v(\tau,\xi-y)\,v(\tau,y)\,dy\Big|_{\tau=0}^{\tau=t} \\ &+ 2\int_0^t d\tau \int_{\mathbf{R}} dy e^{\tau \widetilde{Q}(\xi,y,z)} a_1(\xi,y)\,v(\tau,\xi-y) \\ &\times \int_{\mathbf{R}} a(y,z)\,v(\tau,y-z)\,v(\tau,z)\,dz, \end{aligned} \tag{2.136}$$

where

$$\widetilde{Q}(\xi,y,z) = Q(\xi,y) + Q(y,z),$$

$$\begin{aligned} a_1(\xi,y) &= \frac{a(\xi,y)}{Q(\xi,y)}\psi_1(\xi,y), \\ a_2(\xi,y) &= a(\xi,y)\,\psi_2(\xi,y), \end{aligned}$$

$\psi_2(\xi,y) = 1 - \psi_1(\xi,y)$, and $\psi_1(\xi,y)$ is the characteristic function of the domain

$$\left\{|\xi|^{-\rho} \le |y| \le \frac{|\xi|}{8}, |\xi| \ge 1\right\} \cup \left\{|\xi|^{-\rho} \le |\xi-y| \le \frac{|\xi|}{8}, |\xi| \ge 1\right\}.$$

We integrate again by parts in the last integral in (2.136) to get

$$v(t,\xi) = v(0,\xi) + \int_0^t d\tau \int_{\mathbf{R}} e^{\tau Q(\xi,y)} a_2(\xi,y)\, v(\tau,\xi-y)\, v(\tau,y)\, dy$$
$$+ \int_0^t d\tau \int_{\mathbf{R}^2} e^{\tau \widetilde{Q}(\xi,y,z)} a_4(\xi,y,z)\, v(\tau,\xi-y)\, v(\tau,y-z)\, v(\tau,z)\, dydz$$
$$+ \int_{\mathbf{R}} e^{\tau Q(\xi,y)} a_1(\xi,y)\, v(\tau,\xi-y)\, v(\tau,y)\, dy \Big|_{\tau=0}^{\tau=t}$$
$$+ \int_{\mathbf{R}^2} e^{\tau \widetilde{Q}(\xi,y,z)} a_3(\xi,y,z)\, v(\tau,\xi-y)\, v(\tau,y-z)\, v(\tau,z)\, dydz \Big|_{\tau=0}^{\tau=t}$$
$$+ 3\int_0^t d\tau \int_{\mathbf{R}^2} dydz\; e^{\tau \widetilde{Q}(\xi,y,z)} a_3(\xi,y,z)\, v(\tau,\xi-y)\, v(\tau,y-z)$$
$$\times \int_{\mathbf{R}} a(z,q)\, e^{\tau Q(z,q)} v(\tau,z-q)\, v(\tau,q)\, dq, \tag{2.137}$$

where

$$a_3(\xi,y,z) = \frac{2\psi_3(\xi,y,z)}{\widetilde{Q}(\xi,y,z)} a_1(\xi,y)\, a(y,z),$$
$$a_4(\xi,y,z) = 2\psi_4(\xi,y,z)\, a_1(\xi,y)\, a(y,z)$$

$\psi_4(\xi,y,z) = 1 - \psi_3(\xi,y,z)$, and $\psi_3(\xi,y,z)$ is the characteristic function of the domain

$$\left\{|\xi|^{-\rho} \le |\xi-y| \le \frac{|\xi|}{8}, |\xi| \ge 1\right\} \times \left(\left\{|\xi|^{-\rho} \le |\xi-y+z| \le \frac{|\xi|}{8}\right\}\right.$$
$$\left.\cup \left\{|\xi|^{-\rho} \le |\xi-z| \le \frac{|\xi|}{8}\right\}\right).$$

Returning to the solution u we write equation (2.137) in the following manner

$$(\partial_t + \mathcal{L})(u - \mathcal{N}_1(u) - \mathcal{N}_3(u,u)) = \mathcal{N}_2(u) + \mathcal{N}_4(u) + 3\mathcal{N}_3(u, \mathcal{N}(u)), \tag{2.138}$$

where we denote

$$\mathcal{N}_j(u) = \overline{\mathcal{F}}_{\xi\to x} \int_{\mathbf{R}} a_j(\xi,y)\, \widehat{u}(\tau,\xi-y)\, \widehat{u}(\tau,y)\, dy,\ j = 1,2,$$
$$\mathcal{N}_3(u,\phi) = \overline{\mathcal{F}}_{\xi\to x} \int_{\mathbf{R}^2} a_3(\xi,y,z)\, \widehat{u}(\tau,\xi-y)\, \widehat{u}(\tau,y-z)\, \widehat{\phi}(\tau,z)\, dydz,$$
$$\mathcal{N}_4(u) = \overline{\mathcal{F}}_{\xi\to x} \int_{\mathbf{R}^2} a_4(\xi,y,z)\, \widehat{u}(\tau,\xi-y)\, \widehat{u}(\tau,y-z)\, \widehat{u}(\tau,z)\, dydz.$$

By the Duhamel formula we write (2.138) as the integral equation

$$u = \mathcal{N}_1(u) + \mathcal{N}_3(u,u) + \mathcal{G}(t)(u_0 - \mathcal{N}_1(u_0) - \mathcal{N}_3(u_0,u_0))$$
$$+ \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}_2(u) + \mathcal{N}_4(u) + 3\mathcal{N}_3(u,\mathcal{N}(u)))\, d\tau. \tag{2.139}$$

Now we apply Theorem 1.17. Denote

$$\mathbf{X}=\{u\in\mathbf{C}\left((0,\infty);\mathbf{A}^{0,\infty}\cap\mathbf{B}^{\sigma,1}\right):\|u\|_{\mathbf{X}}\le C\varepsilon\},$$

where

$$\begin{aligned}\|\phi\|_{\mathbf{X}}=&\sup_{\rho\in[\lambda,\beta]}\sup_{t>0}\langle t\rangle^{\frac{\rho+1}{\delta}}\|\phi(t)\|_{\mathbf{A}^{\rho,1}}+\sup_{\rho\in[0,\beta]}\sup_{t>0}\langle t\rangle^{\frac{\rho}{\delta}}\|\phi(t)\|_{\mathbf{A}^{\rho,\infty}}\\&+\sup_{1\le p\le\infty}\sup_{t>0}\langle t\rangle^{\frac{\beta}{\delta}+\frac{1}{\delta p}}\|\phi(t)\|_{\mathbf{B}^{\sigma,p}},\end{aligned}$$

and $\theta>1$, $\beta\ge\alpha$ are such that $\min\left(\frac{\beta}{\delta},\frac{\alpha+1}{\delta}\right)\ge\theta>1$. By condition (2.129) we have the estimate

$$\begin{aligned}|Q(\xi,y)|&\ge|L(\xi)-L(\xi-y)-L(y)|\ge C\langle\xi\rangle^{\rho}\min(|y|,|\xi-y|)\\&\ge C\langle\xi\rangle^{\rho-\gamma\rho}\min\left(|y|^{1-\gamma},|\xi-y|^{1-\gamma}\right)\end{aligned}$$

in the domain

$$\left\{|\xi|^{-\rho}\le|y|\le\frac{|\xi|}{8},|\xi|\ge1\right\}\cup\left\{|\xi|^{-\rho}\le|\xi-y|\le\frac{|\xi|}{8},|\xi|\ge1\right\},$$

where $\gamma>0$ is small. Therefore, by condition (2.131) we get

$$\begin{aligned}\langle\xi\rangle^{\sigma}|a_1(\xi,y)|&\le C\frac{(\{\xi-y\}^{\alpha}\langle\xi-y\rangle^{\sigma}+\{y\}^{\alpha}\langle y\rangle^{\sigma})}{\langle\xi\rangle^{\rho-\gamma\rho-\sigma}\min\left(|y|^{1-\gamma},|\xi-y|^{1-\gamma}\right)}\\&\le C\left(\{\xi-y\}^{\beta}\{y\}^{\lambda}+\{\xi-y\}^{\lambda}\{y\}^{\beta}\right)\langle\xi-y\rangle^{\sigma}\langle y\rangle^{\sigma}\end{aligned}$$

for $|\xi|\ge1$, $y\in\mathbf{R}$, with $\lambda=\gamma-1>-1$ since $\rho>\sigma$. In the same manner we find

$$\langle\xi\rangle^{\sigma}|a_2(\xi,y)|\le C\left(\{\xi-y\}^{\beta}\{y\}^{\lambda}+\{\xi-y\}^{\lambda}\{y\}^{\beta}\right)\langle\xi-y\rangle^{\sigma}\langle y\rangle^{\sigma},$$

for $|\xi|\ge1$, $y\in\mathbf{R}$, and

$$\langle\xi\rangle^{\sigma}|a_2(\xi,y)|\le C\left(\{\xi-y\}^{\alpha}\langle\xi-y\rangle^{\sigma}+\{y\}^{\alpha}\langle y\rangle^{\sigma}\right),$$

for $|\xi|\le1$, $y\in\mathbf{R}$. Therefore,

$$\|\mathcal{N}_1(u)\|_{\mathbf{A}^{0,p}}=0,$$

and by Lemma 2.65 we get

$$\begin{aligned}\|\mathcal{N}_1(u)\|_{\mathbf{B}^{\sigma,p}}+\|\mathcal{N}_2(u)\|_{\mathbf{B}^{\sigma,p}}&\le C\left(\|u\|_{\mathbf{A}^{\lambda,1}}+\|u\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|u\|_{\mathbf{A}^{\beta,p}}+\|u\|_{\mathbf{B}^{\sigma,p}}\right)\\&\le\langle t\rangle^{-\frac{\beta}{\delta}-\frac{1}{\delta p}}\|u\|_{\mathbf{X}}^2\le C\langle t\rangle^{-\theta-\frac{1}{\delta p}}\|u\|_{\mathbf{X}}^2,\end{aligned}$$

$$\begin{aligned}\|\mathcal{N}_2(u)\|_{\mathbf{A}^{0,p}} &\le C\left(\|u\|_{\mathbf{A}^{0,1}} + \|u\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|u\|_{\mathbf{A}^{\alpha,p}} + \|u\|_{\mathbf{B}^{\sigma,p}}\right) \\ &\le C\langle t\rangle^{-\frac{\alpha+1}{\delta}-\frac{1}{\delta p}}\|u\|_{\mathbf{X}}^2 \le C\langle t\rangle^{-\theta-\frac{1}{\delta p}}\|u\|_{\mathbf{X}}^2\end{aligned}$$

since $\min\left(\frac{\beta}{\delta}, \frac{\alpha+1}{\delta}\right) \ge \theta > 1$. Thus

$$\|\mathcal{N}_1(u)\|_{\mathbf{X}} + \|\mathcal{N}_2(u)\|_{\mathbf{Y}} \le C\|u\|_{\mathbf{X}}^2, \tag{2.140}$$

where

$$\|\phi\|_{\mathbf{Y}} = \sup_{1\le p\le\infty}\sup_{t>0}\langle t\rangle^{\theta+\frac{1}{\delta p}}\|\phi(t)\|_{\mathbf{A}^{0,p}} + \sup_{1\le p\le\infty}\sup_{t>0}\langle t\rangle^{\theta+\frac{1}{\delta p}}\|\phi(t)\|_{\mathbf{B}^{\sigma-\nu_1,p}}.$$

Now we estimate $a_3(\xi,y,z)$ and $a_4(\xi,y,z)$. By condition (2.129) we have estimate

$$\begin{aligned}&\left|\widetilde{Q}(\xi,y,z)\right| \ge |\operatorname{Im} L(\xi) - \operatorname{Im} L(y-z) - \operatorname{Im} L(\xi-y) - \operatorname{Im} L(z)| \\ &\ge C\langle\xi\rangle^{\rho}\min\left(|\xi-y+z|, |\xi-z|\right) \\ &\ge C\langle\xi\rangle^{\rho-\gamma\rho}\min\left(|\xi-y+z|^{1-\gamma}, |\xi-z|^{1-\gamma}\right)\end{aligned}$$

in the domain

$$\begin{aligned}&\left\{|\xi|^{-\rho} \le |\xi-y| \le \frac{|\xi|}{8}, |\xi|\ge 1\right\} \\ &\times\left(\left\{|\xi|^{-\rho} \le |\xi-y+z| \le \frac{|\xi|}{8}\right\} \cup \left\{|\xi|^{-\rho} \le |\xi-z| \le \frac{|\xi|}{8}\right\}\right),\end{aligned}$$

where $\gamma > 0$ is small. Therefore by condition (2.131) we obtain

$$\begin{aligned}&\langle\xi\rangle^{\sigma-\nu_1}|a_3(\xi,y,z)| \le C\langle\xi\rangle^{\sigma-\nu_1}\left|\psi_3(\xi,y,z)\,\frac{a_1(\xi,y)\,a(y,z)}{\widetilde{Q}(\xi,y,z)}\right| \\ &\le C\frac{\langle\xi\rangle^{3\sigma-2\rho+2\gamma\rho-\nu_1}\left(\{y-z\}^{\alpha} + \{z\}^{\alpha}\right)}{|\xi-y|^{1-\gamma}\min\left(|\xi-y+z|^{1-\gamma}, |\xi-z|^{1-\gamma}\right)} \\ &\le C\{\xi-y\}^{\lambda}\left(\{\xi-y+z\}^{\lambda} + \{\xi-z\}^{\lambda}\right)\left(\{y-z\}^{\alpha} + \{z\}^{\alpha}\right)\end{aligned}$$

for $|\xi| \ge 1$, $y,z \in \mathbf{R}$, since $3\sigma - 2\rho + 2\gamma\rho - \nu_1 \le 0$. By an identical approach we have

$$\begin{aligned}&\langle\xi\rangle^{\sigma}|a_3(\xi,y,z)| + \langle\xi\rangle^{\sigma}|a_4(\xi,y,z)| \\ &\le C\{\xi-y\}^{\lambda}\left(\{\xi-y+z\}^{\lambda} + \{\xi-z\}^{\lambda}\right) \\ &\times\left(\{y-z\}^{\alpha} + \{z\}^{\alpha}\right)\langle y-z\rangle^{\sigma}\langle z\rangle^{\sigma}\end{aligned}$$

for $|\xi| \ge 1$, $y,z \in \mathbf{R}$, since $\rho > \sigma$. Thus by Lemma 2.65 we get

$$\begin{aligned}
&\|\mathcal{N}_3(u,\mathcal{N}(u))\|_{\mathbf{B}^{\sigma-\nu_1,p}} \\
&\le C\left(\|u\|_{\mathbf{A}^{0,\infty}} + \|u\|_{\mathbf{B}^{0,\infty}} + \|u\|_{\mathbf{B}^{0,1}}\right)\left(\|u\|_{\mathbf{A}^{0,p}} + \|u\|_{\mathbf{B}^{0,p}}\right) \\
&\times\left(\|\mathcal{N}(u)\|_{\mathbf{A}^{0,\infty}} + \|\mathcal{N}(u)\|_{\mathbf{B}^{0,\infty}} + \|\mathcal{N}(u)\|_{\mathbf{B}^{0,1}}\right) \\
&\le \langle t\rangle^{-\theta-\frac{1}{\delta p}} \|u\|_{\mathbf{X}}^2 \|\mathcal{N}(u)\|_{\mathbf{Y}} \le \langle t\rangle^{-\theta-\frac{1}{\delta p}} \|u\|_{\mathbf{X}}^4;
\end{aligned}$$

hence

$$\|\mathcal{N}_3(u,\mathcal{N}(u))\|_{\mathbf{Y}} \le C\|u\|_{\mathbf{X}}^4.$$

In the same manner

$$\begin{aligned}
&\|\mathcal{N}_3(u,u)\|_{\mathbf{B}^{\sigma,p}} + \|\mathcal{N}_4(u)\|_{\mathbf{B}^{\sigma,p}} \\
&\le C\left(\|u\|_{\mathbf{A}^{\beta+\lambda,1}} + \|u\|_{\mathbf{B}^{0,1}}\right)\left(\|u\|_{\mathbf{A}^{0,p}} + \|u\|_{\mathbf{B}^{\sigma,p}}\right) \\
&\times\left(\|u\|_{\mathbf{A}^{0,\infty}} + \|u\|_{\mathbf{B}^{0,\infty}} + \|u\|_{\mathbf{B}^{\sigma,1}}\right) \\
&+ C\left(\|u\|_{\mathbf{A}^{0,\infty}} + \|u\|_{\mathbf{B}^{0,\infty}} + \|u\|_{\mathbf{B}^{0,1}}\right)\left(\|u\|_{\mathbf{A}^{\beta,p}} + \|u\|_{\mathbf{B}^{\sigma,p}}\right) \\
&\times\left(\|u\|_{\mathbf{A}^{0,\infty}} + \|u\|_{\mathbf{B}^{0,\infty}} + \|u\|_{\mathbf{B}^{\sigma,1}}\right) \\
&+ C\left(\|u\|_{\mathbf{A}^{0,\infty}} + \|u\|_{\mathbf{B}^{0,\infty}} + \|u\|_{\mathbf{B}^{0,1}}\right)\left(\|u\|_{\mathbf{A}^{0,p}} + \|u\|_{\mathbf{B}^{\sigma,p}}\right) \\
&\times\left(\|u\|_{\mathbf{A}^{\beta,\infty}} + \|u\|_{\mathbf{B}^{0,\infty}} + \|u\|_{\mathbf{B}^{\sigma,1}}\right) \le \langle t\rangle^{-\theta-\frac{1}{\delta p}} \|u\|_{\mathbf{X}}^3;
\end{aligned}$$

hence,

$$\|\mathcal{N}_3(u,u)\|_{\mathbf{X}} + \|\mathcal{N}_4(u)\|_{\mathbf{Y}} \le C\|u\|_{\mathbf{X}}^3. \tag{2.141}$$

Therefore applying Lemma 2.62 and Lemma 2.64 via (2.140) to (2.141) we get the estimate

$$\begin{aligned}
&\|\mathcal{N}_1(u)\|_{\mathbf{X}} + \|\mathcal{N}_3(u,u)\|_{\mathbf{X}} \\
&+ \|\mathcal{G}(t)(u_0 - \mathcal{N}_1(u_0) - \mathcal{N}_3(u_0,u_0))\|_{\mathbf{X}} \\
&+ \left\|\int_0^t \mathcal{G}(t-\tau)\left(\mathcal{N}_2(u) + \mathcal{N}_4(u) + \mathcal{N}_3(u,\mathcal{N}(u))\right) d\tau\right\|_{\mathbf{X}} \\
&\le C\|u_0\|_{\mathbf{A}^{0,\infty}\cap\mathbf{B}^{\sigma,1}\cap\mathbf{D}^{0,\sigma}} + C\|u_0\|_{\mathbf{A}^{0,\infty}\cap\mathbf{B}^{\sigma,1}\cap\mathbf{D}^{0,\sigma}}^2 \\
&+ C\|u_0\|_{\mathbf{A}^{0,\infty}\cap\mathbf{B}^{\sigma,1}\cap\mathbf{D}^{0,\sigma}}^3 \\
&+ C\|u\|_{\mathbf{X}}^2 + C\|u\|_{\mathbf{X}}^3 + C\|u\|_{\mathbf{X}}^4 \le C\varepsilon.
\end{aligned}$$

Similarly we consider the differences

$$\begin{aligned}
&\left\|\int_0^t \mathcal{G}(t-\tau)\sum_{j=2,4}\left(\mathcal{N}_j(w) - \mathcal{N}_j(v)\right) + \mathcal{N}_3(w,\mathcal{N}(w)) - \mathcal{N}_3(v,\mathcal{N}(v))\right) d\tau\right\|_{\mathbf{X}} \\
&\le C\|w-v\|_{\mathbf{X}}\left(\|w\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}^2 + \|w\|_{\mathbf{X}}^3 + \|v\|_{\mathbf{X}} + \|v\|_{\mathbf{X}}^2 + \|v\|_{\mathbf{X}}^3\right)
\end{aligned}$$

to see that the conditions of Theorem 1.17 are valid. Hence there exists a unique solution $u(t,x) \in \mathbf{X}$.

To prove the estimate of the solution in the norm $\mathbf{D}^{\alpha,\sigma}$ we use the usual integral representation for the Cauchy problem (2.124)

$$u(t) = \mathcal{G}(t) u_0 - \int_0^t \mathcal{G}(t-\tau) \mathcal{N}(u)(\tau) \, d\tau. \tag{2.142}$$

We define two norms

$$\|\phi\|_{\mathbf{X}_1} = \sup_{\rho=0,\alpha} \sup_{t>0} \langle t \rangle^{\frac{\rho-\gamma}{\delta}} \|\phi(t)\|_{\mathbf{D}^{\rho,\sigma}},$$
$$\|\phi\|_{\mathbf{Y}_1} = \sup_{t>0} \langle t \rangle^{\theta - \frac{\gamma}{\delta}} \|\phi(t)\|_{\mathbf{D}^{0,0}},$$

here $\gamma \in (0, \min(1, \delta))$ is such that $\gamma < \alpha$ if $\alpha > 0$. By employing Lemma 2.62 and Lemma 2.64 we have the estimate

$$\begin{aligned}\|u\|_{\mathbf{X}_1} &\le \|\mathcal{G}(t) u_0\|_{\mathbf{X}_1} + \left\| \int_0^t \mathcal{G}(t-\tau) \mathcal{N}(u)(\tau) \, d\tau \right\|_{\mathbf{X}_1} \\ &\le C\varepsilon + C \|\mathcal{N}(u)\|_{\mathbf{Y}_1} + C \|\mathcal{N}(u)\|_{\mathbf{Y}} \le C\varepsilon,\end{aligned}$$

since by Lemma 2.65 and Lemma 2.66 we find

$$\|\mathcal{N}(u)\|_{\mathbf{Y}_1} + \|\mathcal{N}(u)\|_{\mathbf{Y}} \le C \left(\|u\|_{\mathbf{X}} + \|u\|_{\mathbf{X}_1} \right)^2.$$

Now applying Lemma 2.63 to integral equation (2.142) we find the asymptotics (2.132) with the following coefficient

$$\begin{aligned} A &= \widehat{u}_0(0) - \int_0^\infty \widehat{\mathcal{N}(u(\tau))}(0) \, d\tau = \frac{1}{\sqrt{2\pi}} \int u_0(x) \, dx \\ &- \frac{1}{\sqrt{2\pi}} \int_0^\infty d\tau \int \mathcal{N}(u(\tau))(x) \, dx. \end{aligned}$$

Thus we see that the large time asymptotics has a quasi linear character, that is the nonlinearity alters only the coefficient of the main term of the asymptotic formula. Theorem 2.59 is proved.

2.7 A system of nonlinear equations

Consider a system of nonlinear nonlocal evolution equations

$$u_t + \mathcal{N}(u) + \mathcal{L}u = 0, \ x \in \mathbf{R}^n, \ t > 0 \tag{2.143}$$

with initial data $u(0,x) = \widetilde{u}(x), \, x \in \mathbf{R}^n$, where the unknown function $u(t,x)$ is a vector $u = \{u_j\}|_{j=1,\ldots,m}$. The linear part of system (2.143) is a pseudodifferential operator defined by the Fourier transformation as follows

$$\mathcal{L}u = \overline{\mathcal{F}}_{\xi \to x} L(\xi) \mathcal{F}_{x \to \xi} u,$$

where the symbol $L(\xi)$ is a matrix $L = \{L_{jk}\}|_{j,k=1,\ldots,m}$. The nonlinearity $\mathcal{N}(u)$ is a quadratic pseudodifferential operator

$$\mathcal{N}(u) = \overline{\mathcal{F}}_{\xi \to x} \sum_{k,l=1}^{m} \int_{\mathbf{R}^n} a^{kl}(t,\xi,y)\, \hat{u}_k(t,\xi - y)\, \hat{u}_l(t,y)\, dy;$$

here the symbols $a^{kl}(t,\xi,y)$ are vectors $a^{kl} = \left\{a_j^{kl}\right\}\big|_{j=1,\dots,m}$.

We suppose that the symbols $a^{kl}(t,\xi,y)$ are continuous vector-functions with respect to time $t > 0$ and the operators $\mathcal{N}$ and $\mathcal{L}$ have a finite order, that is the symbols $a^{kl}(t,\xi,y)$ and $L(\xi)$ grow with respect to y and ξ no faster than a power of some order κ

$$\left|a^{kl}(t,\xi,y)\right| \le C\left(\langle\xi\rangle^{\kappa} + \langle y\rangle^{\kappa}\right), \quad |L(\xi)| \le C\langle\xi\rangle^{\kappa}$$

where $C > 0$. The absolute value for vectors $\left|a^{kl}\right|$ and matrix $|L|$ we understand as maximum of their components: $\left|a^{kl}\right| = \max_{j=1,..,m}\left|a_j^{kl}\right|$, $|L| = \max_{j,k=1,..,m}|L_{jk}|$.

Model system (2.143) combines many famous equations. For example, let the solution $u(t,x)$ be a real-valued vector-function, $m = n$, and the linear part be a Laplacian $\mathcal{L}u = -\Delta u$. Also consider the nonlinearity of the form

$$\mathcal{N}(u) = (u \cdot \nabla)\, u + \nabla(-\Delta)^{-1} \sum_{k,l=1}^{n} \nabla_k \nabla_l u_k u_l,$$

that is the symbols

$$a_j^{kl}(t,\xi,y) = i\xi_k \delta_{jl} - i\xi_j \frac{\xi_k \xi_l}{|\xi|^2},$$

where $\delta_{jj} = 1$ and $\delta_{jl} = 0$ if $j \neq l$. Let the initial data obey the restriction $(\nabla \cdot \widetilde{u}) = 0$; then the solution $u(t,x)$ for all $t \ge 0$ also satisfy this restriction $(\nabla \cdot u) = 0$. Thus from (2.143) we obtain the famous Navier-Stokes system of equations

$$\begin{cases} u_t + (u \cdot \nabla)\, u + \nabla h - \Delta u = 0, \\ (\nabla \cdot u) = 0. \end{cases} \tag{2.144}$$

System (2.143) also contains the shallow water system of equations (see Whitham [1999])

$$\begin{cases} \eta_t + (\nabla \cdot v\eta) = 0, \\ v_t + (v \cdot \nabla)\, v + \nabla\eta = 0, \end{cases}$$

which contains the nonlinearity but does not take into account the dispersion; here $\eta(t,x)$ is the free surface of water, $v(t,x)$ is the velocity vector and the spatial dimension $n = 2$. If attention is restricted to only the simplest dispersion term, then system (2.143) leads to the well-known Boussinesq equations (see Whitham [1970])

$$\begin{cases} \eta_t + (\nabla \cdot v\eta) + \frac{1}{3}\Delta(\nabla \cdot v) = 0, \\ v_t + (v \cdot \nabla)\, v + \nabla\eta = 0. \end{cases}$$

System (2.143) goes over into the system of Dobrokhotov (see Dobrokhotov [1987]), in the first approximation for the nonlinearity

$$\begin{cases} \eta_t + (\nabla \cdot v\eta) + \mathcal{B}v = 0, \\ v_t + (v \cdot \nabla) v + \nabla \eta = 0, \end{cases}$$

where the operator $\mathcal{B}v = \sum_{j=1}^{n} \mathcal{B}_j v_j$ has the symbols $B_j(\xi) = \frac{i\xi_j}{|\xi|} \tanh |\xi|$ and corresponds to the exact potential theory of water waves. Note that the Boussinesq system is a long wave approximation of the Dobrokhotov system. Another particular case of system (2.143) is the one-dimensional in spatial variable x system of equations proposed by Broer (see Broer [1975]), Kaup (see Kaup [1975])

$$\begin{cases} \eta_t + (\eta v)_x - \beta \eta_{xx} - \alpha v_{xxx} = 0, \\ v_t + v v_x + \eta_x + \beta v_{xx} = 0 \end{cases}$$

as well as others (see Naumkin and Shishmarev [1994b]). Note that in the one dimensional case $n = m = 1$ system (2.143) contains in particular the Whitham equation (see Whitham [1999])

$$u_t + u u_x + \mathcal{L}u = 0,$$

and thus many important one-dimensional equations, for example, the famous Korteweg-de Vries equation, Burgers, Benjamin-Ono and others. Thus, the system (2.143) deserves a serious study.

Suppose for simplicity that the eigenvalues $\lambda_j(\xi)$ of the matrix $L(\xi)$ are distinct for $\xi \in \mathbf{R}^n \setminus \{0\}$ and order them according to the increase of the real parts. We make this supposition to simplify the form of the fundamental Cauchy matrix only. For example, if $L(\xi)$ has a diagonal form $L(\xi) = \|\delta_{jk}\lambda_j(\xi)\|_{j,k=1,\ldots,m}$, then the eigenvalues $\lambda_j(\xi)$ can be arbitrary. Let the $m \times m$ matrix $Q(\xi) = \|Q_{jk}(\xi)\|_{j,k=1,\ldots,m}$ diagonalize the matrix $L(\xi)$, that is $Q^{-1}(\xi) L(\xi) Q(\xi) = \|\lambda_j(\xi)\delta_{jk}\|_{j,k=1,\ldots,m}$, where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ otherwise. Consider the system of ordinary differential equations with constant coefficients depending on a parameter $\xi \in \mathbf{R}^n$

$$\frac{d}{dt}\widehat{u}(t,\xi) + L(\xi)\widehat{u}(t,\xi) = 0. \tag{2.145}$$

Multiplying system (2.145) by $Q^{-1}(\xi)$ from the left and changing $\widehat{u}(t,\xi) = Q(\xi) v(t,\xi)$ we diagonalize system (2.145)

$$\frac{d}{dt} v_j(t,\xi) = -\lambda_j(\xi) v_j(t,\xi);$$

hence, integrating with respect to time $t \geq 0$ we find

$$v_j(t,\xi) = e^{-t\lambda_j(\xi)} v_j(0,\xi).$$

Returning to the solution $\widehat{u}(t,\xi)$ we get

$$\widehat{u}_k(t,\xi) = \sum_{j=1}^{m} e^{-t\lambda_j(\xi)} \sum_{l=1}^{m} Q_{kj}(\xi) \left(Q^{-1}(\xi)\right)_{jl} \widehat{u}_l(0,\xi);$$

therefore

$$\widehat{u}(t,\xi) = e^{-tL(\xi)}\widehat{u}(0,\xi),$$

where the fundamental Cauchy matrix has the form (see Gelfand and Shilov [1968])

$$e^{-tL(\xi)} = \sum_{j=1}^{m} e^{-t\lambda_j(\xi)} P_j(\xi) \tag{2.146}$$

with matrices

$$P_j(\xi) = \left\| Q_{kj}(\xi) \left(Q^{-1}(\xi)\right)_{jl} \right\|_{k,l=1,\dots,m}. \tag{2.147}$$

We rewrite the Cauchy problem (2.143) in the form of the integral equation

$$u(t) = \mathcal{G}(t)\widetilde{u} - \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau, \tag{2.148}$$

where the Green operator $\mathcal{G}(t)\psi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-tL(\xi)}\hat{\psi}(\xi)\right).$

2.7.1 Local existence and smoothing effect

Consider the fundamental Cauchy matrix

$$e^{-tL(\xi)} = \sum_{j=1}^{m} e^{-t\lambda_j(\xi)} P_j(\xi)$$

for the system of ordinary differential equations with constant coefficients depending on a parameter $\xi \in \mathbf{R}^n$

$$\frac{d}{dt}\widehat{u}(t,\xi) + L(\xi)\widehat{u}(t,\xi) = 0.$$

Here $P_j(\xi)$ are $m \times m$ matrices and $\lambda_j(\xi)$ are eigenvalues of the matrix $L(\xi)$.

Let the linear operator $\mathcal{L}$ satisfy the dissipation condition expressed in terms of the eigenvalues of the matrix $L(\xi)$

$$\operatorname{Re}\lambda_j(\xi) \ge \mu|\xi|^{\nu}, \tag{2.149}$$

for all $|\xi| \ge 1$, where $\mu > 0$, $\nu \ge 0$ and

$$\left|\partial_\xi^r \lambda_j(\xi)\right| \le C\{\xi\}^{\delta-|r|}\langle\xi\rangle^{\nu}, \quad \left|\partial_\xi^r P_j(\xi)\right| \le C \tag{2.150}$$

for all $\xi \in \mathbf{R}^n$, $|r| = 0,1$, where $\delta > 0$. Suppose that the symbols of the nonlinearity $\mathcal{N}$ are such that

$$\sum_{|r|=0}^{1} \left|\partial_\xi^r a^{kl}(t,\xi,y)\right| \le C\langle\xi-y\rangle^{\sigma} + C\langle y\rangle^{\sigma} \tag{2.151}$$

for all $\xi, y \in \mathbf{R}^n$, $t > 0$, $k, l = 1, ..., m$, where $\sigma \in [0, \nu]$. Denote

$$\|\varphi\|_{\mathbf{X}_1} = \sup_{1\le p\le\infty} \left\| \langle \cdot \rangle^{s+\frac{\nu}{p}} E(t,\cdot)\, \widehat{\varphi} \right\|_{\mathbf{L}^1_\xi \mathbf{L}^p_t},$$
$$\|\varphi\|_{\mathbf{X}_2} = \sup_{1\le p\le\infty} \left\| (1 + |\partial_\xi|^\omega) \langle \cdot \rangle^{s_- + \frac{\nu}{p}} E(t,\cdot)\, \widehat{\varphi} \right\|_{\mathbf{L}^\infty_\xi \mathbf{L}^p_t},$$

where $E(t,\xi) = e^{\frac{\mu}{2} t \langle \xi \rangle^{\nu_1}}$, $\nu_1 = \min(1,\nu)$, $\omega \in [0,1)$, $s > S = \frac{1}{2}(\sigma - \nu)$, $s_- = \min(s, 0)$, here $\mathbf{L}^q_\xi \mathbf{L}^p_t \equiv \mathbf{L}^q(\mathbf{R}^n; \mathbf{L}^p(0,T))$, with $T \in (0,1]$, $1 \le p, q \le \infty$, and we define the majorant of the fractional derivative of order $\omega \in (0,1)$ as

$$|\partial_\xi|^\omega \phi(\xi) \equiv \int_{\mathbf{R}^n} |\phi(\xi - y) - \phi(\xi)|\, |y|^{-n-\omega}\, dy.$$

Theorem 2.67. *Let the linear operator $\mathcal{L}$ satisfy the dissipation condition (2.149), (2.150) with $\nu \ge 0$ and the nonlinear operator $\mathcal{N}$ satisfy estimates (2.151) with $\sigma \in [0,\nu]$. We take the initial data $\widetilde{u} \in \mathbf{H}^{s'}(\mathbf{R}^n) \cap \mathbf{H}^{s_-,\omega'}(\mathbf{R}^n)$, where $s' > \frac{n}{2} + s$, $s > S$, $\omega' > \frac{n}{2} + \omega$. If $\sigma = \nu > 0$ we suppose additionally that the norm $\|\widetilde{u}\|_{\mathbf{H}^{\frac{n}{2}+s'}}$ is small. Then for some time $T > 0$ there exists a unique solution $u(t,x)$ to the Cauchy problem (2.143) such that $u \in \mathbf{C}([0,T]; \mathbf{X}_1 \cap \mathbf{X}_2)$. In the case of $\nu > 0$ we also have a smoothing property $u \in \mathbf{C}^1((0,T]; \mathbf{H}^\infty(\mathbf{R}^n))$.*

We can apply Theorem 2.67 to the Navier-Stokes system.

Preliminary lemmas

We define the Green operator $\mathcal{G}$ for the Cauchy problem for the linear system of equations

$$\begin{cases} u_t + \mathcal{L}u = f, \ t > 0, x \in \mathbf{R}^n, \\ u(0,x) = \widetilde{u}, \ x \in \mathbf{R}^n. \end{cases} \tag{2.152}$$

Using the Fourier transformation we can formally represent the Green operator as

$$\mathcal{G}(t)\psi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-tL(\xi)}\hat{\psi}(\xi)\right) = \int_{\mathbf{R}^n} G(t, x-y)\,\psi(y)\,dy,$$

where the kernel

$$G(t,x) = \sum_{j=1}^m \overline{\mathcal{F}}_{\xi\to x} e^{-t\lambda_j(\xi)} P_j(\xi).$$

Therefore the solution of problem (2.152) can be written by the Duhamel's integral

$$u(t) = \mathcal{G}(t)\,\widetilde{u} + \int_0^t \mathcal{G}(t-\tau)\, f(\tau)\, d\tau.$$

In the following lemma we prove the smoothing property for the Green operator $\mathcal{G}(t)$. By the result of Lemma 2.50 we have

Lemma 2.68. *Let the linear operator $\mathcal{L}$ satisfy dissipation conditions (2.149) through (2.150) with $\nu \geq 0$. Then the following estimates*

$$\|\mathcal{G}(t)\phi\|_{\mathbf{X}_1} \leq C\|\phi\|_{\mathbf{H}^{s',0}}, \quad \|\mathcal{G}(t)\phi\|_{\mathbf{X}_2} \leq C\|\phi\|_{\mathbf{H}^{s_-,\omega'}}$$

and

$$\left\|\int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau\right\|_{\mathbf{X}_j} \leq C\|\psi\|_{\mathbf{Y}_j}, \ j=1,2$$

are valid, provided that the right-hand sides are bounded, where $s' > \frac{n}{2}+s$, $\omega' > \omega + \frac{n}{2}$, $s_- = \min(s,0)$.

Now we estimate the nonlinearity in the norms

$$\|\varphi\|_{\mathbf{Y}_1} = \|\langle\cdot\rangle^s E(t,\cdot)\widehat{\varphi}\|_{\mathbf{L}^1_\xi\mathbf{L}^1_t},$$
$$\|\varphi\|_{\mathbf{Y}_2} = \|(1+|\partial_\xi|^\omega)\langle\cdot\rangle^{s_-} E(t,\cdot)\widehat{\varphi}\|_{\mathbf{L}^\infty_\xi\mathbf{L}^1_t}$$

where $E(t,\xi) = e^{\frac{\mu}{2}t\langle\xi\rangle^{\nu_1}}$, $\nu_1 = \min(1,\nu)$, $\omega \in [0,1)$, $s > S = \frac{1}{2}(\sigma-\nu)$; here $\mathbf{L}^q_\xi\mathbf{L}^p_t \equiv \mathbf{L}^q(\mathbf{R}^n;\mathbf{L}^p(0,T))$, $1 \leq p,q \leq \infty$. Denote

$$\mathcal{N}(\varphi,\phi) \equiv \sum_{k,l=1}^m \overline{\mathcal{F}}_{\xi\to x}\int_{\mathbf{R}^n} a^{kl}(t,\xi,y)\,\widehat{\varphi}_k(t,\xi-y)\,\widehat{\phi}_l(t,y)\,dy.$$

In the same way as in the proof of Lemma 2.51 we get

Lemma 2.69. *Let the symbols $a^{kl}(t,\xi,y)$ of the nonlinear operator $\mathcal{N}(\varphi,\phi)$ satisfy condition (2.151) with $0 \leq \sigma \leq \nu$, $\nu \geq 0$. Then the inequalities*

$$\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{Y}_j} \leq CT^\vartheta\|\varphi\|_{\mathbf{X}_j}\|\phi\|_{\mathbf{X}_1}, \ j=1,2$$

are valid for any functions $\varphi,\phi \in \mathbf{X}_1 \cap \mathbf{X}_2$ with $s > S = \frac{1}{2}(\sigma-\nu)$, where $\vartheta = 1$ if $\nu = 0$ and $\vartheta = \min\left(1-\frac{\sigma}{\nu}, \frac{s-S}{\nu}\right)$ if $\nu > 0$.

Proof of Theorem 2.67

Via the Green operator $\mathcal{G}(t)$ of the linear Cauchy problem (2.152) we write the Cauchy problem for nonlinear system (2.143) in the form of the integral equation

$$u(t) = \mathcal{G}(t)\widetilde{u} - \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau. \tag{2.153}$$

By virtue of Lemma 2.68 and Lemma 2.69 we get

$$\|\mathcal{G}\widetilde{u}\|_{\mathbf{X}_1\cap\mathbf{X}_2} \leq C\|\widetilde{u}\|_{\mathbf{H}^{s',0}\cap\mathbf{H}^{s_-,\omega'}},$$

and

$$\left\| \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(v) - \mathcal{N}(\tilde{v}))\,d\tau \right\|_{\mathbf{X}_1 \cap \mathbf{X}_2}$$
$$\leq C \|\mathcal{N}(v) - \mathcal{N}(\tilde{v})\|_{\mathbf{Y}_1 \cap \mathbf{Y}_2} \leq CT^{\vartheta} \|v\|_{\mathbf{X}_1} \|v - \tilde{v}\|_{\mathbf{X}_1 \cap \mathbf{X}_2}$$

where we denote $\|v\|_{\mathbf{X}_1 \cap \mathbf{X}_2} = \|v\|_{\mathbf{X}_1} + \|v\|_{\mathbf{X}_2}$. The norms $\|v\|_{\mathbf{Y}_1 \cap \mathbf{Y}_2}$ and $\|v\|_{\mathbf{H}^{s',0} \cap \mathbf{H}^{s_-,\omega'}}$ are defined similarly. Here $s > S = \frac{1}{2}(\sigma - \nu)$, $s_- = \min(0, s)$, $\omega \in [0, 1)$, $\vartheta = 1$ if $\nu = 0$ and $\vartheta = \min\left(1 - \frac{\sigma}{\nu}, \frac{s-S}{\nu}\right)$ if $\nu > 0$. Hence because of Theorems 1.9 and 1.11 we see that there exists a sufficiently small time $T = T(\|\widetilde{u}\|_{\mathbf{H}^{s',0}})$ if $\vartheta > 0$; otherwise the norm $\|\widetilde{u}\|_{\mathbf{H}^{s',0}}$ is sufficiently small, such that there exists a unique solution $u(t,x) \in \mathbf{C}([0,T]; \mathbf{X}_1 \cap \mathbf{X}_2)$ of the Cauchy problem (2.143). By the definition of the norm $\mathbf{X}_1$ we have the smoothing property for the case of $\nu > 0$

$$\sup_{t \in [T_0, T]} \|v(t)\|_{\mathbf{H}^k} = \sup_{t \in [T_0, T]} \left\| \langle \xi \rangle^k \hat{v}(t, \xi) \right\|_{\mathbf{L}^2_\xi}$$
$$\leq C(k, T_0) \left\| \sup_{t \in [T_0, T]} E(t, \xi) \langle \xi \rangle^s |\hat{v}(t, \xi)| \right\|_{\mathbf{L}^1_\xi} \leq C \|v\|_{\mathbf{X}_1},$$

for all $k \geq 0$, where $T_0 \in (0, T]$. The derivatives with respect to time $t > 0$ can be estimated directly from equation (2.143), since the symbols $a^{kl}(t, \xi, y)$ are continuous in time. Therefore, in the case of $\nu > 0$ the solution $u(t,x) \in \mathbf{C}^1\left((0,T]; \mathbf{H}^{\infty,0}(\mathbf{R}^n)\right)$. Theorem 2.67 is proved.

2.7.2 Global existence and asymptotic behavior

In this section we consider a rather general class of nonlinearities in the Cauchy problem (2.143); however, we have to assume the smallness condition on the initial data to obtain the global existence of solutions. Suppose that the symbols of the nonlinear operator $\mathcal{N}$ are such that

$$\left| \partial_\xi^r a^{kl}(t, \xi, y) \right| \leq C \{\xi - y\}^{\alpha - |r|} \langle \xi - y \rangle^\sigma + C \{y\}^{\alpha - |r|} \langle y \rangle^\sigma \tag{2.154}$$

for all $\xi \in \mathbf{R}^n$, $y \in \mathbf{R}^n$, $t > 0$, where $\sigma \geq 0$, $\alpha \geq 1$, $|r| = 0, 1$. Note that condition (2.154) is general enough, and the examples mentioned above satisfy this assumption.

Let the linear operator $\mathcal{L}$ satisfy the dissipation condition which in terms of the eigenvalues of the matrix $L(\xi)$ has the form

$$\operatorname{Re} \lambda_j(\xi) \geq \mu \{\xi\}^\delta \langle \xi \rangle^\nu \tag{2.155}$$

for all $\xi \in \mathbf{R}^n$, where $\mu > 0$, $\nu \geq 0$, $\delta > 0$.

To find the asymptotic formulas for the solution we assume that the eigenvalues of the symbol $L(\xi)$ has the following asymptotic representation in the origin

$$\lambda_j(\xi) = \mu_j |\xi|^{\delta} + O\left(|\xi|^{\delta+\gamma}\right) \tag{2.156}$$

for all $|\xi| \leq 1$, where $\mu_j > 0$, $\gamma > 0$. Also we suppose that the symbol is smooth $L(\xi) \in \mathbf{C}^1(\mathbf{R}^n)$ and has the estimate

$$\left|\partial_{\xi}^{r} \lambda_j(\xi)\right| \leq C \{\xi\}^{\delta-|r|} \langle\xi\rangle^{\nu} \tag{2.157}$$

for all $\xi \in \mathbf{R}^n \setminus \{0\}$, $|r| = 0, 1$ and

$$\sum_{|r|=0}^{1} \left|\partial_{\xi}^{r} P_j(\xi)\right| \leq C \tag{2.158}$$

for all $\xi \in \mathbf{R}^n$.

Denote $S = \frac{1}{2}(\sigma - \nu)$ as the critical order for the local existence and $\delta_c = n + \alpha$ denotes the critical order related to the large time asymptotic behavior. Let

$$\|\varphi(t)\|_{\mathbf{A}_{s,\rho}} = \|\{\cdot\}^{\rho} \langle\cdot\rangle^{s} \mathcal{F}_{x\to\xi}\varphi(t)\|_{\mathbf{L}^1_{\xi}},$$

and

$$\|\varphi(t)\|_{\mathbf{B}_{s,\rho,\omega}} = \|(1 + |\partial_{\xi}|^{\omega}) \{\cdot\}^{\rho} \langle\cdot\rangle^{s} \mathcal{F}_{x\to\xi}\varphi(t)\|_{\mathbf{L}^{\infty}_{\xi}},$$

where $s > S$, $\rho \geq 0$, $\omega \geq 0$.

Theorem 2.70. *Let the linear operator $\mathcal{L}$ satisfy conditions (2.155) to (2.158) with $\delta < n + \alpha$. Suppose that the nonlinear operator $\mathcal{N}$ satisfies estimates (2.154) with $\sigma \in [0, \nu)$ if $\nu > 0$ or $\sigma = 0 = \nu$. Let the initial data $\widetilde{u} \in \mathbf{H}^{s'}(\mathbf{R}^n) \cap \mathbf{H}^{s_-,\omega'}(\mathbf{R}^n)$ and be sufficiently small, where $s' > \frac{n}{2} + s$, $s > S$, $\omega' > \frac{n}{2} + \omega$, $\omega \in (0,1)$, $s_- = \min(s, 0)$. Then there exists a unique solution $u(t,x) \in \mathbf{C}^0\left([0,\infty); \mathbf{A}^{s,0} \cap \mathbf{B}^{0,0,\omega}\right)$ of the Cauchy problem (2.143). In the case $\nu > 0$ we also have a smoothing property $u(t,x) \in \mathbf{C}^1((0,\infty); \mathbf{H}^{\infty}(\mathbf{R}^n))$. Moreover, there exists a unique constant vector A, such that the solution u has the following asymptotics*

$$u(t,x) = t^{-\frac{n}{\delta}} G\left(xt^{-\frac{1}{\delta}}\right) A + O\left(t^{-\frac{n}{\delta}-\gamma}\right) \tag{2.159}$$

for large time $t > 0$ uniformly with respect to $x \in \mathbf{R}^n$, where $\gamma > 0$,

$$G(x) = \sum_{j=1}^{m} P_j(0) \overline{\mathcal{F}}_{\xi\to x} e^{-\mu_j |\xi|^{\delta}}.$$

As an example we apply Theorem 2.70 to the system of equation for surface waves with dissipation

$$\begin{cases} \eta_t + (\nabla \cdot v\eta) - \Delta\eta + \frac{1}{3}\Delta(\nabla \cdot v) = 0, \\ v_t + (v \cdot \nabla) v + \nabla\eta - \Delta v = 0 \end{cases}$$

and to the Dobrokhotov's system of equations with dissipation

$$\begin{cases} \eta_t + (\nabla \cdot v\eta) - \Delta\eta + \mathcal{B}v = 0, \\ v_t + (v \cdot \nabla)\, v + \nabla\eta - \Delta v = 0. \end{cases}$$

We see that for small initial data $(\widetilde{\eta}, \widetilde{v}) \in \left(\mathbf{H}^{s,0}(\mathbf{R}^n) \cap \mathbf{H}^{0,\omega}(\mathbf{R}^n)\right)^3$, with $s > \frac{1}{2}$, $\omega > 1$, there exists a unique solution

$$(\eta(t,x), v(t,x)) \in \Big(\mathbf{C}^\infty\left((0,\infty); \mathbf{H}^{\infty,0}(\mathbf{R}^n)\right) \\ \cap \mathbf{C}^0\left([0,\infty); \mathbf{H}^{s,0}(\mathbf{R}^n) \cap \mathbf{H}^{0,\omega}(\mathbf{R}^n)\right)\Big)^3$$

of the Cauchy problem satisfying the large time asymptotics (2.159) with the heat kernel $G(x)$.

Before proving Theorem 2.70 we state some estimates in the next subsection.

Preliminary lemmas

In the next lemma we give large time decay estimates for the Green function $\mathcal{G}(t)$ in the norms

$$\|\varphi(t)\|_{\mathbf{A}^{s,\rho}} = \|\{\cdot\}^\rho \langle\cdot\rangle^s \widehat{\varphi}(t,\cdot)\|_{\mathbf{L}^1_\xi},$$

and

$$\|\varphi(t)\|_{\mathbf{B}^{s,\rho,\omega}} = \|(1 + |\partial_\xi|^\omega)\{\cdot\}^\rho \langle\cdot\rangle^s \widehat{\varphi}(t,\cdot)\|_{\mathbf{L}^\infty_\xi},$$

where $s \in \mathbf{R}$, $\rho \geq 0$, $\omega \geq 0$. By Lemma 1.38 we obtain

Lemma 2.71. *Let the linear operator $\mathcal{L}$ satisfy dissipation conditions (2.155), (2.157) and (2.158). Then the estimates are valid*

$$\|\mathcal{G}(t)\psi(\tau)\|_{\mathbf{A}^{s,\rho}} \leq C\langle t\rangle^{-\frac{\rho}{\delta}}\{t\}^{-\frac{s}{\nu}}\|\psi(\tau)\|_{\mathbf{A}^{0,0}},$$

$$\|\mathcal{G}(t)\psi(\tau)\|_{\mathbf{A}^{s,\rho}} \leq C\langle t\rangle^{-\frac{\rho+n}{\delta}}\{t\}^{-\frac{s}{\nu}}\|\psi(\tau)\|_{\mathbf{B}^{0,0,0}}$$

and

$$\|\mathcal{G}(t)\psi(\tau)\|_{\mathbf{B}^{s,\rho,\omega}} \leq C\langle t\rangle^{-\frac{\rho}{\delta}}\{t\}^{-\frac{s}{\nu}}\|\psi(\tau)\|_{\mathbf{B}^{0,0,\omega}} \\ + C\langle t\rangle^{\frac{\omega-\rho}{\delta}}\{t\}^{-\frac{s}{\nu}}\|\psi(\tau)\|_{\mathbf{B}^{0,0,0}}$$

for all $0 \leq \tau \leq t$, where $s \geq 0$, $\rho \geq 0$, $\delta > 0$, $\omega \in [0,1)$ is such that $\omega < \delta$ if $\rho = 0$ and $\omega < \min(\rho,\delta)$ if $\rho > 0$. In the case of $\nu = 0$ we take $s = 0$.

By Lemma 1.39 in the next lemma we find the asymptotic formulas for the linear Cauchy problem (2.152).

Lemma 2.72. *Let the linear operator $\mathcal{L}$ satisfy dissipation conditions (2.155) and (2.158) and asymptotic representation (2.156). Then for any $\phi \in \mathbf{A}^{s,0} \cap \mathbf{B}^{0,0,\omega}$ and $\psi \in \mathbf{C}\left((0,\infty);\mathbf{A}^{s,0} \cap \mathbf{B}^{0,0,\omega}\right)$, where $s > \frac{n}{2}$, $\omega \in (0,\delta)$ we have the asymptotic representation as $t \geq 1$ uniformly with respect to $x \in \mathbf{R}^n$*

$$\mathcal{G}(t)\phi = t^{-\frac{n}{\delta}} G\left(xt^{-\frac{1}{\delta}}\right)\hat{\phi}(0) + O\left(t^{-\frac{n+\gamma_1}{\delta}}\left(\|\phi\|_{\mathbf{A}^{0,0}} + \|\phi\|_{\mathbf{B}^{0,0,\omega}}\right)\right),$$

where $\gamma_1 = \min(\gamma,\omega) > 0,$ and

$$\int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau = t^{-\frac{n}{\delta}} G\left(xt^{-\frac{1}{\delta}}\right)\int_0^\infty \hat{\psi}(\tau,0)\,d\tau$$
$$+ O\left(t^{-\frac{n+\gamma_2}{\delta}} \sup_{\tau\in(0,t)} \langle\tau\rangle^{\theta}\left(\langle\tau\rangle^{\frac{n}{\delta}}\|\psi(\tau)\|_{\mathbf{A}^{0,0}} + \|\psi(\tau)\|_{\mathbf{B}^{0,0,\omega}}\right)\right),$$

where the matrix-function $G(x) = \sum_{j=1}^m \overline{\mathcal{F}}_{\xi\to x} e^{-\mu_j|\xi|^\delta} P_j(0)$, $\theta \in (1,2)$, $\gamma_2 = \min(\gamma_1,(\theta-1)\delta)$.

In the same manner as in the proof of Lemma 2.57 we obtain estimates of the nonlinearity in the norms $\mathbf{A}^{s,\rho}$ and $\mathbf{B}^{s,\rho,\omega}$.

Lemma 2.73. *Let the nonlinear operator $\mathcal{N}$ satisfy condition (2.154). Then the inequalities*

$$\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{A}^{0,0}} \leq C\|\varphi\|_{\mathbf{A}^{\sigma,\alpha}}\|\phi\|_{\mathbf{A}^{0,0}} + C\|\varphi\|_{\mathbf{A}^{0,0}}\|\phi\|_{\mathbf{A}^{\sigma,\alpha}},$$
$$\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{B}^{0,0,0}} \leq C\|\varphi\|_{\mathbf{B}^{0,0,0}}\|\phi\|_{\mathbf{A}^{\sigma,\alpha}} + C\|\phi\|_{\mathbf{B}^{0,0,0}}\|\varphi\|_{\mathbf{A}^{\sigma,\alpha}}$$

and

$$\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{B}^{0,0,\omega}} \leq C\|\varphi\|_{\mathbf{B}^{\sigma,\alpha,\omega}}\|\phi\|_{\mathbf{A}^{0,0}} + C\|\varphi\|_{\mathbf{B}^{0,0,\omega}}\|\phi\|_{\mathbf{A}^{\sigma,\alpha}}$$
$$+ C\|\varphi\|_{\mathbf{B}^{\sigma,\alpha-\omega_1,0}}\|\phi\|_{\mathbf{A}^{0,0}} + C\|\varphi\|_{\mathbf{B}^{0,0,0}}\|\phi\|_{\mathbf{A}_{\sigma,\alpha-\omega_1}}$$

are valid, provided that the right-hand sides are bounded, where $\omega \in (0,1)$, $\alpha \geq \omega_1 > \omega$.

Proof of Theorem 2.70

By virtue of the local existence Theorem 2.67 we can consider the Cauchy problem (2.143) for all $t > t_1 > 0$ taking the initial data at a time $t_1 > 0$ $\widetilde{u} = u(t_1) \in \mathbf{H}^{s',0}(\mathbf{R}^n) \cap \mathbf{H}^{0,\omega'}(\mathbf{R}^n)$ with $s' > \frac{n}{2} + s$, $\omega' > \frac{n}{2} + \omega$, due to the smoothing effect in the case of $\nu > 0$. Via the Green operator $\mathcal{G}(t)$ of the linear Cauchy problem (2.152) we write the nonlinear Cauchy problem (2.143) as the integral equation (changing $t - t_1 \to t$ if it is necessary)

$$u(t) = \mathcal{G}(t)\widetilde{u} - \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau. \tag{2.160}$$

We apply Theorem 1.17. Denote

$$\|\phi\|_{\mathbf{X}} = \sup_{s\in[0,\sigma]} \sup_{\rho\in[0,\alpha]} \sup_{t>0} \{t\}^{\frac{s}{\nu}} \langle t\rangle^{\frac{\rho}{\delta}} \|\phi(t)\|_{\mathbf{A}^{s,\rho}} + \sup_{s\in[0,\sigma]} \sup_{\rho\in[0,\alpha]} \sup_{t>0} \{t\}^{\frac{s}{\nu}} \langle t\rangle^{\frac{\rho-\omega}{\delta}} \|\phi(t)\|_{\mathbf{B}^{s,\rho,\omega}} .$$

Using Lemma 2.71 for $s \in [0,\sigma]$, $\rho \in [0,\alpha]$,we get

$$\|\mathcal{G}(t)\widetilde{u}\|_{\mathbf{A}^{s,\rho}} \le C\{t\}^{-\frac{s}{\nu}} \|\widetilde{u}\|_{\mathbf{A}^{0,0}} ,$$

and

$$\left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau\right\|_{\mathbf{A}^{s,\rho}} \le C\int_0^t d\tau\,\{t-\tau\}^{-\frac{s}{\nu}} \|\mathcal{N}(u)(\tau)\|_{\mathbf{A}^{0,0}} ,$$

for $t\in[0,1]$,where $\mathcal{N}(v) \equiv \mathcal{N}(v,v)$. Further we have

$$\|\mathcal{G}(t)\widetilde{u}\|_{\mathbf{A}^{s,\rho}} \le C\langle t\rangle^{-\frac{n+\rho}{\delta}} \|\widetilde{u}\|_{\mathbf{B}^{0,0,\omega}} ,$$

$$\left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau\right\|_{\mathbf{A}^{s,\rho}} \le C\int_0^{\frac{t}{2}} d\tau\,\langle t-\tau\rangle^{-\frac{\rho+n}{\delta}} \|\mathcal{N}(u)(\tau)\|_{\mathbf{B}^{0,0,0}} + C\int_{\frac{t}{2}}^t d\tau\,\langle t-\tau\rangle^{-\frac{\rho}{\delta}} \{t-\tau\}^{-\frac{s}{\nu}} \|\mathcal{N}(u)(\tau)\|_{\mathbf{A}^{0,0}}$$

and

$$\|\mathcal{G}(t)\widetilde{u}\|_{\mathbf{B}^{s,\rho,\omega}} \le C\langle t\rangle^{\frac{\omega-\rho}{\delta}} \|\widetilde{u}\|_{\mathbf{B}^{0,0,\omega}} ,$$

$$\begin{aligned}&\left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau\right\|_{\mathbf{B}^{s,\rho,\omega}} \\ &\le C\int_0^t d\tau\,\langle t-\tau\rangle^{\frac{\omega-\rho}{\delta}} \{t-\tau\}^{-\frac{s}{\nu}} \|\mathcal{N}(u)(\tau)\|_{\mathbf{B}^{0,0,0}} \\ &+ C\int_0^t d\tau\,\langle t-\tau\rangle^{-\frac{\rho}{\delta}} \{t-\tau\}^{-\frac{s}{\nu}} \|\mathcal{N}(u)(\tau)\|_{\mathbf{B}^{0,0,\omega}}\end{aligned}$$

for $t \ge 1$. By Lemma 2.73 we obtain

$$\|\mathcal{N}(u)(\tau)\|_{\mathbf{A}^{0,0}} \le C\|u(\tau)\|_{\mathbf{A}^{\sigma,\alpha}} \|u(\tau)\|_{\mathbf{A}^{0,0}} \le C\varepsilon^2\{\tau\}^{-\frac{\sigma}{\nu}} \langle\tau\rangle^{-\frac{\alpha}{\delta}-\frac{2n}{\delta}} \|u\|_{\mathbf{X}}^2 ,$$

$$\begin{aligned}&\|\mathcal{N}(u)(\tau)\|_{\mathbf{B}^{0,0,0}} \le C\|u(\tau)\|_{\mathbf{A}^{\sigma,\alpha}} \|u(\tau)\|_{\mathbf{B}^{0,0,0}} \\ &\le C\varepsilon^2\{\tau\}^{-\frac{\sigma}{\nu}} \langle\tau\rangle^{-\frac{n+\alpha}{\delta}} \|u\|_{\mathbf{X}}^2 ,\end{aligned}$$

and

$$\begin{aligned}
&\|\mathcal{N}(u)(\tau)\|_{\mathbf{B}^{0,0,\omega}} \leq C\|u(\tau)\|_{\mathbf{B}^{\sigma,\alpha,\omega}}\|u(\tau)\|_{\mathbf{A}^{0,0}} \\
&+ C\|u(\tau)\|_{\mathbf{B}^{\sigma,\alpha-\omega_1,0}}\|u(\tau)\|_{\mathbf{A}^{0,0}} \\
&+ C\|u(\tau)\|_{\mathbf{B}^{0,0,\omega}}\|u(\tau)\|_{\mathbf{A}^{\sigma,\alpha}} + C\|u(\tau)\|_{\mathbf{B}^{0,0,0}}\|u(\tau)\|_{\mathbf{A}^{\sigma,\alpha-\omega_1}} \\
&\leq C\varepsilon^2\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{\frac{\omega_1-n-\alpha}{\delta}}\|u\|_{\mathbf{X}}^2;
\end{aligned}$$

hence,

$$\begin{aligned}
\left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau\right\|_{\mathbf{A}^{s,\rho}} &\leq C\|u\|_{\mathbf{X}}^2\int_0^t d\tau\,\{t-\tau\}^{-\frac{s}{\nu}}\{\tau\}^{-\frac{\sigma}{\nu}} \\
&\leq C\varepsilon\{t\}^{-\frac{s}{\nu}}\|u\|_{\mathbf{X}}^2
\end{aligned}$$

for all $t \in [0,1]$. In addition we have for all $t \geq 1$

$$\begin{aligned}
&\left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau\right\|_{\mathbf{A}^{s,\rho}} \\
&\leq \|u\|_{\mathbf{X}}^2\left(\int_0^{\frac{t}{2}} d\tau\,\langle t-\tau\rangle^{-\frac{\rho+n}{\delta}}\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{-\frac{n+\alpha}{\delta}}\right. \\
&\left.+\int_{\frac{t}{2}}^t d\tau\,\langle t-\tau\rangle^{-\frac{\rho}{\delta}}\{t-\tau\}^{-\frac{s}{\nu}}\langle\tau\rangle^{-\frac{\alpha}{\delta}-\frac{2n}{\delta}}\right) \leq C\|u\|_{\mathbf{X}}^2\langle t\rangle^{-\frac{n+\rho}{\delta}}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau\right\|_{\mathbf{B}^{s,\rho,\omega}} \\
&\leq C\|u\|_{\mathbf{X}}^2\left(\int_0^t d\tau\,\langle t-\tau\rangle^{\frac{\omega-\rho}{\delta}}\{t-\tau\}^{-\frac{s}{\nu}}\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{-\frac{n+\alpha}{\delta}}\right. \\
&\left.+\int_0^t d\tau\,\langle t-\tau\rangle^{-\frac{\rho}{\delta}}\{t-\tau\}^{-\frac{s}{\nu}}\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{\frac{\omega_1-n-\alpha}{\delta}}\right) \leq C\|u\|_{\mathbf{X}}^2\langle t\rangle^{\frac{\omega-\rho}{\delta}}
\end{aligned}$$

since $0 < \delta < n+\alpha$ and $\omega_1 > \omega$ is taken sufficiently close to ω. In the same manner we estimate a difference

$$\begin{aligned}
&\left\|\int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(u_1)(\tau)-\mathcal{N}(u_2)(\tau))\,d\tau\right\|_{\mathbf{X}} \\
&\leq C\|u_1-u_2\|_{\mathbf{X}}(\|u_1\|_{\mathbf{X}}+\|u_2\|_{\mathbf{X}}).
\end{aligned}$$

Hence via Theorem 1.17 there exists a unique solution

$$u(t,x) \in \mathbf{C}\left([t_1,\infty), \mathbf{H}^{s'}(\mathbf{R}^n) \cap \mathbf{H}^{0,\omega'}(\mathbf{R}^n)\right).$$

Now applying Lemma 2.72 and Theorem2.4 we find the asymptotics (2.159) with coefficient

$$A = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \widetilde{u}(x)\, dx - (2\pi)^{-\frac{n}{2}} \int_0^\infty d\tau \int_{\mathbf{R}^n} \mathcal{N}(u)(\tau, x)\, dx.$$

Note that in view of the estimates for the solution $u(t, x)$, the coefficient A can be calculated approximately with any desired accuracy via the integral equation (2.160). Theorem 2.70 is proved.

2.7.3 Large initial data

In Theorem 2.70 we did not take into account a special character of the nonlinearity and in fact we applied the methods of the linear theory, so its applications to particular systems gives us rough results. In this section we intend to remove the smallness condition for the initial data. As we know the condition of strong dissipation (2.155) prevents the effect of blow up for solutions (see, e.g. Naumkin and Shishmarev [1994b], Naumkin and Shishmarev [1996]) so that any large classical solutions can exist globally in time. In particular, this fact is due to some special symmetry of the nonlinearity of these equations, allowing us to easily estimate the $\mathbf{L}^2$ norm of the solution. We now write this symmetry property in the following form

$$\operatorname{Re} \int_{\mathbf{R}^n} (v \cdot \mathcal{N}(v))\, dx = 0 \tag{2.161}$$

for any vector-function $v \in \mathbf{C}_0^\infty(\mathbf{R}^n)$. Note that the symmetry property (2.161) is fulfilled for the Navier-Stokes system.

Now we can state the result analogous to Theorem 2.70 without any restriction on the size of the initial data; however, the critical value δ_c is now shifted by the value $\frac{n}{2}$.

We suppose that the symbol of the nonlinear operator $\mathcal{N}$ is such that

$$\left|\partial_\xi^r a^{kl}(t, \xi, y)\right| \le C \{\xi\}^{\alpha - |r|} \langle \xi \rangle^\sigma \tag{2.162}$$

for all $\xi \in \mathbf{R}^n$, $y \in \mathbf{R}^n$, $t > 0$, where $\sigma \ge 0$, $\alpha \ge 1$, $|r| = 0, 1$.

Theorem 2.74. *Let the linear operator $\mathcal{L}$ satisfy conditions (2.155) through (2.158) with $\delta \le \alpha + \frac{n}{2}$. Suppose that the nonlinear operator $\mathcal{N}$ satisfies estimates (2.162) with $\sigma \in [0, \nu)$, $\nu > 0$ and the symmetry property (2.161). Let the initial data $\widetilde{u} \in \mathbf{H}^{s',0}(\mathbf{R}^n) \cap \mathbf{H}^{s_-,\omega'}(\mathbf{R}^n)$, with $s' > \frac{n}{2} + s$, $s > S$, $\omega' > \frac{n}{2}$, $s_- = \min(s, 0)$, $\nu > \frac{n}{3} + \frac{2\sigma}{3}$. Then there exists a unique global solution $u(t, x) \in \mathbf{C}^0\left([0, \infty); \mathbf{A}^{s,0} \cap \mathbf{B}^{0,0,\omega}\right) \cap \mathbf{C}^1\left((0, \infty)); \mathbf{H}^{\infty,0}(\mathbf{R}^n)\right)$ of the Cauchy problem (2.143). Moreover, the solution has asymptotics (2.159).*

For example, we can apply Theorem 2.74 to the Navier-Stokes system with large initial data in space dimension $n = 2$. The global existence of "large" solutions to the Navier-Stokes system was studied extensively by many authors (see Ladyženskaja et al. [1967], Lions [1969], Temam [1979] and references cited there in). Large time decay estimates of solutions in different norms were obtained in papers Ladyzhenskaya [1963], Giga and Kambe [1988], Schonbek [1986], Schonbek [1991], Wiegner [1987]. Our Theorem 2.74 gives us the following asymptotics as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$

$$u(t,x) = t^{-1} G\left(xt^{-\frac{1}{2}}\right) \int_{\mathbf{R}^n} \widetilde{u}(x)\,dx + O\left(t^{-1-\gamma}\right),$$

where $\gamma > 0$, $G(x)$ is the heat kernel. Note that the main term of the asymptotics has the same form as that of the linear case.

Before proving Theorem 2.74 we consider some estimates.

Preliminary lemmas

Suppose that the symbols of the nonlinear operator $\mathcal{N}$ are such that

$$\left|\partial_\xi^r a^{kl}(t,\xi,y)\right| \le C\{\xi\}^{\alpha-|r|}\langle\xi\rangle^\sigma \tag{2.163}$$

for all $\xi \in \mathbf{R}^n$, $y \in \mathbf{R}^n$, $t > 0$, where $\sigma \ge 0$, $\alpha \ge 1$, $|r| = 0, 1$. We use the norm

$$\|\varphi(t)\|_{\mathbf{A}^{s,\rho,p}} = \|\{\cdot\}^\rho \langle\cdot\rangle^s \widehat{\varphi}(t,\cdot)\|_{\mathbf{L}^p_\xi}.$$

Note that $\|\cdot\|_{\mathbf{A}^{s,\rho}} = \|\cdot\|_{\mathbf{A}^{s,\rho,1}}$.

Lemma 2.75. *Let the nonlinear operator $\mathcal{N}$ satisfy condition (2.163). Then the estimate*

$$\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{A}^{s,\rho,p}} \le C\|\varphi\|_{\mathbf{A}^{s+\sigma,\rho+\alpha,q}}\|\phi\|_{\mathbf{A}^{0,0,r}} + C\|\varphi\|_{\mathbf{A}^{0,0,q}}\|\phi\|_{\mathbf{A}^{s+\sigma,\rho+\alpha,r}}$$

is true provided that the right-hand side is bounded, where $s \ge -\sigma$, $\rho \ge -\alpha$, $1 \le p,q,r \le \infty$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1$.

Proof. By virtue of condition (2.163) via the Young inequality we obtain

$$\begin{aligned}
&\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{A}^{s,\rho,p}} \\
&\le C\sum_{k,l=1}^m \left\|\{\cdot\}^\rho\langle\cdot\rangle^s \int_{\mathbf{R}^n} a^{kl}(t,\cdot,y)\,\widehat{\varphi}_k(t,\cdot-y)\,\widehat{\phi}_l(t,y)\,dy\right\|_{\mathbf{L}^p_\xi} \\
&\le C\sum_{k,l=1}^m \left\|\int_{\mathbf{R}^n} \{\cdot\}^{\rho+\alpha}\langle\cdot\rangle^{s+\sigma}\left|\widehat{\varphi}_k(t,\cdot-y)\,\widehat{\phi}_l(t,y)\right|dy\right\|_{\mathbf{L}^p_\xi} \\
&\le C\sum_{k,l=1}^m \left(\left\|\{\cdot\}^{\rho+\alpha}\langle\cdot\rangle^{s+\sigma}\widehat{\varphi}_k\right\|_{\mathbf{L}^q_\xi}\left\|\widehat{\phi}_l\right\|_{\mathbf{L}^r_\xi}\right. \\
&\left.+\|\widehat{\varphi}_k\|_{\mathbf{L}^q_\xi}\left\|\{\cdot\}^{\rho+\alpha}\langle\cdot\rangle^{s+\sigma}\widehat{\phi}_l\right\|_{\mathbf{L}^r_\xi}\right) \\
&= C\|\varphi\|_{\mathbf{A}^{s+\sigma,\rho+\alpha,q}}\|\phi\|_{\mathbf{A}^{0,0,r}} + C\|\varphi\|_{\mathbf{A}^{0,0,q}}\|\phi\|_{\mathbf{A}^{s+\sigma,\rho+\alpha,r}},
\end{aligned}$$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1$. Lemma 2.75 is proved.

Proof of Theorem 2.74

We can easily see that there exists a unique solution

$$u(t,x) \in \mathbf{C}\left((0,\infty);\mathbf{H}^{s',0}(\mathbf{R}^n) \cap \mathbf{H}^{0,\omega'}(\mathbf{R}^n)\right)$$

to the Cauchy problem (2.143), where $s' > \frac{n}{2} + s$, $\omega' > \frac{n}{2} + \omega$. If the initial data $\widetilde{u}$ can be taken from $\mathbf{L}^2$ (for example, when $\nu > n - \sigma$), then the global existence of solutions follows through a standard prolongation procedure, in view of the estimate

$$\|u\|_{\mathbf{L}^2}^2 \le \|\widetilde{u}\|_{\mathbf{L}^2}^2$$

(see (2.164) below). For the global existence of solutions to the Navier-Stokes system in two dimensional case see Ladyzhenskaya [1963].

Multiplying equation (2.143) by u, then integrating with respect to $x \in \mathbf{R}^n$, using property (2.161), we get

$$\begin{aligned} \frac{d}{dt}\|u\|_{\mathbf{L}^2}^2 &= -2\mathrm{Re}\int_{\mathbf{R}^n}(u\cdot\mathcal{N}(u) + u\cdot\mathcal{L}u)\,dx \\ &= -2\int_{\mathbf{R}^n}\mathrm{Re}\,L(\xi)\,|\widehat{u}(t,\xi)|^2\,d\xi \le 0; \end{aligned} \tag{2.164}$$

hence we see that the norm $\|u(t)\|_{\mathbf{L}^2}$ is bounded and monotonically decreases for all $t \ge 0$. We now prove that

$$\|u(t)\|_{\mathbf{L}^2} \to 0$$

as $t \to \infty$. On the contrary we assume that there exists $\varepsilon_1 > 0$ such that

$$\|u(t)\|_{\mathbf{L}^2} \ge \varepsilon_1 \text{ for all } t > 0.$$

We take $\varrho \in (0,1)$ and write the estimate

$$\begin{aligned} &\int_{\mathbf{R}^n}\mathrm{Re}\,L(\xi)\,|\widehat{u}(t,\xi)|^2\,d\xi \ge \mu\int_{\mathbf{R}^n:|\xi|\ge\varrho}\{\xi\}^{\delta}\langle\xi\rangle^{\nu}\,|\widehat{u}(t,\xi)|^2\,d\xi \\ &\ge \mu\varrho^{\delta}\|u(t)\|_{\mathbf{L}^2}^2 - \mu\varrho^{\delta+n}\sup_{|\xi|<\varrho}|\widehat{u}(t,\xi)|^2 \ge \mu\varrho^{\delta}\|u(t)\|_{\mathbf{L}^2}^2 - \mu\varrho^{\delta+n}\|u(t)\|_{\mathbf{A}^{0,0,\infty}}^2. \end{aligned}$$

Since $\|u(t)\|_{\mathbf{L}^2} = \|u(t)\|_{\mathbf{A}^{0,0,2}}$ is bounded we get via Lemma 2.75

$$\|\mathcal{N}(u)(\tau)\|_{\mathbf{A}^{-\sigma,-\alpha,\infty}} \le C\|u(t)\|_{\mathbf{L}^2}^2 \le C;$$

hence

$$\begin{aligned} &\|u(t)\|_{\mathbf{A}^{0,0,\infty}} \le \|\mathcal{G}(t)\widetilde{u}\|_{\mathbf{A}^{0,0,\infty}} + \left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau\right\|_{\mathbf{A}^{0,0,\infty}} \\ &\le C + C\int_0^t d\tau\,\{t-\tau\}^{-\frac{\sigma}{\nu}}\langle t-\tau\rangle^{-\frac{\alpha}{\delta}}\|\mathcal{N}(u)(\tau)\|_{\mathbf{A}^{-\sigma,-\alpha,\infty}} \\ &\le C + C\int_0^t \{t-\tau\}^{-\frac{\sigma}{\nu}}\langle t-\tau\rangle^{-\frac{\alpha}{\delta}}\,d\tau \le C\langle t\rangle^{1-\frac{\alpha}{\delta}}. \end{aligned}$$

Therefore we obtain

$$\int_{\mathbf{R}^n} \operatorname{Re} L(\xi) |\widehat{u}(t,\xi)|^2 \, d\xi \geq \mu \varrho^\delta \left(\varepsilon_1 - C\varrho^n \langle t \rangle^{2-\frac{2\alpha}{\delta}} \right)$$

and

$$\frac{d}{dt} \|u\|_{\mathbf{L}^2}^2 \leq -\mu \varrho^\delta \left(\varepsilon_1 - C\varrho^n \langle t \rangle^{2-\frac{2\alpha}{\delta}} \right). \tag{2.165}$$

We now choose $\varrho = \varrho(t)$ such that

$$\varrho(t) = o\left(\langle t \rangle^{\frac{2\alpha}{\delta n} - \frac{2}{n}} \right)$$

as $t \to \infty$ and

$$\int_0^\infty \varrho^\delta(t) \, dt = \infty.$$

For example, we can take $\varrho(t) = \langle t \rangle^{\frac{2\alpha}{\delta n} - \frac{2}{n}} (\log(2+t))^{-\frac{1}{\delta}}$, since now $\delta \leq \alpha + \frac{n}{2}$ by the condition of Theorem 2.74. Inequality (2.165) yields

$$\frac{d}{dt} \|u\|_{\mathbf{L}^2}^2 \leq -\frac{\mu}{2} \varrho^\delta \varepsilon_1$$

for all $t \geq T_1 > 0$, where $T_1 > 0$ is sufficiently large. Integrating the last estimate with respect to $t > T_1$ we see that the norm $\|u(t)\|_{\mathbf{L}^2}^2$ becomes negative after some time. The contradiction obtained proves that $\|u(t)\|_{\mathbf{L}^2} \to 0$ for $t \to \infty$. Integration of (2.164) with respect to $t \geq 0$ gives us

$$\sup_{t \geq 0} \|u(t)\|_{\mathbf{L}^2}^2 + 2\mu \int_0^\infty d\tau \int_{\mathbf{R}^n} \{\xi\}^\delta \langle \xi \rangle^\nu |\widehat{u}(t,\xi)|^2 \, d\xi \leq \|\widetilde{u}\|_{\mathbf{L}^2}^2 ;$$

hence we see that for any $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that the norm

$$\|u(T_\varepsilon)\|_{\mathbf{H}^{\nu/2}} < \varepsilon.$$

We take $T_\varepsilon > 0$ as the initial time and consider the Cauchy problem (2.143) with initial data $\widetilde{u} = u(T_\varepsilon)$ such that the norm

$$\|\widetilde{u}\|_{\mathbf{A}_{s,0}} \leq C \|\widetilde{u}\|_{\mathbf{H}^{\nu/2}}$$

is sufficiently small, where we suppose that $\frac{\nu}{2} > \frac{n}{2} + s$ and $s > S = \frac{1}{2}(\sigma - \nu)$.Via the Green operator $\mathcal{G}(t)$ of the linear Cauchy problem (2.152) we write the nonlinear Cauchy problem (2.143) as the integral equation (2.160) with initial data $\widetilde{u}$. Define a ball

$$\mathbf{Y} = \Bigg\{ \phi \in \mathbf{S}' : \sup_{s \in [0,\sigma]} \sup_{\rho \in [0,\alpha]} \sup_{p \in [1,2]} \sup_{t > 0} \{t\}^{\frac{s}{\nu}} \langle t \rangle^{\frac{\rho}{\delta} + \frac{n}{\delta p} - \frac{n}{2\delta}} \|\phi(t)\|_{\mathbf{A}^{s,\rho,p}} \leq \varepsilon;$$
$$\sup_{s \in [0,\sigma]} \sup_{\rho \in [0,\alpha]} \sup_{p \in [1,\infty]} \sup_{t > 0} \{t\}^{\frac{s}{\nu}} \langle t \rangle^{\frac{\rho}{\delta} + \frac{n}{\delta p}} \|\phi(t)\|_{\mathbf{A}^{s,\rho,p}}$$
$$+ \sup_{s \in [0,\sigma]} \sup_{\rho \in [0,\alpha]} \sup_{t > 0} \{t\}^{\frac{s}{\nu}} \langle t \rangle^{\frac{\rho - \omega}{\delta}} \|\phi(t)\|_{\mathbf{B}^{s,\rho,\omega}} \leq C \Bigg\}.$$

We define the transformation for $v \in \mathbf{Y}$

$$\mathcal{M}(v)(t) = \mathcal{G}(t)\widetilde{u} + \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v)(\tau)\,d\tau.$$

Now we prove that $\mathcal{M}$ is a contraction mapping in $\mathbf{Y}$.

By Lemma 2.71 we have for $s \in [0,\sigma]$, $\rho \in [0,\alpha]$, $p \in [1,2]$

$$\begin{aligned}
&\|\mathcal{M}(v)(t)\|_{\mathbf{A}^{s,\rho,p}} \le \|\mathcal{G}(t)\widetilde{u}\|_{\mathbf{A}^{s,\rho,p}} + \left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v)(\tau)\,d\tau\right\|_{\mathbf{A}^{s,\rho,p}} \\
&\le C\{t\}^{-\frac{s}{\nu}}\langle t\rangle^{-\frac{\rho}{\delta}+\frac{n}{\delta}\left(\frac{1}{2}-\frac{1}{p}\right)}\|\widetilde{u}\|_{\mathbf{L}^2} \\
&+C\int_0^{\frac{t}{2}} d\tau\,\langle t-\tau\rangle^{-\frac{\rho+\alpha}{\delta}-\frac{n}{p\delta}+\frac{n}{2\delta}}\{t-\tau\}^{-\frac{s+\sigma}{\nu}}\|\mathcal{N}(v)(\tau)\|_{\mathbf{A}^{-\sigma,-\alpha,2}} \\
&+C\int_{\frac{t}{2}}^t d\tau\,\{t-\tau\}^{-\frac{\sigma}{\nu}}\|\mathcal{N}(v)(\tau)\|_{\mathbf{A}^{s-\sigma,\rho,p}}.
\end{aligned}$$

Since $v \in \mathbf{Y}$, by virtue of Lemma 2.73 we get

$$\|\mathcal{N}(v)(\tau)\|_{\mathbf{A}^{-\sigma,-\alpha,2}} \le C\|v(\tau)\|_{\mathbf{A}^{0,0,1}}\|v(\tau)\|_{\mathbf{A}^{0,0,2}} \le C\varepsilon^2\langle\tau\rangle^{-\frac{n}{2\delta}}$$

and

$$\begin{aligned}
\|\mathcal{N}(v)(\tau)\|_{\mathbf{A}^{s-\sigma,\rho,p}} &\le C\|v(\tau)\|_{\mathbf{A}^{s,\rho+\alpha,1}}\|v(\tau)\|_{\mathbf{A}^{0,0,p}} \\
&\le C\varepsilon^2\{\tau\}^{-\frac{s}{\nu}}\langle\tau\rangle^{-\frac{\rho+\alpha}{\delta}-\frac{n}{\delta p}};
\end{aligned}$$

therefore,

$$\begin{aligned}
&\|\mathcal{M}(v)(t)\|_{\mathbf{A}^{s,\rho,p}} \le C\varepsilon\{t\}^{-\frac{s}{\nu}}\langle t\rangle^{-\frac{\rho}{\delta}-\frac{n}{\delta p}+\frac{n}{2\delta}} \\
&+C\varepsilon^2\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-\frac{\rho+\alpha}{\delta}-\frac{n}{p\delta}+\frac{n}{2\delta}}\{t-\tau\}^{-\frac{s+\sigma}{\nu}}\langle\tau\rangle^{-\frac{n}{2\delta}}\,d\tau \\
&+C\varepsilon^2\int_{\frac{t}{2}}^t\{t-\tau\}^{-\frac{\sigma}{\nu}}\{\tau\}^{-\frac{s}{\nu}}\langle\tau\rangle^{-\frac{\rho+\alpha}{\delta}-\frac{n}{\delta p}}\,d\tau \\
&\le C\varepsilon\{t\}^{-\frac{s}{\nu}}\langle t\rangle^{-\frac{\rho}{\delta}-\frac{n}{\delta p}+\frac{n}{2\delta}}.
\end{aligned}$$

In the same manner we obtain for $s \in [0,\sigma]$, $\rho \in [0,\alpha]$, $p \in [1,\infty]$

$$\begin{aligned}
&\|\mathcal{M}(v)(t)\|_{\mathbf{A}^{s,\rho,p}} \le \|\mathcal{G}(t)\widetilde{u}\|_{\mathbf{A}^{s,\rho,p}} + \left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v)(\tau)\,d\tau\right\|_{\mathbf{A}^{s,\rho,p}} \\
&\le C\{t\}^{-\frac{s}{\nu}}\langle t\rangle^{-\frac{\rho}{\delta}-\frac{n}{\delta p}}\|\widetilde{u}\|_{\mathbf{A}^{0,0,\infty}} \\
&+C\int_0^{\frac{t}{2}} d\tau\,\langle t-\tau\rangle^{-\frac{\rho+\alpha}{\delta}-\frac{n}{p\delta}}\{t-\tau\}^{-\frac{s+\sigma}{\nu}}\|\mathcal{N}(v)(\tau)\|_{\mathbf{A}^{-\sigma,-\alpha,\infty}} \\
&+C\int_{\frac{t}{2}}^t d\tau\,\{t-\tau\}^{-\frac{\sigma}{\nu}}\|\mathcal{N}(v)(\tau)\|_{\mathbf{A}^{s-\sigma,\rho,p}}.
\end{aligned}$$

Since $v \in \mathbf{Y}$ and $\delta \leq \alpha + \frac{n}{2}$ in view of Lemma 2.73 we have

$$\|\mathcal{N}(v)(\tau)\|_{\mathbf{A}^{-\sigma,-\alpha,\infty}} \leq C \|v(\tau)\|_{\mathbf{A}^{0,0,2}} \|v(\tau)\|_{\mathbf{A}^{0,0,2}} \leq C\varepsilon \langle\tau\rangle^{-\frac{n}{2\delta}}$$

and

$$\begin{aligned}\|\mathcal{N}(v)(\tau)\|_{\mathbf{A}^{s-\sigma,\rho,p}} &\leq C \|v(\tau)\|_{\mathbf{A}^{s,\rho+\alpha,1}} \|v(\tau)\|_{\mathbf{A}^{0,0,p}} \\ &\leq C\varepsilon \{\tau\}^{-\frac{s}{\nu}} \langle\tau\rangle^{-\frac{\rho+\alpha}{\delta}-\frac{n}{\delta p}-\frac{n}{2\delta}};\end{aligned}$$

hence

$$\begin{aligned}&\|\mathcal{M}(v)(t)\|_{\mathbf{A}^{s,\rho,p}} \leq C\{t\}^{-\frac{s}{\nu}} \langle t\rangle^{-\frac{\rho}{\delta}-\frac{n}{\delta p}} \\ &+C\varepsilon \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{\rho+\alpha}{\delta}-\frac{n}{p\delta}} \{t-\tau\}^{-\frac{s+\sigma}{\nu}} \langle\tau\rangle^{-\frac{n}{2\delta}} d\tau \\ &+C\varepsilon \int_{\frac{t}{2}}^{t} \{t-\tau\}^{-\frac{\sigma}{\nu}} \{\tau\}^{-\frac{s}{\nu}} \langle\tau\rangle^{-\frac{\rho+\alpha}{\delta}-\frac{n}{\delta p}-\frac{n}{2\delta}} d\tau \\ &\leq C\{t\}^{-\frac{s}{\nu}} \langle t\rangle^{-\frac{\rho}{\delta}-\frac{n}{\delta p}}.\end{aligned}$$

Finally for $s \in [0,\sigma]$, $\rho \in [0,\alpha]$ we get

$$\begin{aligned}&\|\mathcal{M}(v)(t)\|_{\mathbf{B}^{s,\rho,\omega}} \leq \|\mathcal{G}(t)\widetilde{u}\|_{\mathbf{B}^{s,\rho,\omega}} + \left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v)(\tau)\,d\tau\right\|_{\mathbf{B}^{s,\rho,\omega}} \\ &\leq C\{t\}^{-\frac{s}{\nu}} \langle t\rangle^{\frac{\omega-\rho}{\delta}} \|\widetilde{u}\|_{\mathbf{B}^{0,0,\omega}} \\ &+C\int_0^t d\tau \langle t-\tau\rangle^{\frac{\omega-\rho}{\delta}} \{t-\tau\}^{-\frac{s}{\nu}} \|\mathcal{N}(v)(\tau)\|_{\mathbf{A}^{0,0,\infty}} \\ &+C\int_0^t d\tau \langle t-\tau\rangle^{-\frac{\rho}{\delta}} \{t-\tau\}^{-\frac{s}{\nu}} \|\mathcal{N}(v)(\tau)\|_{\mathbf{B}^{0,0,\omega}}.\end{aligned}$$

Estimating by Lemma 2.73

$$\begin{aligned}&\|\mathcal{N}(v)(\tau)\|_{\mathbf{A}^{0,0,\infty}} \leq C \|v(\tau)\|_{\mathbf{A}^{s,\sigma,2}} \|v(\tau)\|_{\mathbf{A}^{0,0,2}} \\ &\leq C\sqrt{\varepsilon}\{\tau\}^{-\frac{\sigma}{\nu}} \langle\tau\rangle^{-\frac{\alpha}{\delta}-\frac{3n}{4\delta}}\end{aligned}$$

and

$$\begin{aligned}&\|\mathcal{N}(v)(\tau)\|_{\mathbf{B}^{0,0,\omega}} \\ &\leq C\|v(\tau)\|_{\mathbf{B}^{\sigma,\alpha,\omega}} \|v(\tau)\|_{\mathbf{A}^{0,0}} + C\|v(\tau)\|_{\mathbf{B}^{\sigma,\alpha-\omega_1,0}} \|v(\tau)\|_{\mathbf{A}^{0,0}} \\ &+ C\|v(\tau)\|_{\mathbf{B}^{0,0,\omega}} \|v(\tau)\|_{\mathbf{A}^{\sigma,\alpha}} + C\|v(\tau)\|_{\mathbf{B}^{0,0,0}} \|v(\tau)\|_{\mathbf{A}^{\sigma,\alpha-\omega_1}} \\ &\leq C\sqrt{\varepsilon}\{\tau\}^{-\frac{\sigma}{\nu}} \langle\tau\rangle^{\frac{\omega-\alpha}{\delta}-\frac{n}{2\delta}},\end{aligned}$$

we find

$$\begin{aligned}
&\|\mathcal{M}(v)(t)\|_{\mathbf{B}^{s,\rho,\omega}} \\
&\le C\{t\}^{-\frac{s}{\nu}}\langle t\rangle^{\frac{\omega-\rho}{\delta}} + C\sqrt{\varepsilon}\int_0^t \langle t-\tau\rangle^{\frac{\omega-\rho}{\delta}}\{t-\tau\}^{-\frac{s}{\nu}}\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{-\frac{\alpha}{\delta}-\frac{3n}{4\delta}}\,d\tau \\
&+C\sqrt{\varepsilon}\int_0^t \langle t-\tau\rangle^{-\frac{\rho}{\delta}}\{t-\tau\}^{-\frac{s}{\nu}}\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{\frac{\omega-\alpha}{\delta}-\frac{n}{2\delta}}\,d\tau \\
&\le C\{t\}^{-\frac{s}{\nu}}\langle t\rangle^{\frac{\omega-\rho}{\delta}}.
\end{aligned}$$

In the same manner we estimate a difference

$$\|u_1-u_2\|_{\mathbf{Y}} \le \frac{1}{2}\|v_1-v_2\|_{\mathbf{Y}}.$$

Therefore the transformation $\mathcal{M}$ is a contraction mapping in $\mathbf{Y}$; hence there exists a unique solution $u(t,x)\in\mathbf{C}\left([t_1,\infty),\mathbf{H}^{s'}(\mathbf{R}^n)\cap\mathbf{H}^{0,\omega'}(\mathbf{R}^n)\right)$. Applying Lemma 2.72 to integral equation (2.160) we find the asymptotics (2.159) of the solution. Theorem 2.74 is proved.

2.8 Comments

Section 2.1.

Large time asymptotic behavior of solutions for the nonlinear heat equations and convection-diffusion type equations was studied by many authors, see, e.g. papers Duro and Carpio [2001], Duro and Zuazua [2000], Kamin and Peletier [1986], Zhang [2001], and others Dix [1992], Dix [1997], Giga and Kambe [1988], Gmira and Véron [1984], Il′in and Oleĭnik [1960], Kamin and Peletier [1985], Naumkin and Shishmarev [1993b], Naumkin and Shishmarev [1993a], Naumkin and Shishmarev [1994b], Naumkin and Shishmarev [1996], Zuazua [1995].

Section 2.2.

Equation (2.26) with $\alpha=2,\ b=0$ is a nonlinear heat equation $u_t-u_{xx}+a|u|^{\sigma}u=0$; it was studied in paper Kamin and Peletier [1986] in the supercritical case $\sigma>2$. Nonlinear dissipative equations with a derivative of a fractional order in the principal part were studied extensively (see, Bardos et al. [1979], Biler et al. [1998], Hayashi et al. [2000], Hayashi et al. [2004a], Komatsu [1984] and references cited therein). Blow-up in finite time of positive solutions to the Cauchy problem

$$\left\{u_t-u_{xx}-u^{1+\sigma}=0,\ u(0,x)=u_0(x)>0\right. \tag{2.166}$$

was proven in papers Fujita [1966], Weissler [1981] for the case $0<\sigma<2,\ \alpha=2$, in papers Hayakawa [1973], Kobayashi et al. [1977] for the case $\sigma=2,\ \alpha=2$, and in paper Shlesinger et al. [1995] for the case $0<\sigma\le\alpha\le 2$. Global in time existence of small solutions to (2.166) was proven in Fujita [1966] for the supercritical case $\sigma>2$. In Escobedo and Zuazua [1991] large time behavior of solutions to the convection-diffusion equation was studied for the case $\sigma\ge 1$ without smallness condition on the data. They showed that when $u_0\in\mathbf{L}^1$, solutions of (2.26) behave like the heat kernel if $\sigma>1$ and the corresponding self-similar solutions if $\sigma=1$. Large time behavior of solutions to the problem (2.26) with $L=-\partial_x^2+|\partial_x|^{\alpha}$, $1<\alpha<2$, was studied in Biler et al. [2000], Biler et al. [2001a]. Similar results to those of paper Escobedo and Zuazua [1997] were obtained in Biler et al. [2000] for the supercritical case $\sigma > \alpha - 1$ and in Biler et al. [2001a] for the critical case $\sigma = \alpha - 1$. Their method is based on the $\mathbf{L}^1(\mathbf{R})$ - contraction property of the semigroup $\exp(-t|\partial_x|^\alpha)$ if $1 < \alpha < 2$ (see Bardos et al. [1979]). In paper Naumkin [1992], the asymptotics of solutions to the Ott-Sudan-Ostrovsky equation for large t and x was considered by application of the Fourier transformation and by use of the perturbation theory with respect to the small parameter, characterizing smallness of the initial data. In this section we developed another approach, based on the detailed investigation of the Green function of the linear theory. Also we removed the restrictions on the size of the initial data when considering the special form of the nonlinearity.

Section 2.3.

The power $\rho > 1 + \frac{2}{n}$ is supercritical for large time since it is known that solutions of (2.49) with nonlinearity $N(u) = |u|^\rho$, $1 < \rho \leq 1 + \frac{2}{n}$, blow-up in a finite time, when the initial data are positive. See Li and Zhou [1995], Nishihara [2003], Zuazua [1993].

In papers Kawashima et al. [1995], Matsumura [1976/77] it was shown that under the condition

$$\langle \xi \rangle^\delta \widehat{u}_0(\xi), \langle \xi \rangle^{\delta-1} \widehat{u}_1(\xi) \in \mathbf{L}^2$$

with $\delta > \frac{n}{2}$, the Fourier transform of a solution to the linearized problem corresponding to the damped wave equation decays exponentially in time and behaves like a solution of the linear wave equation in the high - frequency part $|\xi| \geq \frac{1}{2}$. In the low - frequency part $|\xi| \leq \frac{1}{2}$ the solution resembles that of the linear heat equation. Therefore some regularity assumptions on the data are required to get $\mathbf{L}^\infty$ time decay estimates of solutions in the high - frequency part for the case of higher space dimensions. This fact is an obstacle for considering the problem with fractional order nonlinearities in $\mathbf{L}^\infty$ function spaces in higher space dimensions $n \geq 4$. In the one dimensional case $n = 1$, the global existence in time of small solutions can be obtained by the method of paper Matsumura [1976/77].

When the initial data are in the Sobolev space $u_0 \in \mathbf{W}_1^1(\mathbf{R}^n) \cap \mathbf{W}_\infty^1(\mathbf{R}^n)$, $u_1 \in \mathbf{L}^1(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n)$ and $n = 3$, the Cauchy problem (2.49) for the damped wave equation was considered in Nishihara [2003]. By making use of the fundamental solution of the linear problem, the global existence of small solutions and large time decay estimates $\|u\|_{L^q} \leq Ct^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}$ were proven for $1 \leq q \leq \infty$ and $n = 3$. Later these requirements on the initial data were relaxed in Ono [2003] as follows $u_0 \in \mathbf{L}^1(\mathbf{R}^n) \cap \mathbf{H}^1(\mathbf{R}^n)$, $u_1 \in \mathbf{L}^1(\mathbf{R}^n) \cap \mathbf{L}^2(\mathbf{R}^n)$, under the additional assumptions on ρ and q such that $\rho \leq 5$, $q \leq 6$ for the space dimension $n = 3$ and $q < \infty$ for the two dimensional case $n = 2$. In the case of higher dimensions $n = 4, 5$, the global existence and $\mathbf{L}^q(\mathbf{R}^n)$ - time decay estimates for $\rho \leq q \leq \frac{\rho}{\rho-1}$ were obtained via Fourier analysis in paper Narazaki [2004], when the power of the nonlinearity ρ is such that $1 + \frac{2}{n} < \rho \leq \frac{n+2}{n-2}$ and the initial data are small enough and satisfy $u_0 \in \mathbf{W}_1^1(\mathbf{R}^n) \cap \mathbf{W}_{\frac{\rho}{\rho-1}}^1(\mathbf{R}^n) \cap \mathbf{H}^2(\mathbf{R}^n)$, $u_1 \in \mathbf{L}^1(\mathbf{R}^n) \cap \mathbf{L}^{\frac{\rho}{\rho-1}}(\mathbf{R}^n) \cap \mathbf{H}^1(\mathbf{R}^n)$.

Applying energy type estimates obtained in papers Matsumura [1976/77] and Kawashima et al. [1995] it was proved in Karch [2000b] that solutions of the nonlinear damped wave equation in the supercritical cases $1 + \frac{4}{n} < \rho \leq \frac{n}{n-2}$, if $n = 3$ and $1 + \frac{4}{n} < \rho < \infty$, if $n = 1, 2$, with arbitrary initial data $u_0 \in \mathbf{H}^1(\mathbf{R}^n) \cap \mathbf{L}^1(\mathbf{R}^n)$,

$u_1 \in \mathbf{L}^2(\mathbf{R}^n) \cap \mathbf{L}^1(\mathbf{R}^n)$ (that is without smallness assumption on the initial data) have the same large time asymptotics as the linear heat equation $\mathcal{L} = \partial_t - \Delta$:

$$\|u(t) - \theta G_0(t)\|_{\mathbf{L}^p} = o\left(t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}\right)$$

as $t \to \infty$, where $2 \le p < \frac{2n}{n-2}$ for $n = 3$, $2 \le p < \infty$ for $n = 2$ and $2 \le p \le \infty$ for $n = 1$. Here $G_0(t,x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ is the heat kernel, θ is a constant.

In the case of higher space dimensions the global existence and energy decay estimates of solutions to the Cauchy problem for the damped wave equation were proved in paper Todorova and Yordanov [2001] for $\rho > 1 + \frac{2}{n}$ and for sufficiently small initial data having a compact support. The material of this section was taken from paper Hayashi et al. [2004d].

Section 2.4.

A wide spectrum of physical processes lead to the Sobolev type equations (see, Gabov [1998], Gabov and Sveshnikov [1990], Korpusov et al. [1999], Korpusov et al. [2000], Korpusov and Sveshnikov [2003], Naumkin and Shishmarev [1994b], Sobolev [1954]). Application of semigroup approach to the general theory of singular Sobolev type equations was developed in paper Sviridyuk and Fëdorov [1995]. The Sobolev type equation with undefined or uninvertable operator at the highest derivative with respect to time was studied in Egorov et al. [2000]. The degenerate Sobolev type equations were investigated from the abstract point of view in Favini and Yagi [1999]. Many results for systems of equations which are not resolved with respect to the highest time derivative were obtained in Demidenko and Uspenskiĭ [1998]. For the local solvability of the Sobolev type equations see Gajewski et al. [1974]. Sobolev type equations with two nonlinearities were considered in Stefanelli [2002]. Sobolev type equations with monotone non linear terms were studied in Showalter [1997]. In papers Kozhanov [1994], Kozhanov [1999] the blow-up phenomena and global in time existence of bounded solutions to Sobolev type equations were studied. For investigation of the blow-up of solutions to various classes of nonlinear parabolic equations we refer to Fujita [1966], Levine [1973], Mitidieri and Pokhozhaev [2001]. For the large time asymptotics we refer to [Shishmarev]. The results of this section were published in Kaĭkina et al. [2005].

Section 2.5.

Last years the large time asymptotics of solutions to the Cauchy problems for dissipative equations was extensively studied. For the sharp time decay estimates for solutions see papers Amick et al. [1989], Biler [1984], Bona et al. [1999], Bona and Luo [2001], Karch [2000a], Lax [1957], Liu [1985], Pego [1986], Schonbek [1986], Schonbek [1991], Strauss [1981], Vishik [1992], Zhang [1995] and the literature cited there in. Other works concerning specific equations and proving large time asymptotic representations of solutions include Dix [1992], Dix [1997], Giga and Kambe [1988], Gmira and Véron [1984], Il′in and Oleĭnik [1960], Kamin and Peletier [1985], Naumkin and Shishmarev [1993b], Naumkin and Shishmarev [1993a], Naumkin and Shishmarev [1996], Naumkin and Shishmarev [1994b], Zuazua [1995]. This section follows the method of paper Cardiel and Naumkin [2002].

Section 2.6.

The large time asymptotic formulas for solutions of dissipative nonlinear nonlocal equations in the case of weak dissipation and strong dispersion were obtained in Naumkin and Shishmarev [1993a] via the perturbation theory with respect to the parameter, which characterizes the smallness of the initial data. In this section we develop another approach based on the construction of solutions by the contraction mapping principle. Also we use a special transformation of the equation similar to the Shatah (see Shatah [1985]) method of normal forms to consider the case of weak dissipation and strong dispersion. The results of this section were taken from paper Kaĭkina et al. [2003].

Section 2.7.

Some other functional-analytic methods were applied to dissipative equations in Dix [1997], Il′in and Oleĭnik [1960], Kato [1984], Naumkin and Shishmarev [1991], Naumkin and Shishmarev [1994b], Naumkin and Shishmarev [1995], Zuazua [1995]. The main results of this section were published in paper Kaĭkina et al. [2004a].

3

Critical Nonconvective Equations

In this chapter we study large time asymptotic behavior of solutions to the Cauchy problem for different nonlinear dissipative equations in the critical case: when the time decay rate of the nonlinear term is balanced with that of the linear part of the equation. Here we consider the nonconvective type of nonlinearities. The typical example of the nonconvective equation is the cubic nonlinear heat equation $u_t + u^3 - u_{xx} = 0$, $x \in \mathbf{R}, t > 0$. The character of the large time behavior of solutions for nonconvective equations is similar to the linear case with logarithmic correction in the decay rate. Taking into account some additional properties such as the maximum principle, positivity of initial data or weighted energy type estimates we will remove the smallness condition on the initial data.

3.1 General approach

Now we give a general approach for obtaining the large time asymptotic representation of solutions to the Cauchy problem (1.7)

$$\begin{cases} u_t + \mathcal{N}(u) + \mathcal{L}u = 0, \ x \in \mathbf{R}^n, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^n, \end{cases} \tag{3.1}$$

in the case of critical nonlinearity $\mathcal{N}(u)$ of nonconvective type. We fix a metric space $\mathbf{Z}$ of functions defined on $\mathbf{R}^n$ and a complete metric space $\mathbf{X}$ of functions defined on $[0,\infty) \times \mathbf{R}^n$. We denote as above by $G_0 \in \mathbf{X}$ the asymptotic kernel for the Green operator $\mathcal{G}$ in spaces $\mathbf{X}$, $\mathbf{Z}$ with a continuous linear functional f (see Definition 2.1.)

Definition 3.1. *We call the nonlinearity $\mathcal{N}$ in equation (3.1) critical nonconvective if the estimate is true*

$$\int_0^t \operatorname{Re} f(\mathcal{N}(\theta G_0(\tau)))\, d\tau \geq \eta \theta^{\sigma+1} \log(1+t) \tag{3.2}$$

for all $t > 0$, $\theta > 0$ with some positive constants η and σ.

Now we prove the global existence and obtain large time asymptotics of small solutions to the Cauchy problem (3.1) with a critical nonlinearity of nonconvective type. Define the function

$$g(t) = 1 + \eta\theta^{\sigma} \log(1+t)$$

with some positive constants η, θ and σ.

Theorem 3.2. *Assume that the linear operator $\mathcal{L}$ is such that $f(\mathcal{L}(u)) = 0$ for any $u \in \mathbf{X}$. Let the nonlinearity $\mathcal{N}(u)$ in equation (3.1) be critical nonconvective. Assume that*

$$e^{z}\mathcal{N}(ue^{-z}) = e^{-\sigma \operatorname{Re} z}\mathcal{N}(u) \tag{3.3}$$

for any $z \in \mathbf{C}$ and $u \in \mathbf{X}$, where $\sigma > 0$. Suppose that the estimates are valid

$$\begin{aligned} &|\nu(t) f(\mathcal{N}(v(t)) - \mathcal{N}(w(t)))| \\ &\leq C\{t\}^{-\alpha}\langle t\rangle^{-1}\|\nu(t)(v(t)-w(t))\|_{\mathbf{X}}\left(\|v\|_{\mathbf{X}}^{\sigma} + \|w\|_{\mathbf{X}}^{\sigma}\right), \end{aligned} \tag{3.4}$$

for all $t > 0$ and for any $v, w \in \mathbf{X}$, $\nu(t) > 0$, where $\alpha < 1$, $\sigma > 0$, and

$$\begin{aligned} &\left\| g(t)\int_0^t g^{-1}(\tau)\,|\mathcal{G}(t-\tau)(\mathcal{K}(v(\tau)) - \mathcal{K}(w(\tau)))|\,d\tau \right\|_{\mathbf{X}} \\ &\leq C\|v-w\|_{\mathbf{X}}\left(\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}\right)^{\sigma}\left(1 + \frac{\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}}{\theta}\right) \end{aligned} \tag{3.5}$$

for any $v, w \in \mathbf{X}$ such that $f(v) = f(w) = \theta > 0$, where $\sigma > 0$ and $\mathcal{K}(v) = \mathcal{N}(v) - \frac{v}{\theta} f(\mathcal{N}(v))$. Let the initial data $u_0 \in \mathbf{Z}$ have a small norm $\|u_0\|_{\mathbf{Z}} \leq \varepsilon$ and the mean value $\theta \equiv |f(u_0)| \geq C\varepsilon > 0$ with some $C > 0$. Then there exists a unique global solution $u \in \mathbf{X}$ to the Cauchy problem (3.1) satisfying the time decay estimate

$$\left\| g^{\frac{1}{\sigma}} u \right\|_{\mathbf{X}} \leq C\varepsilon.$$

Moreover if we assume that

$$1 + \frac{\sigma}{\theta}\int_0^t \operatorname{Re} f(\mathcal{N}(\theta G_0(\tau)))\,d\tau = g(t) + O(\log g(t)) \tag{3.6}$$

for $t \to \infty$, then the asymptotics is true

$$\left\| \left(u - \theta g^{-\frac{1}{\sigma}} e^{i\psi} G_0\right) \frac{g^{1+\frac{1}{\sigma}}}{\log\log(4+t)} \right\|_{\mathbf{X}} \leq C,$$

where $\psi(t)$ satisfies the asymptotic estimate

$$\psi(t) = \widetilde{\eta}\log\log t + O(1)$$

for $t \to \infty$, with some constant $\widetilde{\eta} \in \mathbf{R}$.

Proof. We change the dependent variable $u(t,x) = v(t,x)\,e^{-\varphi(t)+i\psi(t)}$ in equation (3.1). Then in view of condition (3.3) we get the following equation for the new unknown function $v(t,x)$

$$v_t + \mathcal{L}v + e^{-\sigma\varphi}\mathcal{N}(v) - (\varphi' - i\psi')v = 0.$$

We now choose the auxiliary functions $\varphi(t)$ and $\psi(t)$ by the following condition

$$f\left(e^{-\sigma\varphi}\mathcal{N}(v) - (\varphi' - i\psi')v\right) = 0;$$

then in view of the condition $f(\mathcal{L}(v)) = 0$ we obtain a conservation law

$$\frac{d}{dt}f(v(t)) = f(v_t(t)) = 0.$$

Hence $f(v(t)) = f(v_0)$ for all $t > 0$. We also choose $\varphi(0) = 0$ and $\psi(0) = \arg f(u_0)$ so that

$$f(v(t)) = f(v_0) = |f(u_0)| = \theta > 0$$

and

$$\varphi' = \frac{1}{\theta}e^{-\sigma\varphi}\operatorname{Re} f(\mathcal{N}(v)),\ \psi' = -\frac{1}{\theta}e^{-\sigma\varphi}\operatorname{Im} f(\mathcal{N}(v)).$$

Thus we obtain the Cauchy problem for the new dependent variable $v(t,x)$

$$\begin{cases} v_t + \mathcal{L}v = -e^{-\sigma\varphi}\left(\mathcal{N}(v) - \frac{v}{\theta}f(\mathcal{N}(v))\right),\ t > 0, x \in \mathbf{R}^n, \\ v(0,x) = v_0(x) \equiv u_0(x)\,e^{-i\arg f(u_0)},\ x \in \mathbf{R}^n. \end{cases} \tag{3.7}$$

We denote $h_v(t) = e^{\sigma\varphi(t)}$, then we get

$$\frac{d}{dt}h_v(t) = \frac{\sigma}{\theta}\operatorname{Re} f(\mathcal{N}(v)),\ h_v(0) = 1;$$

integration with respect to time therefore yields

$$h_v(t) = 1 + \frac{\sigma}{\theta}\int_0^t \operatorname{Re} f(\mathcal{N}(v(\tau)))\,d\tau.$$

Now the integral equation associated with (3.7) can be written as

$$v(t) = \mathcal{G}(t)\,v_0 - \int_0^t \mathcal{G}(t-\tau)\,\mathcal{K}(v(\tau))\,\frac{d\tau}{h_v(\tau)}, \tag{3.8}$$

where the nonlinearity

$$\mathcal{K}(v(\tau)) = \mathcal{N}(v(\tau)) - \frac{v(\tau)}{\theta}f(\mathcal{N}(v(\tau)))$$

and the functional

$$h_v(t) = 1 + \frac{\sigma}{\theta} \int_0^t \operatorname{Re} f(\mathcal{N}(v(\tau)))\, d\tau.$$

We now prove the existence of the solution $v(t,x)$ for integral equation (3.8) by the contraction mapping principle. We define the transformation $\mathcal{M}(w)$ by the formulas

$$\mathcal{M}(w) = \mathcal{G}(t)\, v_0 - \int_0^t \mathcal{G}(t-\tau)\, \mathcal{K}(w(\tau)) \frac{d\tau}{h_w(\tau)}, \tag{3.9}$$

and

$$h_w(t) = 1 + \frac{\sigma}{\theta} \int_0^t \operatorname{Re} f(\mathcal{N}(w(\tau)))\, d\tau$$

for any $w \in \mathbf{B}$, where

$$\mathbf{B} = \Big\{ w \in \mathbf{X} : \ f(w) = \theta, \ \ \|w\|_{\mathbf{X}} \le C\varepsilon, \\ \|\log(2+t)(w - \theta G_0(t))\|_{\mathbf{X}} \le C\varepsilon, \ \sup_{t>0} \frac{g(t)}{h_w(t)} \le 3 \Big\}.$$

First we check that the mapping $\mathcal{M}$ transforms the set $\mathbf{B}$ into itself. Since

$$f(\mathcal{K}(w)) = f(\mathcal{N}(w)) - \frac{1}{\theta} f(\mathcal{N}(w))\, f(w) = 0,$$

we see that

$$f(\mathcal{M}(w)) = f(\mathcal{G}(t)\, v_0) = f(v_0) = \theta.$$

By the definition of the asymptotic kernel (see (2.3)) we have

$$\|\log(2+t)(\mathcal{G}(t)\, v_0 - \theta G_0(t))\|_{\mathbf{X}} \\ \le C \left\| \langle t \rangle^{\gamma} (\mathcal{G}(t)\, v_0 - \theta G_0(t) \right\|_{\mathbf{X}} \le C \|v_0\|_{\mathbf{Z}} \le C\varepsilon;$$

furthermore by the condition of the theorem we get

$$\left\| \log(2+t) \left| \int_0^t \mathcal{G}(t-\tau)\, \mathcal{K}(w(\tau)) \frac{d\tau}{h_w(\tau)} \right| \right\|_{\mathbf{X}} \\ \le 3 \left\| \log(2+t) \int_0^t g^{-1}(\tau) \left| \mathcal{G}(t-\tau)\, \mathcal{K}(w(\tau)) \right| d\tau \right\|_{\mathbf{X}} \\ \le \frac{C}{\eta \theta^{\sigma}} \|w\|_{\mathbf{X}}^{\sigma+1} \left(1 + \frac{\|w\|_{\mathbf{X}}}{\theta} \right) \le C\varepsilon.$$

In particular, we see that

$$\|\mathcal{M}(w)\|_{\mathbf{X}} \le \|\mathcal{G}(t)\, v_0\|_{\mathbf{X}} + \left\| \int_0^t \mathcal{G}(t-\tau)\, \mathcal{K}(w(\tau)) \frac{d\tau}{h_w(\tau)} \right\|_{\mathbf{X}} \\ \le C\varepsilon + C \left\| \int_0^t g^{-1}(\tau) \left| \mathcal{G}(t-\tau)\, \mathcal{K}(w(\tau)) \right| d\tau \right\|_{\mathbf{X}} \le C\varepsilon + C\varepsilon^{\sigma+1} \le C\varepsilon,$$

and

$$\begin{aligned}
&\|\log(2+t)(\mathcal{M}(w)-\theta G_0(t))\|_{\mathbf{X}} \\
&\leq \|\log(2+t)(\mathcal{G}(t)v_0-\theta G_0(t))\|_{\mathbf{X}} \\
&+\left\|\log(2+t)\int_0^t \mathcal{G}(t-\tau)\mathcal{K}(w(\tau))\frac{d\tau}{h_w(\tau)}\right\|_{\mathbf{X}} \leq C\varepsilon.
\end{aligned}$$

It remains to prove the estimate

$$h_{\mathcal{M}(w)}(t)=1+\frac{\sigma}{\theta}\int_0^t \operatorname{Re} f(\mathcal{N}(\mathcal{M}(w)))\,d\tau \geq \frac{1}{3}g(t)$$

for all $t>0$. We have by condition (3.2)

$$\begin{aligned}
1+\frac{\sigma}{\theta}\int_0^t \operatorname{Re} f(\mathcal{N}(\mathcal{M}(w)))\,d\tau &= 1+\frac{\sigma}{\theta}\int_0^t \operatorname{Re} f(\mathcal{N}(\theta G_0(\tau)))\,d\tau \\
+\frac{\sigma}{\theta}\int_0^t \operatorname{Re} f(\mathcal{N}(\mathcal{M}(w))-\mathcal{N}(\theta G_0(\tau)))\,d\tau &\geq \frac{1}{2}g(t)+R(t), \qquad (3.10)
\end{aligned}$$

where

$$R(t)=\frac{\sigma}{\theta}\int_0^t \operatorname{Re} f(\mathcal{N}(\mathcal{M}(w))-\mathcal{N}(\theta G_0(\tau)))\,d\tau.$$

By estimate (3.4), we have

$$\begin{aligned}
&|R(t)| \leq C\|\log(2+t)(\mathcal{M}(w)-\theta G_0(t))\|_{\mathbf{X}}\left(\|\mathcal{M}(w)\|_{\mathbf{X}}^{\sigma}+\|\theta G_0\|_{\mathbf{X}}^{\sigma}\right) \\
&\times\int_0^t \{\tau\}^{-\alpha}\langle\tau\rangle^{-1}\frac{d\tau}{\log(2+\tau)} \leq C\varepsilon^{\sigma+1}\log\log(4+\tau).
\end{aligned}$$

Therefore by virtue of (3.10) we find that

$$h_{\mathcal{M}(w)}(t) \geq \frac{1}{2}g(t)-C\varepsilon^{\sigma+1}\log\log(4+\tau) \geq \frac{1}{3}g(t)$$

for all $t>0$. Thus we see that $\mathcal{M}$ transforms $\mathbf{B}$ into itself. Now by virtue of (3.5) let us estimate the difference

$$\begin{aligned}
&\|\mathcal{M}(v)-\mathcal{M}(w)\|_{\mathbf{X}} \\
&=\left\|\int_0^t \mathcal{G}(t-\tau)\left(\mathcal{K}(v(\tau))\frac{1}{h_v(\tau)}-\mathcal{K}(w(\tau))\frac{1}{h_w(\tau)}\right)d\tau\right\|_{\mathbf{X}} \\
&\leq C\left\|\int_0^t g^{-1}(\tau)\left|\mathcal{G}(t-\tau)(\mathcal{K}(v(\tau))-\mathcal{K}(w(\tau)))\right|d\tau\right\|_{\mathbf{X}} \\
&+C\left\|\int_0^t g^{-1}(\tau)\left|\mathcal{G}(t-\tau)\mathcal{K}(w(\tau))\right|\frac{|h_v(\tau)-h_w(\tau)|}{g(\tau)}d\tau\right\|_{\mathbf{X}} \\
&\leq C\|v-w\|_{\mathbf{X}}\left(\varepsilon^{\sigma}+\frac{1}{\theta}\left\|\int_0^t g^{-1}(\tau)\left|\mathcal{G}(t-\tau)\mathcal{K}(w(\tau))\right|d\tau\right\|_{\mathbf{X}}\right) \\
&\leq C\varepsilon^{\sigma}\left(1+\frac{\varepsilon}{\theta}\right)\|v-w\|_{\mathbf{X}} \leq \frac{1}{2}\|v-w\|_{\mathbf{X}},
\end{aligned}$$

where in view of (3.4) with $\nu(t) = 1$ we used the estimate

$$\frac{|h_v(\tau) - h_w(\tau)|}{g(\tau)} \le \frac{C}{\theta g(t)} \left| \int_0^t f(\mathcal{N}(v(\tau)) - \mathcal{N}(w(\tau)))\, d\tau \right|$$
$$\le \frac{C\varepsilon^\sigma \log(1+t)}{\theta g(t)} \|v - w\|_{\mathbf{X}} \le \frac{C}{\theta} \|v - w\|_{\mathbf{X}}.$$

Therefore $\mathcal{M}$ is a contraction mapping in the closed set $\mathbf{B}$ of a complete metric space $\mathbf{X}$. Hence there exists a unique global solution $v \in \mathbf{B}$ to the Cauchy problem (3.7) such that

$$\|v\|_{\mathbf{X}} \le C\varepsilon, \ \ \|\log(2+t)(v - \theta G_0(t))\|_{\mathbf{X}} \le C\varepsilon, \ h_v(t) \ge \frac{1}{3} g(t).$$

Using the relation $u(t,x) = v(t,x) e^{i\psi(t)} h_v^{-\frac{1}{\sigma}}(t)$ we obtain the existence of the solution to the Cauchy problem (3.1), satisfying the following time decay estimates

$$\left\| g^{\frac{1}{\sigma}} u \right\|_{\mathbf{X}} \le C\varepsilon.$$

We now prove the asymptotics for the solution. By condition (3.6) and representation (3.10) we have

$$h_v(t) = 1 + \frac{\sigma}{\theta} \int_0^t \operatorname{Re} f(\mathcal{N}(v))\, d\tau$$
$$= 1 + \frac{\sigma}{\theta} \int_0^t \operatorname{Re} f(\mathcal{N}(\theta G_0(\tau)))\, d\tau$$
$$+ \frac{\sigma}{\theta} \int_0^t \operatorname{Re} f(\mathcal{N}(v) - \mathcal{N}(\theta G_0(\tau)))\, d\tau$$
$$= g(t) + O(\log\log(4+t)).$$

Then via formulas $u(t,x) = e^{-\varphi(t) + i\psi(t)} v(t,x) = h_v^{-\frac{1}{\sigma}}(t) e^{i\psi(t)} v(t,x)$ we find the estimate

$$\left\| \frac{g^{1+\frac{1}{\sigma}}}{\log\log(4+t)} \left(u - \theta G_0 e^{-i\psi} g^{-\frac{1}{\sigma}} \right) \right\|_{\mathbf{X}} \le C,$$

since

$$\left\| \frac{g^{1+\frac{1}{\sigma}}}{\log\log(4+t)} \left(\theta G_0 e^{-i\psi} \left(h_v^{-\frac{1}{\sigma}} - g^{-\frac{1}{\sigma}} \right) \right) \right\|_{\mathbf{X}}$$
$$\le C \sup_{t>0} \left| \frac{g^{1+\frac{1}{\sigma}}}{\log\log(4+t)} \left(h_v^{-\frac{1}{\sigma}} - g^{-\frac{1}{\sigma}} \right) \right| \|G_0\|_{\mathbf{X}} \le C.$$

Also we have

$$\begin{aligned}\psi(t) &= \arg f(u_0) - \frac{1}{\theta}\int_0^t h_v^{-1}(\tau)\operatorname{Im} f(\mathcal{N}(v(\tau)))\,d\tau \\ &= \arg f(u_0) - \frac{1}{\theta}\int_0^t g^{-1}(\tau)\operatorname{Im} f(\mathcal{N}(\theta G_0(\tau)))\,d\tau \\ &+ O\left(\int_0^t \{\tau\}^{-\alpha}\langle\tau\rangle^{-1}\frac{\log\log(4+\tau)\,d\tau}{\log^2(2+\tau)}\right) \\ &= \widetilde{\eta}\log\log t + O(1)\end{aligned}$$

for large $t \to \infty$, with some constant $\widetilde{\eta} \in \mathbf{R}$. This completes the proof of Theorem 3.2.

Example 3.3. **Large time asymptotics for the critical nonlinear heat equation**

Consider the Cauchy problem

$$\begin{cases} u_t - \Delta u = \lambda |u|^\sigma u, \; x \in \mathbf{R}^n, \; t > 0, \\ u(0,x) = u_0(x), \; x \in \mathbf{R}^n \end{cases} \tag{3.11}$$

for the nonlinear heat equation in the critical case $\sigma = \frac{2}{n}$.

Denote $g(t) = 1 + \theta^{\frac{2}{n}}\eta\log(1+t)$, $\eta = \frac{|\lambda|}{2\pi n}\left(1+\frac{2}{n}\right)^{-\frac{n}{2}}$,

$$\theta = \left|\int_{\mathbf{R}^n} u_0(x)\,dx\right|.$$

Theorem 3.4. *Let* $\sigma = \frac{2}{n}$, $\lambda < 0$. *Let the initial data* $u_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^p(\mathbf{R}^n)$, $a \in (0,1]$, $p > 2$, *have a small norm* $\|u_0\|_{\mathbf{L}^{1,a}} + \|u_0\|_{\mathbf{L}^p} \le \varepsilon$, *and the mean value* $\theta \ge C\varepsilon > 0$ *with some* $C > 0$. *Then the Cauchy problem (3.11) has a unique global solution*

$$u(t,x) \in \mathbf{C}\left([0,\infty);\mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^p(\mathbf{R}^n)\right) \cap \mathbf{C}\left((0,\infty);\mathbf{L}^\infty(\mathbf{R}^n)\right)$$

satisfying the asymptotics

$$u(t) = \theta G_0(t)\, g^{-\frac{n}{2}}(t) + O\left(\left(t^{-\frac{n}{2}}\log^{-\frac{n}{2}-1} t\right)\ \log\log t\right)$$

for large time $t \to \infty$, *uniformly with respect to* $x \in \mathbf{R}^n$.

The Green operator $\mathcal{G}(t)$ of the linear heat equation has the form

$$\mathcal{G}(t)\phi = \int_{\mathbf{R}^n} G(t,x-y)\,\phi(y)\,dy,$$

where the heat kernel $G(t,x)$ is

$$G(t,x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$

By Lemma 1.28 we have the estimates

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^p} \leq C\|\phi\|_{\mathbf{L}^p} \tag{3.12}$$

and

$$\left\||\cdot|^b \left(\mathcal{G}(t)\phi - \vartheta G_0(t)\right)\right\|_{\mathbf{L}^p} \leq Ct^{\frac{n}{2}\left(\frac{1}{p}-1\right)+\frac{b-a}{2}}\|\phi\|_{\mathbf{L}^{1,a}} \tag{3.13}$$

for all $t > 0$, where $1 \leq p \leq \infty$, $0 \leq b \leq a$, $\vartheta = \int_{\mathbf{R}^n}\phi(x)\,dx$ and the asymptotic kernel is

$$G_0(t,x) = (4\pi(t+1))^{-\frac{n}{2}} e^{-\frac{|x|^2}{4(t+1)}}.$$

To apply Theorem 3.2 we choose as above the space

$$\mathbf{Z} = \left\{\phi \in \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)\right\}$$

with $a \in (0,1]$ and $p > 1$ and the space

$$\mathbf{X} = \left\{\phi \in \mathbf{C}([0,\infty);\mathbf{Z}) \cap \mathbf{C}((0,\infty);\mathbf{L}^\infty(\mathbf{R}^n)) : \ \|\phi\|_{\mathbf{X}} < \infty\right\},$$

where the norm

$$\begin{aligned}\|\phi\|_{\mathbf{X}} &= \sup_{t\geq 0}\left(\langle t\rangle^{-\frac{a}{2}}\|\phi(t)\|_{\mathbf{L}^{1,a}} + \langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{p}\right)}\|\phi(t)\|_{\mathbf{L}^p}\right)\\ &\quad + \sup_{t>0}\{t\}^{\frac{n}{2p}}\langle t\rangle^{\frac{n}{2}}\|\phi(t)\|_{\mathbf{L}^\infty}.\end{aligned}$$

Definition 3.5. *We say that the triad* $(\mathbf{X},\mathbf{Y},\mathcal{G})$ *is concordant if for any* ϕ *such that* $f(\phi) = 0$ *the following inequality*

$$\left\|\log(2+t)\int_0^t \mathcal{G}(t-\tau)\phi(\tau)\frac{d\tau}{\log(2+\tau)}\right\|_{\mathbf{X}} \leq C\|\phi\|_{\mathbf{Y}}$$

is valid, provided that the right-hand side is finite.

Define the norm

$$\begin{aligned}\|\phi\|_{\mathbf{Y}} &= \sup_{t>0}\{t\}^{\frac{2}{p}}\langle t\rangle\left(\langle t\rangle^{-\frac{a}{2}}\|\phi(t)\|_{\mathbf{L}^{1,a}}\right.\\ &\quad\left. + \langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{p}\right)}\|\phi(t)\|_{\mathbf{L}^p} + \{t\}^{\frac{n}{2p}}\langle t\rangle^{\frac{n}{2}}\|\phi(t)\|_{\mathbf{L}^\infty}\right).\end{aligned}$$

Let us prove that the triad $(\mathbf{X},\mathbf{Y},\mathcal{G})$ is concordant. In view of the estimate $g^{-1}(\tau) \leq 1$ and estimates of the local existence Theorem 1.13 we get

$$\begin{aligned}&\left\|\int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)\phi(\tau)\,d\tau\right\|_{\mathbf{L}^\infty}\\ &+\left\|\int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)\phi(\tau)\,d\tau\right\|_{\mathbf{L}^{1,a}}\\ &\leq C\|\phi\|_{\mathbf{Y}} \leq C\|\phi\|_{\mathbf{Y}}\,g^{-1}(t)\end{aligned} \tag{3.14}$$

for all $0 \le t \le 1$. We now consider $t > 1$. We have the estimate $\langle t\rangle^{-\frac{a}{4}} \le Cg^{-1}(t)$ and

$$\sup_{\tau\in[\sqrt{t},t]} g^{-1}(\tau) \le C\left(1+\theta^{\frac{2}{n}}\eta\log\left(1+\sqrt{t}\right)\right)^{-1}$$
$$\le C\left(1+\frac{\theta^{\frac{2}{n}}\eta}{2}\log(1+t)\right)^{-1} \le Cg^{-1}(t); \tag{3.15}$$

hence by virtue of (3.12), (3.13) we obtain

$$\left\|\int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)\phi(\tau)\,d\tau\right\|_{\mathbf{L}^q}$$
$$\le C\int_0^{\sqrt{t}}(t-\tau)^{\frac{n}{2}\left(\frac{1}{q}-1\right)-\frac{a}{2}}\langle\tau\rangle^{\frac{a}{2}-1}\{\tau\}^{-\frac{2}{p}}d\tau\sup_{\tau>0}\{\tau\}^{\frac{2}{p}}\langle\tau\rangle^{1-\frac{a}{2}}\|\phi(\tau)\|_{\mathbf{L}^{1,a}}$$
$$+Cg^{-1}(t)\int_{\sqrt{t}}^{\frac{t}{2}}(t-\tau)^{\frac{n}{2}\left(\frac{1}{q}-1\right)-\frac{a}{2}}\langle\tau\rangle^{\frac{a}{2}-1}\{\tau\}^{-\frac{2}{p}}d\tau\sup_{\tau>0}\{\tau\}^{\frac{2}{p}}\langle\tau\rangle^{1-\frac{a}{2}}\|\phi(\tau)\|_{\mathbf{L}^{1,a}}$$
$$+Cg^{-1}(t)\int_{\frac{t}{2}}^{t}\tau^{\frac{n}{2}\left(\frac{1}{q}-1\right)-1}d\tau\sup_{\tau>1}\tau^{1-\frac{n}{2}\left(\frac{1}{q}-1\right)}\|\phi(\tau)\|_{\mathbf{L}^q}$$
$$\le Ct^{\frac{n}{2}\left(\frac{1}{q}-1\right)}\left(t^{-\frac{a}{4}}+g^{-1}(t)\right)\|\phi\|_{\mathbf{Y}} \le Cg^{-1}(t)\,t^{\frac{n}{2}\left(\frac{1}{q}-1\right)}\|\phi\|_{\mathbf{Y}}$$

for all $t > 4$, where $1 \le q \le \infty$. Similarly we have via (3.12) and (3.13)

$$\left\|\int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)\phi(\tau)\,d\tau\right\|_{\mathbf{L}^{1,a}}$$
$$\le C\int_0^{\sqrt{t}}\langle\tau\rangle^{\frac{a}{2}-1}\{\tau\}^{-\frac{2}{p}}d\tau\sup_{\tau>0}\{\tau\}^{\frac{2}{p}}\langle\tau\rangle^{1-\frac{a}{2}}\|\phi(\tau)\|_{\mathbf{L}^{1,a}}$$
$$+Cg^{-1}(t)\int_{\sqrt{t}}^{t}\langle\tau\rangle^{\frac{a}{2}-1}\{\tau\}^{-\frac{2}{p}}d\tau\sup_{\tau>1}\tau^{1-\frac{a}{2}}\|\phi(\tau)\|_{\mathbf{L}^{1,a}}$$
$$\le C\varepsilon\left(t^{\frac{a}{4}}+g^{-1}(t)\,t^{\frac{a}{2}}\right)\|\phi\|_{\mathbf{Y}} \le Cg^{-1}(t)\,t^{\frac{a}{2}}\|\phi\|_{\mathbf{Y}}$$

for all $t > 1$. Hence the triad $(\mathbf{X},\mathbf{Y},\mathcal{G})$ is concordant.

By a direct computation we have

$$\int_0^t \operatorname{Re} f\left(\mathcal{N}(\theta G_0(\tau))\right)d\tau = |\lambda|\,\theta^{\sigma+1}\int_0^t\int_{\mathbf{R}^n}G_0^{\sigma+1}(\tau,x)\,dxd\tau$$
$$=|\lambda|\,\theta^{\sigma+1}(4\pi)^{-\frac{n}{2}-1}\int_0^t\int_{\mathbf{R}^n}e^{-\frac{|x|^2}{4(\tau+1)}\left(1+\frac{2}{n}\right)}dx\,(\tau+1)^{-\frac{n}{2}-1}d\tau$$
$$=|\lambda|\,\theta^{\sigma+1}(4\pi)^{-1}\left(1+\frac{2}{n}\right)^{-\frac{n}{2}}\int_0^t\frac{d\tau}{1+\tau}$$
$$=\frac{n}{2}\eta\theta^{\sigma+1}\log(1+t)$$

and

$$e^{z}\mathcal{N}\left(ue^{-z}\right)=e^{z}\left|ue^{-z}\right|^{\sigma}ue^{-z}=e^{-\sigma z}\mathcal{N}\left(u\right);$$

so conditions (3.2), (3.3) and (3.6) are fulfilled. Since by interpolation inequality (1.4)

$$\begin{aligned}&\log\left(2+t\right)\left\|\mathcal{N}\left(v\left(t\right)\right)-\mathcal{N}\left(w\left(t\right)\right)\right\|_{\mathbf{L}^{1}}\\&\leq C\log\left(2+t\right)\left(\left\|v\left(t\right)\right\|_{\mathbf{L}^{\infty}}^{\sigma}+\left\|w\left(t\right)\right\|_{\mathbf{L}^{\infty}}^{\sigma}\right)\left\|v\left(t\right)-w\left(t\right)\right\|_{\mathbf{L}^{1}}\\&\leq C\left\{t\right\}^{-\frac{1}{p}}\left\langle t\right\rangle^{-1}\left\|\log\left(2+t\right)\left(v-w\right)\right\|_{\mathbf{X}}\left(\left\|v\right\|_{\mathbf{X}}^{\sigma}+\left\|w\right\|_{\mathbf{X}}^{\sigma}\right),\end{aligned}$$

condition (3.4) is true. By estimates (1.24) we have

$$\begin{aligned}&\left\|\mathcal{K}\left(v\right)-\mathcal{K}\left(w\right)\right\|_{\mathbf{Y}}\leq\left\|\mathcal{N}\left(v\right)-\mathcal{N}\left(w\right)\right\|_{\mathbf{Y}}\\&+\frac{1}{\theta}\left(\left\|v\right\|_{\mathbf{X}}+\left\|w\right\|_{\mathbf{X}}\right)\sup_{t>0}\left\{t\right\}^{\frac{1}{p}}\left\langle t\right\rangle\left\|\mathcal{N}\left(v\left(t\right)\right)-\mathcal{N}\left(w\left(t\right)\right)\right\|_{\mathbf{L}^{1}}\\&+\frac{1}{\theta}\left\|v-w\right\|_{\mathbf{X}}\sup_{t>0}\left\{t\right\}^{\frac{1}{p}}\left\langle t\right\rangle\left(\left\|\mathcal{N}\left(v\left(t\right)\right)\right\|_{\mathbf{L}^{1}}+\left\|\mathcal{N}\left(w\left(t\right)\right)\right\|_{\mathbf{L}^{1}}\right)\\&\leq C\left\|v-w\right\|_{\mathbf{X}}\left(\left\|v\right\|_{\mathbf{X}}^{\sigma}+\left\|w\right\|_{\mathbf{X}}^{\sigma}\right)\left(1+\frac{1}{\theta}\left\|v\right\|_{\mathbf{X}}+\frac{1}{\theta}\left\|w\right\|_{\mathbf{X}}\right).\end{aligned}$$

Since the triad $(\mathbf{X},\mathbf{Y},\mathcal{G})$ is concordant we see that condition (3.5) is fulfilled. Now applying Theorem 3.2 we easily get the results of Theorem 3.4.

Example 3.6. **The case of odd solutions to the nonlinear heat equation**

Now let us consider problem (3.11) with some special initial data $u_0(x)$, which are odd functions in $\mathbf{R}^n$, that is

$$u_0\left(x_1,...,-x_j,...,x_n\right)=-u_0\left(x_1,...,x_j,...,x_n\right),$$

for every $j=1,2,...,n$. In this case the solutions $u(t,x)$ will also be odd functions with respect to $x\in\mathbf{R}^n$. Now the critical value is $\sigma=\frac{1}{n}$.

Define the space $\mathbf{Z}$

$$\mathbf{Z}=\left\{\phi\in\mathbf{L}^{1,a}\left(\mathbf{R}^n\right)\cap\mathbf{L}^{p}\left(\mathbf{R}^n\right):\ \phi\text{ is odd function in }\mathbf{R}^n\right\},$$

where now $a\in(n,n+1]$, and $p>1$ and the space

$$\begin{aligned}\mathbf{X}=\{\phi\in\mathbf{C}\left([0,\infty);\mathbf{Z}\right)\cap\mathbf{C}\left((0,\infty);\mathbf{L}^{\infty}\left(\mathbf{R}^n\right)\right):\\\phi\text{ is odd function in }\mathbf{R}^n\text{ and }\left\|\phi\right\|_{\mathbf{X}}<\infty\},\end{aligned}$$

where the norm

$$\begin{aligned}\left\|\phi\right\|_{\mathbf{X}}&=\sup_{t\geq0}\left(\left\langle t\right\rangle^{-\frac{a-n}{2}}\left\|\phi\left(t\right)\right\|_{\mathbf{L}^{1,a}}+\left\langle t\right\rangle^{n-\frac{n}{2p}}\left\|\phi\left(t\right)\right\|_{\mathbf{L}^{p}}\right)\\&+\sup_{t>0}\left\{t\right\}^{\frac{n}{2p}}\left\langle t\right\rangle^{n}\left\|\phi\left(t\right)\right\|_{\mathbf{L}^{\infty}}.\end{aligned}$$

Also we define the norm to estimate the nonlinearity

$$\|\phi\|_{\mathbf{Y}} = \sup_{t>0} \{t\}^{\frac{1}{2p}} \langle t \rangle \left(\langle t \rangle^{-\frac{a-n}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} + \langle t \rangle^{n-\frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^p} + \{t\}^{\frac{n}{2p}} \langle t \rangle^n \|\phi(t)\|_{\mathbf{L}^\infty} \right).$$

Denote $g(t) = 1 + \theta^{\sigma+1}\eta \log(1+t)$,

$$\eta = |\lambda| (4\pi)^{-\frac{n+1}{2}} \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4}\left(1+\frac{1}{n}\right)} \prod_{j=1}^{n} |x_j|^{\frac{1}{n}+2} dx > 0,$$

and

$$\theta = \left| \int_{\mathbf{R}^n} u_0(x) \prod_{j=1}^{n} x_j dx \right|.$$

Theorem 3.7. *Let $\sigma = \frac{1}{n}$, $\lambda < 0$. Assume that the initial data u_0 are odd functions in $\mathbf{R}^n$, and $u_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^p(\mathbf{R}^n)$, with $a \in (n, n+1]$, $p > 1$. Also suppose that the norm $\|u_0\|_{\mathbf{L}^{1,a}} + \|u_0\|_{\mathbf{L}^p}$ is sufficiently small and the mean value $\theta \geq C\varepsilon > 0$ with some $C > 0$. Then the Cauchy problem (3.11) has a unique global solution*

$$u(t,x) \in \mathbf{C}\left([0,\infty); \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^p(\mathbf{R}^n)\right) \cap \mathbf{C}((0,\infty); \mathbf{L}^\infty(\mathbf{R}^n))$$

satisfying the asymptotics

$$u(t) = \theta g^{-\frac{n}{2}}(t) \frac{\prod_{j=1}^{n} x_j}{4^n \pi^{\frac{n}{2}} t^{\frac{3n}{2}}} e^{-\frac{|x|^2}{4t}} + O\left(\left(t^{-n} \log^{-\frac{n}{2}-1} t\right) \log\log t\right) \tag{3.16}$$

for large time $t \to \infty$, uniformly with respect to $x \in \mathbf{R}^n$.

Remark 3.8. Note that in the domain $x = O\left(\sqrt{t}\right)$ the main term of the asymptotics (3.16) behaves like $O\left(t^{-n} \log^{-\frac{n}{2}} t\right)$, so the remainder term decays faster. In the domains $x = o\left(\sqrt{t}\right)$ and $\frac{|x|}{\sqrt{t}} \to \infty$ the main term of the asymptotic representation (3.16) decays faster than the remainder term. So formula (3.16) gives only a decay estimate for these cases. Below in Section 3.3 we will obtain uniform asymptotic representations for solutions.

The Green operator $\mathcal{G}(t)$ of the linear heat equation has now the asymptotic kernel

$$G_0(t,x) = \frac{1}{4^n \pi^{\frac{n}{2}} (t+1)^{\frac{3n}{2}}} e^{-\frac{|x|^2}{4(t+1)}} \prod_{j=1}^{n} x_j$$

if we define the functional $f : \mathbf{Z} \to \mathbf{R}$, by

$$f(\phi) = \int_{\mathbf{R}^n} \phi(x) \prod_{j=1}^{n} x_j dx.$$

Using Lemma 1.30 with $\delta = \nu = 2$ we get

$$\left\| |\cdot|^b \left(\mathcal{G}(t)\phi - \vartheta G_0(t)\right)\right\|_{\mathbf{L}^q} \leq C t^{-n+\frac{n}{2q}-\frac{a-b}{2}} \|\phi\|_{\mathbf{L}^{1,a}} \tag{3.17}$$

for all $t > 0$, where

$$\vartheta = \int_{\mathbf{R}^n} \phi(x) \prod_{j=1}^{n} x_j dx,$$

$1 \leq r \leq q \leq \infty$, $\beta \geq 0$, $0 \leq b \leq a$.

Now let us prove that the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ is concordant. In view of the estimate $g^{-1}(\tau) \leq 1$ and estimates of the local existence Theorem 1.13 we get (3.14) for all $0 \leq t \leq 1$. We now consider $t > 1$. We have the estimate $\langle t\rangle^{-\frac{a-n}{4}} \leq C g^{-1}(t)$ and (3.15); hence by virtue of (3.17) we obtain

$$\begin{aligned}
&\left\| \int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)\phi(\tau)d\tau \right\|_{\mathbf{L}^q} \\
&\leq C\int_0^{\sqrt{t}} (t-\tau)^{-n+\frac{n}{2q}-\frac{a-n}{2}} \langle\tau\rangle^{\frac{a-n}{2}-1} \{\tau\}^{-\frac{1}{2p}} d\tau \\
&\qquad \times \sup_{\tau>0} \{\tau\}^{\frac{1}{2p}} \langle\tau\rangle^{1-\frac{a-n}{2}} \|\phi(\tau)\|_{\mathbf{L}^{1,a}} \\
&+ C g^{-1}(t) \int_{\sqrt{t}}^{\frac{t}{2}} (t-\tau)^{-n+\frac{n}{2q}-\frac{a-n}{2}} \langle\tau\rangle^{\frac{a-n}{2}-1} \{\tau\}^{-\frac{1}{2p}} d\tau \\
&\qquad \times \sup_{\tau>0} \{\tau\}^{\frac{1}{2p}} \langle\tau\rangle^{1-\frac{a-n}{2}} \|\phi(\tau)\|_{\mathbf{L}^{1,a}} \\
&+ C g^{-1}(t) \int_{\frac{t}{2}}^{t} \tau^{\frac{n}{2q}-n-1} d\tau \sup_{\tau>1} \tau^{1+n-\frac{n}{2q}} \|\phi(\tau)\|_{\mathbf{L}^q} \\
&\leq C t^{-n+\frac{n}{2q}} \left(t^{-\frac{a-n}{4}} + g^{-1}(t)\right) \|\phi\|_{\mathbf{Y}} \leq C g^{-1}(t) t^{-n+\frac{n}{2q}} \|\phi\|_{\mathbf{Y}}
\end{aligned}$$

for all $t > 1$, where $1 \leq q \leq \infty$. In the same manner we have via (3.12) and (3.13)

$$\begin{aligned}
&\left\| \int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)\phi(\tau)d\tau \right\|_{\mathbf{L}^{1,a}} \\
&\leq C\int_0^{\sqrt{t}} \langle\tau\rangle^{\frac{a-n}{2}-1} \{\tau\}^{-\frac{1}{2p}} d\tau \sup_{\tau>0} \{\tau\}^{\frac{1}{2p}} \langle\tau\rangle^{1-\frac{a-n}{2}} \|\phi(\tau)\|_{\mathbf{L}^{1,a}} \\
&+ C g^{-1}(t) \int_{\sqrt{t}}^{t} \langle\tau\rangle^{\frac{a-n}{2}-1} \{\tau\}^{-\frac{1}{2p}} d\tau \sup_{\tau>1} \tau^{1-\frac{a-n}{2}} \|\phi(\tau)\|_{\mathbf{L}^{1,a}} \\
&\leq C\varepsilon \left(t^{\frac{a-n}{4}} + g^{-1}(t) t^{\frac{a-n}{2}}\right) \|\phi\|_{\mathbf{Y}} \leq C g^{-1}(t) t^{\frac{a-n}{2}} \|\phi\|_{\mathbf{Y}}
\end{aligned}$$

for all $t > 4$. Thus the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ is concordant.

By a direct computation we have

$$
\begin{aligned}
&\int_0^t \operatorname{Re} f\left(\mathcal{N}\left(\theta G_0(\tau)\right)\right) d\tau \\
&= |\lambda|\, \theta^{\sigma+1} \int_0^t \int_{\mathbf{R}^n} \left|G_0(\tau, x)\right|^{\sigma} G_0(t, x) \prod_{j=1}^{n} x_j\, dx\, d\tau \\
&= |\lambda|\, \theta^{\sigma+1} (4\pi)^{-\frac{n+1}{2}} \int_0^t \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4(\tau+1)}(1+\sigma)} \prod_{j=1}^{n} \left|x_j\right|^{\sigma+2} dx\, (\tau+1)^{-\frac{3n+3}{2}} d\tau \\
&= \theta^{\sigma+1} \eta \int_0^t \frac{d\tau}{1+\tau} = \eta \theta^{\sigma+1} \log(1+t);
\end{aligned}
$$

hence conditions (3.2), (3.3) and (3.6) are fulfilled. Since by interpolation inequality (1.4)

$$
\begin{aligned}
&\log(2+t) \left\|\mathcal{N}(v(t)) - \mathcal{N}(w(t))\right\|_{\mathbf{L}^{1,n}} \\
&\leq C \{t\}^{-\frac{1}{p}} \langle t\rangle^{-1} \left\|\log(2+t)(v-w)\right\|_{\mathbf{X}} \left(\|v\|_{\mathbf{X}}^{\sigma} + \|w\|_{\mathbf{X}}^{\sigma}\right),
\end{aligned} \tag{3.18}
$$

condition (3.4) is true. By estimates (1.24) as above we have

$$
\|\mathcal{K}(v) - \mathcal{K}(w)\|_{\mathbf{Y}} \leq C \|v-w\|_{\mathbf{X}} \left(\|v\|_{\mathbf{X}}^{\sigma} + \|w\|_{\mathbf{X}}^{\sigma}\right) \left(1 + \frac{1}{\theta}\left(\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}\right)\right). \tag{3.19}
$$

Since the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ is concordant we see that condition (3.5) is fulfilled. Now applying Theorem 3.2 we readily obtain the results of Theorem 3.7. Theorem 3.7 is proved.

Example 3.9. **Burgers type equations with initial data having zero mean value**

Now we consider the Cauchy problem for the Burgers type equation (1.18) with initial data having zero mean value $\int_{\mathbf{R}^n} u_0(x)\, dx = 0$. Since the nonlinearity is a full derivative the solutions also have a zero mean value for all times. Then we will show that the asymptotics is defined by the first moments of the initial data. So the nonlinearity $(\lambda \cdot \nabla)|u|^{\sigma+1}$ of the Burgers type equation (1.18) behaves for large times like a nonconvective one.

By the rotation in the x - plane we always can transform the Burgers type equations (1.18) to the form, where the vector $\lambda = (-1, 0, ..., 0)$. We now consider the problem

$$
\begin{cases}
u_t - \Delta u - \partial_{x_1}\left(|u|^{\sigma+1}\right) = 0, \ x \in \mathbf{R}^n, \ t > 0, \\
u(0, x) = u_0(x), \ x \in \mathbf{R}^n,
\end{cases} \tag{3.20}
$$

where $\sigma > 0$.

Define the space $\mathbf{Z}$

$$
\mathbf{Z} = \left\{\phi \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^p(\mathbf{R}^n)\right\},
$$

where now $a \in (1,2]$, and $p > 1$ and the space

$$\mathbf{X} = \left\{ \phi \in \mathbf{C}([0,\infty);\mathbf{Z}) \cap \mathbf{C}\left((0,\infty);\mathbf{W}^1_\infty(\mathbf{R}^n)\right) : \ \|\phi\|_{\mathbf{X}} < \infty \right\},$$

where the norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t \geq 0} \left(\langle t \rangle^{-\frac{a-1}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} + \langle t \rangle^{\frac{n+1}{2} - \frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^p} \right)$$
$$+ \sup_{t>0} \left(\{t\}^{\frac{n}{2p}} \langle t \rangle^{\frac{n+1}{2}} \|\phi(t)\|_{\mathbf{L}^\infty} + \{t\}^{\frac{n}{2p}+\frac{1}{2}} \langle t \rangle^{\frac{n}{2}+1} \|\nabla\phi(t)\|_{\mathbf{L}^\infty} \right).$$

Also we consider the norm

$$\|\phi\|_{\mathbf{Y}} = \sup_{t>0} \{t\}^{\frac{n\sigma}{2p}+\frac{1}{2}} \langle t \rangle \left(\langle t \rangle^{-\frac{a-1}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} \right.$$
$$\left. + \langle t \rangle^{\frac{n+1}{2} - \frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^p} + \{t\}^{\frac{n}{2p}} \langle t \rangle^{\frac{n+1}{2}} \|\phi(t)\|_{\mathbf{L}^\infty} \right).$$

We choose the asymptotic kernel

$$G_0(t,x) = \frac{x_1}{(4\pi)^{\frac{n}{2}} (t+1)^{1+\frac{n}{2}}} e^{-\frac{|x|^2}{4(t+1)}}$$

and the functional $f(\phi) \equiv \int_{\mathbf{R}^n} \phi(x) x_1 dx$. Denote $g(t) = 1 + \theta^{\sigma+1} \eta \log(1+t)$,

$$\eta = |\lambda| (4\pi)^{-\frac{n}{2} - \frac{n}{2(n+1)}} \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4}\left(1+\frac{1}{n+1}\right)} |x_1|^{\frac{1}{n+1}+1} dx > 0,$$

and

$$\theta = \int_{\mathbf{R}^n} u_0(x) x_1 dx.$$

Theorem 3.10. *Let $\sigma = \frac{1}{n+1}$. Assume that the initial data $u_0 \in \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, with $a \in (1,2]$ and $p > 1$. Let the mean value $\int_{\mathbf{R}^n} u_0(x) dx = 0$ and the moments $\int_{\mathbf{R}^n} x_j u_0(x) dx = 0$, $j \neq 1$, $\int_{\mathbf{R}^n} x_1 u_0(x) dx > 0$. Suppose that the norm $\|u_0\|_{\mathbf{L}^{1,a}} + \|u_0\|_{\mathbf{L}^p} = \varepsilon$ is sufficiently small and the mean value $\theta \geq C\varepsilon > 0$ with some $C > 0$. Then the Cauchy problem for the Burgers type equation (3.20) has a unique global solution $u(t,x) \in \mathbf{C}\left([0,\infty);\mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^p(\mathbf{R}^n)\right) \cap \mathbf{C}((0,\infty);\mathbf{L}^\infty(\mathbf{R}^n))$ satisfying the asymptotics*

$$u(t) = \theta G_0(t) g^{-\frac{n+1}{2}}(t) + O\left(t^{-\frac{n+1}{2}} \log^{-\frac{n}{2}-1} t \ \log\log t\right)$$

for large time $t \to \infty$, uniformly with respect to $x \in \mathbf{R}^n$.

Using Lemma 1.30 with $\delta = \nu = 2$ we get

$$\left\| |\cdot|^b \left(\mathcal{G}(t)\phi - \vartheta G_0(t)\right) \right\|_{\mathbf{L}^q} \leq C t^{-\frac{n}{2}+\frac{n}{2q}-\frac{a-b}{2}} \|\phi\|_{\mathbf{L}^{1,a}} \tag{3.21}$$

for all $t > 0$, where

$$\vartheta = \int_{\mathbf{R}^n} \phi(x)\, x_1 dx,$$

$1 \le r \le q \le \infty, 1 \le b \le a.$

Let us prove that the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ is concordant. In view of the estimate $g^{-1}(\tau) \le 1$ and estimates of the local existence Theorem 1.13 we get (3.14) for all $0 \le t \le 1$. We now consider $t > 1$. We have the estimate $\langle t \rangle^{-\frac{a-1}{4}} \le Cg^{-1}(t)$ and (3.15); thus by virtue of (3.21) we obtain

$$\begin{aligned}
&\left\| \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau)\, \phi(\tau)\, d\tau \right\|_{\mathbf{L}^q} \\
&\le C \int_0^{\sqrt{t}} (t-\tau)^{-\frac{n}{2}+\frac{n}{2q}-\frac{a}{2}} \langle \tau \rangle^{\frac{a-1}{2}-1} \{\tau\}^{-\frac{1}{2p}} d\tau \\
&\quad \times \sup_{\tau>0} \{\tau\}^{\frac{1}{2p}} \langle \tau \rangle^{1-\frac{a-1}{2}} \|\phi(\tau)\|_{\mathbf{L}^{1,a}} \\
&+ Cg^{-1}(t) \int_{\sqrt{t}}^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}+\frac{n}{2q}-\frac{a}{2}} \langle \tau \rangle^{\frac{a-1}{2}-1} \{\tau\}^{-\frac{1}{2p}} d\tau \\
&\quad \times \sup_{\tau>0} \{\tau\}^{\frac{1}{2p}} \langle \tau \rangle^{1-\frac{a-1}{2}} \|\phi(\tau)\|_{\mathbf{L}^{1,a}} \\
&+ Cg^{-1}(t) \int_{\frac{t}{2}}^{t} \tau^{\frac{n}{2q}-\frac{n+1}{2}-1} d\tau \sup_{\tau>1} \tau^{1+\frac{n+1}{2}-\frac{n}{2q}} \|\phi(\tau)\|_{\mathbf{L}^q} \\
&\le Ct^{-\frac{n+1}{2}+\frac{n}{2q}} \left(t^{-\frac{a-1}{4}} + g^{-1}(t) \right) \|\phi\|_{\mathbf{Y}} \le Cg^{-1}(t)\, t^{-\frac{n+1}{2}+\frac{n}{2q}} \|\phi\|_{\mathbf{Y}}
\end{aligned}$$

for all $t > 1$, where $1 \le q \le \infty$. In the identical manner we have via (3.12) and (3.13)

$$\begin{aligned}
&\left\| \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau)\, \phi(\tau)\, d\tau \right\|_{\mathbf{L}^{1,a}} \\
&\le C \int_0^{\sqrt{t}} \langle \tau \rangle^{\frac{a-1}{2}-1} \{\tau\}^{-\frac{1}{2p}} d\tau \sup_{\tau>0} \{\tau\}^{\frac{1}{2p}} \langle \tau \rangle^{1-\frac{a-1}{2}} \|\phi(\tau)\|_{\mathbf{L}^{1,a}} \\
&+ Cg^{-1}(t) \int_{\sqrt{t}}^{t} \langle \tau \rangle^{\frac{a-1}{2}-1} \{\tau\}^{-\frac{1}{2p}} d\tau \sup_{\tau>1} \tau^{1-\frac{a-1}{2}} \|\phi(\tau)\|_{\mathbf{L}^{1,a}} \\
&\le C\varepsilon \left(t^{\frac{a-1}{4}} + g^{-1}(t)\, t^{\frac{a-1}{2}} \right) \|\phi\|_{\mathbf{Y}} \le Cg^{-1}(t)\, t^{\frac{a-1}{2}} \|\phi\|_{\mathbf{Y}}
\end{aligned}$$

for all $t > 4$. Hence the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ is concordant.

By a direct computation we have

$$\int_0^t \operatorname{Re} f\left(\mathcal{N}\left(\theta G_0(\tau)\right)\right) d\tau = |\lambda|\,\theta^{\sigma+1} \int_0^t \int_{\mathbf{R}^n} \left|G_0(\tau,x)\right|^{\sigma+1} dx d\tau$$
$$= |\lambda|\,\theta^{\sigma+1} (4\pi)^{-\frac{n}{2}(1+\sigma)} \int_0^t \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4(\tau+1)}(1+\sigma)} |x_1|^{\sigma+1} dx\, (\tau+1)^{-(1+\sigma)\left(1+\frac{n}{2}\right)} d\tau$$
$$= |\lambda|\,\theta^{\sigma+1} (4\pi)^{-\frac{n}{2}-\frac{n}{2(n+1)}} \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4}\left(1+\frac{1}{n+1}\right)} |x_1|^{\frac{1}{n+1}+1} dx \int_0^t \frac{d\tau}{1+\tau}$$
$$= \theta^{\sigma+1}\eta \int_0^t \frac{d\tau}{1+\tau} = \eta\theta^{\sigma+1} \log(1+t)\,,$$

so conditions (3.2), (3.3) and (3.6) are fulfilled. By interpolation inequality (1.4) we have (3.18); hence condition (3.4) is true. By estimates (1.24) we have (3.19). Since the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ is concordant we see that condition (3.5) is also fulfilled. Now applying Theorem 3.2 we easily get the results of Theorem 3.10 which is now proved.

3.2 Fractional equations

In this section we study nonlinear equations with fractional power of the negative Laplacian

$$\begin{cases} \partial_t u + \alpha\,(-\Delta)^{\frac{\rho}{2}} u + \beta\,|u|^{\frac{\rho}{n}} u + \nu\,|u|^{\kappa} u = 0,\ x \in \mathbf{R}^n, t > 0, \\ u(0,x) = u_0(x)\,,\ x \in \mathbf{R}^n, \end{cases} \tag{3.22}$$

where $\alpha, \beta, \nu \in \mathbf{C}$, $\operatorname{Re}\alpha > 0$, $\rho > 0$, $\kappa > \frac{\rho}{n} > 0$. Furthermore we assume that $\operatorname{Re}\beta\delta(\alpha,\rho) > 0$ where

$$\delta(\alpha,\rho) = \int_{\mathbf{R}^n} |G(x)|^{\frac{\rho}{n}} G(x)\,dx,$$

and $G(x) = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-\alpha|\xi|^\rho}\right)$. The nonlinearity $|u|^{\frac{\rho}{n}} u$ is critical and $|u|^{\kappa} u$ is asymptotically weak. In this section we prove global in time existence of small solutions to the Cauchy problem (3.22).

3.2.1 Small data

Denote $g(t) = 1 + \theta^{\frac{\rho}{n}}\eta \log(1+t)\,,\ \eta = \rho \operatorname{Re}\beta\delta(\alpha,\rho)\,,\ \theta = (2\pi)^{\frac{n}{2}}\,|\hat{u}_0(0)|$.

We assume that $\operatorname{Re}\beta\delta(\alpha,\rho) > 0$. Note that in the particular case of $\rho = 2$ the condition $\operatorname{Re}\beta\delta(\alpha,\rho) > 0$ takes the form

$$\operatorname{Re}\beta\left(\left(2+\frac{\rho}{n}\right)|\alpha|^2 + \frac{\rho}{n}\alpha^2\right)^{-\frac{n}{2}} = |\beta|\left|\left(2+\frac{\rho}{n}\right)|\alpha|^2 + \frac{\rho}{n}\alpha^2\right|^{-\frac{n}{2}}$$
$$\times\cos\left(\arg\beta - \frac{n}{2}\arctan\frac{\sin(2\arg\alpha)}{1+\frac{2n}{\rho}+\cos(2\arg\alpha)}\right) > 0.$$

In this subsection we prove the following result.

Theorem 3.11. *Assume that* $\operatorname{Re}\alpha > 0,\ 0 < \frac{\rho}{n} < \kappa$ *and* $\operatorname{Re}\beta\delta(\alpha,\rho) > 0.$ *Furthermore let* $u_0 \in \mathbf{L}^\infty(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, $a \in (0, \min(1,\rho))$ *and* $|\hat{u}_0(0)| > 0.$ *Then there exists a positive* ε *such that if* $\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^{1,a}} \le \varepsilon$, $|\hat{u}_0(0)| \ge C\varepsilon$, *then the Cauchy problem (3.22) has a unique global solution* $u(t,x) \in \mathbf{C}\left([0,\infty);\mathbf{L}^{1,a}(\mathbf{R}^n)\right) \cap \mathbf{C}\left((0,\infty);\mathbf{L}^\infty(\mathbf{R}^n)\right)$ *satisfying the asymptotics*

$$\begin{aligned} u(t,x) &= \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}x\right) g^{-\frac{n}{\rho}}(t)\, e^{i\psi(t)} \\ &\quad + O\left(t^{-\frac{n}{\rho}} g^{-1-\frac{n}{\rho}}(t) \log\log t\right) \end{aligned}$$

for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}^n$. *Here* $\psi(t)$ *satisfies the estimate*

$$\begin{aligned} &\left|\psi(t) - \arg\widehat{u}_0(0) + \theta^{\frac{\rho}{n}}\widetilde{\eta}\int_0^t g^{-\frac{n}{\rho}}(\tau)(1+\tau)^{-1}d\tau\right| \\ &\le C\varepsilon^{1+\frac{\rho}{n}}\int_0^t g^{-1-\frac{n}{\rho}}(\tau)\log g(\tau)(1+\tau)^{-1}d\tau, \end{aligned}$$

where $\widetilde{\eta} = \operatorname{Im}\beta\delta(\alpha,\rho)$.

Proof of Theorem 3.11

The Green operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t)\phi = \overline{\mathcal{F}}_{\xi\to x} e^{-\alpha|\xi|^\rho t}\hat{\phi}(\xi) = t^{-\frac{n}{\rho}}\int_{\mathbf{R}^n} G\left(t^{-\frac{1}{\rho}}(x-y)\right)\phi(y)dy \tag{3.23}$$

with a kernel $G(x) = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-\alpha|\xi|^\rho}\right)$.

Using result of Lemma 1.28 we have

Lemma 3.12. *Suppose that the function* $\phi \in \mathbf{L}^\infty(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, *where* $a \in (0,1)$. *Then the estimates*

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^p} \le C\|\phi\|_{\mathbf{L}^p}$$

and

$$\left\||\cdot|^\omega\left(\mathcal{G}(t)\phi - \vartheta t^{-\frac{n}{\rho}}G\left(t^{-\frac{1}{\rho}}(\cdot)\right)\right)\right\|_{\mathbf{L}^p} \le Ct^{\frac{n}{\rho}\left(\frac{1}{p}-1\right)+\frac{\omega-a}{\rho}}\|\phi\|_{\mathbf{L}^{1,a}}$$

are valid for all $t > 0$, *where*

$$1 \le p \le \infty,\ 0 \le \omega \le a,\ \vartheta = \int_{\mathbf{R}^n}\phi(x)\,dx.$$

To apply Theorem 3.2 we choose the space

$$\mathbf{Z} = \left\{\phi \in \mathbf{L}^\infty(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)\right\}$$

with $a \in (0, \min(1, \rho)]$ and the space

$$\mathbf{X} = \{\phi \in \mathbf{C}([0, \infty); \mathbf{Z}) : \ \|\phi\|_{\mathbf{X}} < \infty\},$$

with the norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0} \left(\langle t \rangle^{\frac{n}{\rho}} \|\phi(t)\|_{\mathbf{L}^\infty} + \|\phi(t)\|_{\mathbf{L}^1} + \langle t \rangle^{-\frac{a}{\rho}} \|\phi(t)\|_{\mathbf{L}^{1,a}} \right),$$

where $a \in (0,1)$. Also we define the norm $\|f\|_{\mathbf{Y}} = \|\langle t \rangle f(t)\|_{\mathbf{X}}$.

Denote

$$G_0(t,x) = (t+1)^{-\frac{n}{\rho}} G\left((t+1)^{-\frac{1}{\rho}} x \right).$$

Using Lemma 3.12 we clearly see that according to Definition 2.1 the function $G_0(t,x)$ is the asymptotic kernel for the operator $\mathcal{G}$ in spaces $\mathbf{X}, \mathbf{Z}$ with continuous linear functional

$$f(\phi) = \int_{\mathbf{R}^n} \phi(x)\, dx$$

and $\gamma = \frac{a}{\rho}$.

We now prove that the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ is concordant. In view of the estimate $g^{-1}(\tau) \leq 1$ and estimates of the local existence Theorem 1.13 we get (3.14) for all $0 \leq t \leq 1$. We now consider $t > 1$. We have the estimate $\langle t \rangle^{-\frac{a}{2\rho}} \leq C g^{-1}(t)$ and (3.15). Hence by virtue of Lemma 3.12 we obtain

$$\begin{aligned}
&\left\| \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{L}^p} \\
&\leq C \int_0^{\sqrt{t}} (t-\tau)^{\frac{n}{\rho}\left(\frac{1}{p}-1\right) - \frac{a}{\rho}} \tau^{\frac{a}{\rho}-1} d\tau \sup_{\tau>0} (1+\tau)^{-\frac{a}{\rho}} \|(1+\tau) f(\tau)\|_{\mathbf{L}^{1,a}} \\
&+ C g^{-1}(t) \int_{\sqrt{t}}^{\frac{t}{2}} (t-\tau)^{\frac{n}{\rho}\left(\frac{1}{p}-1\right) - \frac{a}{\rho}} \tau^{\frac{a}{\rho}-1} d\tau \sup_{\tau>0} (1+\tau)^{-\frac{a}{\rho}} \|(1+\tau) f(\tau)\|_{\mathbf{L}^{1,a}} \\
&+ C g^{-1}(t) \int_{\frac{t}{2}}^{t} \tau^{\frac{n}{\rho}\left(\frac{1}{p}-1\right) - 1} d\tau \sup_{\tau>0} (1+\tau)^{-\frac{n}{\rho}\left(\frac{1}{p}-1\right)} \|(1+\tau) f(\tau)\|_{\mathbf{L}^p} \\
&\leq C \left(t^{\frac{n}{\rho}\left(\frac{1}{p}-1\right) - \frac{a}{2\rho}} + g^{-1}(t)\, t^{\frac{n}{\rho}\left(\frac{1}{p}-1\right)} \right) \|(1+t) f(t,x)\|_{\mathbf{X}} \\
&\leq C g^{-1}(t)\, t^{\frac{n}{\rho}\left(\frac{1}{p}-1\right)} \|(1+t) f(t,x)\|_{\mathbf{X}}
\end{aligned}$$

for $1 \leq p \leq \infty$, and

$$
\begin{aligned}
&\left\|\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)f(\tau)\,d\tau\right\|_{\mathbf{L}^{1,a}}\\
&\le C\int_0^{\sqrt{t}}\tau^{\frac{a}{\rho}-1}d\tau\sup_{\tau>0}(1+\tau)^{-\frac{a}{\rho}}\|f(\tau)\|_{\mathbf{L}^{1,a}}\\
&+Cg^{-1}(t)\int_{\sqrt{t}}^t\tau^{\frac{a}{\rho}-1}d\tau\sup_{\tau>0}(1+\tau)^{-\frac{a}{\rho}}\|f(\tau)\|_{\mathbf{L}^{1,a}}\\
&\le C\varepsilon\left(t^{\frac{a}{2\rho}}+g^{-1}(t)\,t^{\frac{a}{\rho}}\right)\|(1+t)\,f(t,x)\|_{\mathbf{X}}\\
&\le Cg^{-1}(t)\,t^{\frac{a}{\rho}}\|(1+t)\,f(t,x)\|_{\mathbf{X}}
\end{aligned}
$$

for all $t>1$. Hence the triad $(\mathbf{X},\mathbf{Y},\mathcal{G})$ is concordant.

By a direct computation we have

$$
\begin{aligned}
&\int_0^t \operatorname{Re} f\left(\mathcal{N}\left(\theta G_0(\tau)\right)\right)d\tau=|\beta|\,\theta^{\frac{\rho}{n}+1}\operatorname{Re}\int_0^t\int_{\mathbf{R}^n}G_0^{\frac{\rho}{n}+1}(\tau,x)\,dxd\tau\\
&+|\nu|\,\theta^{\kappa+1}\operatorname{Re}\int_0^t\int_{\mathbf{R}^n}G_0^{\kappa+1}(\tau,x)\,dxd\tau\ge\theta^{\frac{\rho}{n}}\eta\log(1+t),
\end{aligned}
$$

where $\eta=\rho\operatorname{Re}\beta\delta(\alpha,\rho)$. Finally conditions (3.2) and (3.6) with $g(t)=1+\theta^{\frac{\rho}{n}}\eta\log(1+t)$ are fulfilled; therefore the nonlinearity is critical nonconvective.

Since we cannot directly apply Theorem 3.2 we need to carry out some modifications. As in the proof of Theorem 3.2 we make a change of the dependent variable $u(t,x)=v(t,x)\,e^{-\varphi(t)+i\psi(t)}$. Then for the new function $v(t,x)$ we get the following equation

$$
\partial_t v+\alpha(-\Delta)^{\frac{\rho}{2}}v+\beta e^{-\frac{\rho}{n}\varphi}|v|^{\frac{\rho}{n}}v+\nu e^{-\kappa\varphi}|v|^{\kappa}v-(\varphi'-i\psi')v=0.
$$

Assume that

$$
\int_{\mathbf{R}^n}\left(\beta e^{-\frac{\rho}{n}\varphi}|v|^{\frac{\rho}{n}}v+\nu e^{-\kappa\varphi}|v|^{\kappa}v-(\varphi'-i\psi')v\right)dx=0;
$$

the mean value of the new function $v(t,x)$ then satisfies a conservation law:

$$
\frac{d}{dt}\int_{\mathbf{R}^n}v(t,x)\,dx=0.
$$

Consequently $\widehat{v}(t,0)=\widehat{v}_0(0)$ for all $t>0$. We can choose $\varphi(0)=0$ and $\psi(0)$ such that

$$
\hat{v}_0(0)=|\hat{u}_0(0)|=\theta(2\pi)^{-\frac{n}{2}}>0.
$$

Thus we consider the Cauchy problem for the new dependent variables $(v(t,x),\varphi(t))$

$$
\begin{cases}
\partial_t v+\alpha(-\Delta)^{\frac{\rho}{2}}v=-\beta e^{-\frac{\rho}{n}\varphi}\left(|v|^{\frac{\rho}{n}}-\frac{1}{\theta}\int_{\mathbf{R}^n}|v|^{\frac{\rho}{n}}vdx\right)v\\
\qquad -\nu e^{-\kappa\varphi}\left(|v|^{\kappa}-\frac{1}{\theta}\int_{\mathbf{R}^n}|v|^{\kappa}vdx\right)v,\\
\partial_t\varphi(t)=\frac{1}{\theta}e^{-\frac{\rho}{n}\varphi}\left(\operatorname{Re}\beta\int_{\mathbf{R}^n}|v|^{\frac{\rho}{n}}vdx+e^{\left(\frac{\rho}{n}-\kappa\right)\varphi}\operatorname{Re}\nu\int_{\mathbf{R}^n}|v|^{\kappa}vdx\right),\\
\qquad v(0,x)=v_0(x),\ \varphi(0)=0.
\end{cases}
\tag{3.24}
$$

Denote $h(t) = e^{\frac{\rho}{n}\varphi(t)}$ and write (3.24) as

$$\begin{cases} \partial_t v + \alpha(-\Delta)^{\frac{\rho}{2}} v = F(v,h),\ v(0,x) = v_0(x), \\ \partial_t h = \frac{\rho}{\theta n}\left(\operatorname{Re}\beta \int_{\mathbf{R}^n} |v|^{\frac{\rho}{n}} v dx + h^{1-\frac{\kappa n}{\rho}} \operatorname{Re}\nu \int_{\mathbf{R}^n} |v|^{\kappa} v dx\right),\ h(0) = 1, \end{cases}$$

where

$$F(v,h) = -\beta h^{-1}\left(|v|^{\frac{\rho}{n}} - \frac{1}{\theta}\int_{\mathbf{R}^n} |v|^{\frac{\rho}{n}} v dx\right) v$$
$$- \nu h^{-\frac{\kappa n}{\rho}}\left(|v|^{\kappa} - \frac{1}{\theta}\int_{\mathbf{R}^n} |v|^{\kappa} v dx\right) v.$$

Note that the mean value of the nonlinearity $\widehat{F(v,h)}(t,0) = 0$ for all $t > 0$.

Denote by

$$\mathcal{N}_1 = \beta |u|^{\frac{\rho}{n}} u,\ \mathcal{N}_2 = \nu |u|^{\kappa} u.$$

Now let us check the conditions (3.4) and (3.5) of Theorem 3.2 for each nonlinearity $\mathcal{N}_j$. Since

$$\begin{aligned} &\nu(t) \|\mathcal{N}_j(v(t)) - \mathcal{N}_j(w(t))\|_{\mathbf{L}^1} \\ &\le C\nu(t)\left(\|v(t)\|_{\mathbf{L}^\infty}^{\frac{\rho}{n}} + \|w(t)\|_{\mathbf{L}^\infty}^{\frac{\rho}{n}}\right)\|v(t) - w(t)\|_{\mathbf{L}^1} \\ &\le C\langle t\rangle^{-1}\|\nu(t)(v-w)\|_{\mathbf{X}}\left(\|v\|_{\mathbf{X}}^{\frac{\rho}{n}} + \|w\|_{\mathbf{X}}^{\frac{\rho}{n}}\right), \end{aligned}$$

condition (3.4) is true. Also we have

$$\begin{aligned} &\|\mathcal{K}_j(v) - \mathcal{K}_j(w)\|_{\mathbf{Y}} \le \|\mathcal{N}_j(v) - \mathcal{N}_j(w)\|_{\mathbf{Y}} \\ &+ \frac{1}{\theta}\left(\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}\right)\sup_{t>0}\langle t\rangle \|\mathcal{N}_j(v(t)) - \mathcal{N}_j(w(t))\|_{\mathbf{L}^1} \\ &+ \frac{1}{\theta}\|v - w\|_{\mathbf{X}}\sup_{t>0}\langle t\rangle\left(\|\mathcal{N}_j(v(t))\|_{\mathbf{L}^1} + \|\mathcal{N}_j(w(t))\|_{\mathbf{L}^1}\right) \\ &\le C\|v - w\|_{\mathbf{X}}\left(\|v\|_{\mathbf{X}}^{\frac{\rho}{n}} + \|w\|_{\mathbf{X}}^{\frac{\rho}{n}}\right)\left(1 + \frac{1}{\theta}\|v\|_{\mathbf{X}} + \frac{1}{\theta}\|w\|_{\mathbf{X}}\right), \end{aligned}$$

where $\|w\|_{\mathbf{Y}} = \|\langle t\rangle w\|_{\mathbf{X}}$. Therefore, since the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ is concordant we see that condition (3.5) is fulfilled. Now applying Theorem 3.2 we easily get the results of Theorem 3.11. This completes the proof of Theorem 3.11.

3.2.2 Large data

In this subsection we consider the Cauchy problem for the nonlinear heat equations with fractional power of the negative Laplacian

$$\begin{cases} u_t + (-\Delta)^{\frac{\rho}{2}} u + u^{1+\frac{\rho}{n}} = 0,\ x \in \mathbf{R}^n,\ t > 0, \\ u(0,x) = u_0(x),\ x \in \mathbf{R}^n, \end{cases} \tag{3.25}$$

where $\rho \in (0,2]$. In particular, when $\rho = 2$ equation (3.25) is the nonlinear heat equation. Let $u_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n)$, $u_0(x) \geq 0$ almost everywhere in $\mathbf{R}^n$, $\theta = \int_{\mathbf{R}^n} u_0(x)\,dx > 0$. We are interested in the global existence of solutions to the Cauchy problem (3.25) with critical power of the nonlinearity when the initial data are not small.

Denote

$$g(t) = 1 + \theta^{\frac{\rho}{n}} \eta(\rho) \log(1+t),$$

$$\eta(\rho) = \int_{\mathbf{R}^n} (G(x))^{1+\frac{\rho}{n}}\,dx,$$

where $G(x) = \overline{\mathcal{F}}_{\xi \to x}\left(e^{-|\xi|^\rho}\right)$. We state the result of this subsection.

Theorem 3.13. *Assume that* $u_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n) \cap \mathbf{C}(\mathbf{R}^n)$, $u_0(x) \geq 0$ *in* $\mathbf{R}^n$, $\theta = \int_{\mathbf{R}^n} u_0(x)\,dx > 0$, $0 < a < \min(1,\rho)$. *Then the Cauchy problem (3.25) has a unique global solution*

$$u(t,x) \in \mathbf{C}\left([0,\infty); \mathbf{L}^\infty(\mathbf{R}^n) \cap \mathbf{C}(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)\right)$$

satisfying the time decay estimate

$$u(t,x) = \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}} x\right) g^{-\frac{n}{\rho}}(t) + O\left(t^{-\frac{n}{\rho}} g^{-1-\frac{n}{\rho}}(t) \log g(t)\right)$$

for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}^n$.

Lemmas

In the following lemma we compare the solutions of the following two problems

$$\begin{cases} w_t + (-\Delta)^{\frac{\rho}{2}} w + w^{1+\frac{\rho}{n}} = 0, \ x \in \mathbf{R}^n, \ t > 0, \\ w(0,x) = w_0(x), \ x \in \mathbf{R}^n \end{cases} \tag{3.26}$$

and

$$\begin{cases} v_t + (-\Delta)^{\frac{\rho}{2}} v + \epsilon v^{1+\frac{\rho}{n}} = 0, \ x \in \mathbf{R}^n, \ t > 0, \\ v(0,x) = v_0(x), \ x \in \mathbf{R}^n. \end{cases} \tag{3.27}$$

Lemma 3.14. *Let* $0 < \rho \leq 2$. *Suppose that* $w_0(x), v_0(x) \in \mathbf{L}^\infty(\mathbf{R}^n) \cap \mathbf{C}(\mathbf{R}^n)$, $w_0(x) \leq v_0(x)$ *in* $\mathbf{R}^n$, *and* $0 \leq \epsilon \leq 1$. *Then* $w(t,x) \leq v(t,x)$ *for all* $t > 0$ *and* $x \in \mathbf{R}^n$.

Proof. For the difference $r = v - w$ we get

$$r_t + (-\Delta)^{\frac{\rho}{2}} r = (v-r)^{1+\frac{\rho}{n}} - \epsilon v^{1+\frac{\rho}{n}}, \ x \in \mathbf{R}^n, \ t > 0, \tag{3.28}$$

with initial condition $r(0,x) = v_0(x) - w_0(x) \geq 0$ in $\mathbf{R}^n$. Define

$$R(t) \equiv \inf_{x \in \mathbf{R}^n} r(t,x).$$

We need to prove that $R(t) \geq 0$ for all $t > 0$. On the contrary, suppose that there exists a time $T > 0$ such that $R(T) < 0$. By the continuity of the function $R(t)$ we can find an interval $[T_1, T]$ such that $R(t) \leq 0$ for all $t \in [T_1, T]$ and $R(T_1) = 0$. By Theorem 2.1 from paper Constantin and Escher [1998] there exists a point $\zeta(t) \in \mathbf{R}^n$ such that $R(t) = r(t, \zeta(t))$; moreover $R'(t) = \frac{d}{dt} r(t, \zeta(t))$ almost everywhere on $t \in [T_1, T]$. We have

$$(v(t,\zeta(t)) - R(t))^{1+\frac{\rho}{n}} - \epsilon v^{1+\frac{\rho}{n}}(t,\zeta(t)) \geq 0 \text{ for all } t \in [T_1, T].$$

Representing the operator $(-\Delta)^{\frac{\rho}{2}}$ in the point of maximum $\zeta(t)$ via the Riesz potential (see Stein [1970], Yosida [1995]) we see that

$$(-\Delta)^{\frac{\rho}{2}} r(t,\zeta(t)) = C_\rho \int_{\mathbf{R}^n} |\zeta(t) - y|^{-\rho - n} (r(t,y) - R(t))\, dy \leq 0,$$

where $C_\rho > 0$ is a constant. Therefore by equation (3.28) we get

$$R'(t) \geq 0 \text{ for all } t \in [T_1, T].$$

Integration with respect to time yields $R(T) \geq 0$, which gives a contradiction. Lemma 3.14 is proved.

The next lemma will be used in the proof of Theorem 3.13 to evaluate the large time behavior of the mean value of the nonlinearity in equation (3.25). We use the notation

$$\eta(\rho) = \int_{\mathbf{R}^n} G^{1+\frac{\rho}{n}}(x)\, dx$$

and $g(t) = 1 + \theta^{\frac{\rho}{n}} \eta(\rho) \log(1+t)$, $G(x) = \overline{\mathcal{F}}_{\xi \to x}\left(e^{-|\xi|^\rho}\right)$.

Lemma 3.15. *Assume that $u_0 \in \mathbf{L}^\infty(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, and $\int_{\mathbf{R}^n} u_0(x)\, dx = \theta$. Let function $v(t,x)$ satisfy the estimates*

$$\|v(t)\|_{\mathbf{L}^p} \leq C \langle t \rangle^{-\frac{n}{\rho}\left(1-\frac{1}{p}\right)}$$

and

$$\|v(t) - \mathcal{G}(t) u_0\|_{\mathbf{L}^1} \leq C (\log(2+t))^{-1}$$

for all $t > 0$, $1 \leq p \leq \infty$.

Then the inequality

$$\left| 1 + \frac{\rho}{n\theta} \int_0^t d\tau \int_{\mathbf{R}^n} v^{1+\frac{\rho}{n}}(\tau,x)\, dx - g(t) \right| \leq C \log(g(t)) \tag{3.29}$$

is valid for all $t > 0$.

Proof. We get

$$\left\| \mathcal{G}\left(t\right)u_0 - \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}\left(\cdot\right)\right)\right\|_{\mathbf{L}^1} \leq C t^{-\frac{a}{\rho}},$$

where $\theta = \int_{\mathbf{R}^n} u_0\left(x\right)dx$. We thus find

$$\begin{aligned}
&\left\| v^{1+\frac{\rho}{n}} - \theta^{1+\frac{\rho}{n}} t^{-1-\frac{1}{\rho}} G^{1+\frac{\rho}{n}}\left(t^{-\frac{1}{\rho}}\left(\cdot\right)\right)\right\|_{\mathbf{L}^1} \\
&\leq C\left(\left\|v\left(t\right) - \mathcal{G}\left(t\right)u_0\right\|_{\mathbf{L}^1} + \left\|\mathcal{G}\left(t\right)u_0 - \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}\left(\cdot\right)\right)\right\|_{\mathbf{L}^1}\right) \\
&\times\left(\left\|v\right\|_{\mathbf{L}^\infty}^{\frac{\rho}{n}} + \left\|\mathcal{G}\left(t\right)u_0\right\|_{\mathbf{L}^\infty}^{\frac{\rho}{n}} + \theta^{\frac{\rho}{n}} t^{-1}\left\|G\right\|_{\mathbf{L}^\infty}^{\frac{\rho}{n}}\right) \\
&\leq t^{-1} C\left(\log\left(2+t\right)\right)^{-1} + C t^{-1-\frac{a}{\rho}}
\end{aligned}$$

for all $t > 0$. Since

$$t^{-\frac{n}{\rho}} \int_{\mathbf{R}^n}\left(t^{-\frac{n}{\rho}} G\left(x t^{-\frac{1}{\rho}}\right)\right)^{1+\frac{\rho}{n}} dx = \eta\left(\rho\right),$$

we get

$$\begin{aligned}
&\left|\int_{\mathbf{R}^n} v^{1+\frac{\rho}{n}}\left(t,x\right)dx - \theta^{1+\frac{\rho}{n}} \frac{\eta\left(\rho\right)}{\rho t}\right| \\
&\leq C\left\| v^{1+\frac{\rho}{n}} - \theta^{1+\frac{\rho}{n}} t^{-1-\frac{\rho}{n}} G^{1+\frac{\rho}{n}}\left(t^{-\frac{1}{\rho}}\left(\cdot\right)\right)\right\|_{\mathbf{L}^1} \\
&\leq C t^{-1}\left(\log\left(2+t\right)\right)^{-1} + C t^{-1-\frac{a}{\rho}}
\end{aligned}$$

for all $t > 0$. Therefore

$$\begin{aligned}
&\left|\frac{\rho}{\theta}\int_1^t d\tau \int_{\mathbf{R}^n} v^{1+\frac{\rho}{n}}\left(\tau,x\right)dx - \theta^{\frac{\rho}{n}}\eta\left(\rho\right)\log t\right| \\
&\leq \int_1^t \frac{C d\tau}{\tau\left(1+\theta^{\frac{\rho}{n}}\eta\left(\rho\right)\log(1+\tau)\right)} + C\int_1^t \tau^{-1-\frac{a}{\rho}} d\tau \qquad (3.30) \\
&\leq C\log\left(1+\theta^{\frac{\rho}{n}}\eta\left(\rho\right)\log\left(1+t\right)\right)
\end{aligned}$$

for all $t > 0$. Thus in view of (3.30) we obtain (3.29). Lemma 3.15 is proved.

Proof of Theorem 3.13

We take sufficiently small $\varepsilon > 0$ and consider the following two auxiliary Cauchy problems

$$\begin{cases} U_t + \left(-\Delta\right)^{\frac{\rho}{2}} U + U^{1+\frac{\rho}{n}} = 0,\ x \in \mathbf{R}^n,\ t > 0, \\ U\left(0,x\right) = \varepsilon u_0\left(x\right),\ x \in \mathbf{R}^n \end{cases} \qquad (3.31)$$

and

$$\begin{cases} V_t + (-\Delta)^{\frac{\rho}{2}} V + \varepsilon^{\frac{\rho}{n}} V^{1+\frac{\rho}{n}} = 0, \ x \in \mathbf{R}^n, \ t > 0, \\ V(0,x) = u_0(x), \ x \in \mathbf{R}^n. \end{cases} \tag{3.32}$$

Note that problem (3.32) can be reduced to problem (3.31) by the change $V = \varepsilon^{-1} U$. In addition problem (3.31) has a sufficiently small initial data; so for the functions $U(t,x)$ and $V(t,x)$ the large time asymptotics are known (see Section 3.2). Employing Lemma 3.14 we then get

$$U(t,x) \le u(t,x) \le V(t,x)$$

for all $t > 0$ and $x \in \mathbf{R}^n$. In particular, using the results of Subsection 3.2.1 we have

$$\|u(t)\|_{\mathbf{L}^p} \le C\varepsilon^{-1} \langle t \rangle^{-\frac{n}{\rho}\left(1-\frac{1}{p}\right)} (\log(2+t))^{-\frac{n}{\rho}} \tag{3.33}$$

and

$$\|u(t)\|_{\mathbf{L}^{1,a}} \le C\varepsilon^{-1} \langle t \rangle^{\frac{a}{\rho}} (\log(2+t))^{-\frac{n}{\rho}} \tag{3.34}$$

for all $t > 0$ and $1 \le p \le \infty$.

As in the proof of Theorem 3.2 we change the dependent variable $u(t,x) = v(t,x) e^{-\varphi(t)}$. Then for the new function $v(t,x)$ we get the following equation

$$v_t + (-\Delta)^{\frac{\rho}{2}} v + u^{\frac{\rho}{n}} v - \varphi' v = 0.$$

We assume that $\varphi(0) = 0$ and

$$\int_{\mathbf{R}^n} \left(u^{\frac{\rho}{n}} v - \varphi' v \right) dx = 0,$$

then the mean value of new function $v(t,x)$ satisfies a conservation law:

$$\frac{d}{dt} \int_{\mathbf{R}^n} v(t,x)\, dx = 0.$$

Hence

$$\int_{\mathbf{R}^n} v(t,x)\, dx = \theta = \int_{\mathbf{R}^n} u_0(x)\, dx$$

for all $t > 0$. Now we consider the Cauchy problem for the new dependent variables $(v(t,x), \varphi(t))$

$$\begin{cases} v_t + (-\Delta)^{\frac{\rho}{2}} v = -\left(u^{\frac{\rho}{n}} - \frac{1}{\theta} \int_{\mathbf{R}^n} u^{\frac{\rho}{n}} v dx \right) v \\ \varphi'(t) = \frac{1}{\theta} \int_{\mathbf{R}^n} u^{\frac{\rho}{n}} v dx, \\ v(0,x) = u_0(x), \ \varphi(0) = 0. \end{cases} \tag{3.35}$$

Let us prove the estimate

$$\|v(t)\|_{\mathbf{L}^{1,a}} \le C \langle t \rangle^{\frac{a}{\rho}} \tag{3.36}$$

for all $t > 0$. By estimate (3.33) we have

$$\|u(t)\|_{\mathbf{L}^\infty}^{\frac{\rho}{n}} \le C\varepsilon^{-\frac{\rho}{n}} \langle t\rangle^{-1} (\log(2+t))^{-1},$$

then

$$\varphi'(t) \le \frac{1}{\theta}\int_{\mathbf{R}^n} u^{\frac{\rho}{n}} v dx \le C\varepsilon^{-\frac{\rho}{n}} \langle t\rangle^{-1} (\log(2+t))^{-1},$$

and furthermore

$$\varphi(t) \le C\varepsilon^{-\frac{\rho}{n}} \log(\log(4+t)).$$

Therefore by (3.33) and (3.34) we get

$$\begin{aligned}\|v(t)\|_{\mathbf{L}^{1,a}} &\le e^{\varphi(t)} \|u(t)\|_{\mathbf{L}^{1,a}} \le C\varepsilon^{-1} \langle t\rangle^{\frac{a}{\rho}} (\log(2+t))^{C-\frac{n}{\rho}} \\ &\le C(\varepsilon, T) \langle t\rangle^{\frac{a}{\rho}}\end{aligned}$$

and

$$\begin{aligned}\|v(t)\|_{\mathbf{L}^p} &\le e^{\varphi(t)} \|u(t)\|_{\mathbf{L}^p} \le C\varepsilon^{-1} \langle t\rangle^{-\frac{n}{\rho}\left(1-\frac{1}{p}\right)} (\log(2+t))^{C-\frac{n}{\rho}} \\ &\le C(\varepsilon, T) \langle t\rangle^{-\frac{n}{\rho}\left(1-\frac{1}{p}\right)}\end{aligned}$$

for all $0 < t \le T$. Consider now $t > T$. We use the integral equation associated with (3.35)

$$\begin{aligned}&v(t) = \mathcal{G}(t-T) v(T) \\ &- \int_T^t \mathcal{G}(t-\tau)\left(u^{\frac{\rho}{n}}(\tau) v(\tau) - \frac{v(\tau)}{\theta}\int_{\mathbf{R}^n} u^{\frac{\rho}{n}}(\tau) v(\tau) dx\right) d\tau.\end{aligned} \tag{3.37}$$

By virtue of estimate (3.34) we get

$$\begin{aligned}&\left\|\int_T^t \mathcal{G}(t-\tau)\left(u^{\frac{\rho}{n}}(\tau) v(\tau) - \frac{v(\tau)}{\theta}\int_{\mathbf{R}^n} u^{\frac{\rho}{n}}(\tau) v(\tau) dx\right) d\tau\right\|_{\mathbf{L}^{1,a}} \\ &\le C\int_T^t \left\|u^{\frac{\rho}{n}}(\tau) v(\tau) - \frac{v(\tau)}{\theta}\int_{\mathbf{R}^n} u^{\frac{\rho}{n}}(\tau) v(\tau) dx\right\|_{\mathbf{L}^{1,a}} d\tau \\ &\le C\int_T^t \langle\tau\rangle^{-1} (\log(2+\tau))^{-1} \|v(\tau)\|_{\mathbf{L}^{1,a}} d\tau\end{aligned}$$

for all $t > T$. Therefore in view of (3.37) we find

$$\begin{aligned}&\|v(t)\|_{\mathbf{L}^{1,a}} \le \|\mathcal{G}(t-T) v(T)\|_{\mathbf{L}^{1,a}} \\ &+ C\varepsilon^{-1}\int_T^t \langle\tau\rangle^{-1} (\log(2+\tau))^{-1} \|v(\tau)\|_{\mathbf{L}^{1,a}} d\tau \\ &\le C(\varepsilon, T) \langle t\rangle^{\frac{a}{\rho}} + \varepsilon\int_T^t \langle\tau\rangle^{-1} \|v(\tau)\|_{\mathbf{L}^{1,a}} d\tau\end{aligned}$$

for all $t > T$. Here $\varepsilon > 0$ is small enough, and $T > 0$ is sufficiently large. Application of the Gronwall's lemma yields the estimate

$$\|v(t)\|_{\mathbf{L}^{1,a}} \le C \langle t\rangle^{\frac{a}{\rho}}$$

for all $t > 0$.

Likewise by virtue of estimates (3.33) we get

$$\begin{aligned}
&\|v(t)\|_{\mathbf{L}^p} \le \|\mathcal{G}(t-T)\, v(T)\|_{\mathbf{L}^p} \\
&+ \left\| \int_T^t \mathcal{G}(t-\tau) \left(u^{\frac{\rho}{n}}(\tau) - \frac{1}{\theta} \int_{\mathbf{R}^n} u^{\frac{\rho}{n}}(\tau)\, v(\tau)\, dx \right) v(\tau)\, d\tau \right\|_{\mathbf{L}^p} \\
&\le C(\varepsilon, T) \langle t\rangle^{-\frac{n}{\rho}\left(1-\frac{1}{p}\right)} \\
&+ C\varepsilon^{-1} \int_T^{\frac{t}{2}} (t-\tau)^{-\frac{n}{\rho}\left(1-\frac{1}{p}\right)-\frac{a}{\rho}} \langle \tau\rangle^{\frac{a}{\rho}-1} (\log(2+\tau))^{-1} d\tau \\
&+ C\varepsilon^{-1} \int_{\frac{t}{2}}^t \langle \tau\rangle^{-1} (\log(2+\tau))^{-1} \|v(\tau)\|_{\mathbf{L}^p} d\tau \\
&\le C(\varepsilon, T) \langle t\rangle^{-\frac{n}{\rho}\left(1-\frac{1}{p}\right)} + \varepsilon \int_{\frac{t}{2}}^t \langle \tau\rangle^{-1} \|v(\tau)\|_{\mathbf{L}^p} d\tau
\end{aligned}$$

for all $t > T$. The application of Gronwall's lemma yields the estimate

$$\|v(t)\|_{\mathbf{L}^p} \le C \langle t\rangle^{-\frac{n}{\rho}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$.

Using the integral equation (3.37) we obtain

$$\begin{aligned}
&\|v(t) - \mathcal{G}(t)\, u_0\|_{\mathbf{L}^1} \\
&\le \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{a}{\rho}} \left\| u^{\frac{\rho}{n}}(\tau)\, v(\tau) - \frac{v(\tau)}{\theta} \int_{\mathbf{R}^n} u^{\frac{\rho}{n}}(\tau)\, v(\tau)\, dx \right\|_{\mathbf{L}^{1,a}} d\tau \\
&+ \int_{\frac{t}{2}}^t \left\| u^{\frac{\rho}{n}}(\tau)\, v(\tau) - \frac{v(\tau)}{\theta} \int_{\mathbf{R}^n} u^{\frac{\rho}{n}}(\tau)\, v(\tau)\, dx \right\|_{\mathbf{L}^1} d\tau \\
&\le C \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{a}{\rho}} \langle \tau\rangle^{\frac{a}{\rho}-1} (\log(2+\tau))^{-1} d\tau \\
&+ C \int_{\frac{t}{2}}^t \langle \tau\rangle^{-1} (\log(2+\tau))^{-1} d\tau \le C \log(2+t)^{-1}
\end{aligned} \tag{3.38}$$

for all $t > 0$. Now we apply Lemma 3.15 to find for $h(t) = e^{\frac{\rho}{n}\varphi(t)}$

$$|h(t) - g(t)| \le C \log g(t),$$

for all $t > 0$. Then via formula

$$u(t,x) = e^{-\varphi(t)} v(t,x) = h^{-\frac{n}{\rho}}(t)\, v(t,x)$$

we find the estimates

$$\begin{aligned}
&\left\| u(t) - \theta t^{-\frac{n}{\rho}} e^{-\varphi(t)} G\left(t^{-\frac{1}{\rho}}(\cdot)\right)\right\|_{\mathbf{L}^\infty} \\
&\le \left\| u(t) - e^{-\varphi(t)} \mathcal{G}(t) u_0 \right\|_{\mathbf{L}^\infty} \\
&+ \left\| \mathcal{G}(t) u_0 - \theta t^{-\frac{n}{\rho}} e^{-\varphi(t)} G\left(t^{-\frac{1}{\rho}}(\cdot)\right)\right\|_{\mathbf{L}^\infty} \\
&\le C t^{-\frac{n}{\rho}} g^{-1-\frac{n}{\rho}}(t) + C t^{-\frac{n}{\rho}-\frac{a}{\rho}} \|u_0\|_{\mathbf{L}^{1,a}} \le C t^{-\frac{n}{\rho}} g^{-1-\frac{n}{\rho}}(t)
\end{aligned} \tag{3.39}$$

for all $t > 0$. We also have

$$\begin{aligned}
&\left\| \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right) h^{-\frac{n}{\rho}}(t) - \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right) g^{-\frac{n}{\rho}}(t)\right\|_{\mathbf{L}^\infty} \\
&\le C t^{-\frac{n}{\rho}} g^{-1-\frac{n}{\rho}}(t) \left| h(t) - g(t)\right|,
\end{aligned}$$

and via (3.39) it follows that

$$\begin{aligned}
&\left\| u(t) - \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right) g^{-\frac{n}{\rho}}(t)\right\|_{\mathbf{L}^\infty} \\
&\le C(1+t)^{-\frac{n}{\rho}} g^{-1-\frac{n}{\rho}}(t) \log g(t).
\end{aligned}$$

This completes the proof of Theorem 3.13.

3.3 Asymptotics for large x and t

We study fractional nonlinear equations

$$\begin{cases} u_t + |\partial_x|^\alpha u + |u|^\alpha u = 0, \ x \in \mathbf{R}, t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}, \end{cases} \tag{3.40}$$

where $\alpha > 0$. For simplicity we consider the one dimensional case. Higher dimensional problems also can be considered by our method.

In Section 3.2 for the solutions of the Cauchy problem (3.40) with initial data u_0 such that $\left(1+x^2\right)^{\frac{\beta}{2}} u_0 \in \mathbf{L}^1(\mathbf{R})$, $\beta > 0$ we obtained the following asymptotics

$$u(t,x) = t^{-\frac{1}{\alpha}} (\log t)^{-\frac{1}{\alpha}} \left(G(\xi) + O\left((\log t)^{-1} \log\log t \right)\right) \tag{3.41}$$

for large time $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $\xi = t^{-\frac{1}{\alpha}} x$,

$$G(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{i\xi\eta} e^{-|\eta|^\alpha} d\eta.$$

Observe that formula (3.41) does not explain the behavior of solution $u(t,x)$, when t and x tend to infinity simultaneously, because the main term of asymptotics (3.41) vanishes as $|\xi| \to \infty$. However, the dependence on ξ of the remainder in the right-hand side of (3.41) is not evaluated explicitly.

In this section we fill this gap and calculate the asymptotics of solutions $u(t,x)$ to the Cauchy problem for equation (3.40) as $t \to \infty$ and $\xi = x t^{-\frac{1}{\alpha}} \to \infty$.

Denote the total mass of the initial data $\theta = \int_{\mathbf{R}} u_0(x)\, dx$.

3.3.1 Small initial data

In this subsection we prove the following result

Theorem 3.16. *Suppose that the initial data* $u_0 \in \mathbf{L}^{\infty,\alpha+1}(\mathbf{R}) \cap \mathbf{C}(\mathbf{R})$ *have a sufficiently small norm* $\|u_0\|_{\mathbf{L}^{\infty,\alpha+1}}$ *and* $\theta = \int_{\mathbf{R}} u_0(x)\,dx > 0$. *Then there exists a unique solution* $u \in \mathbf{C}\left([0,\infty);\mathbf{L}^{\infty,\alpha+1}(\mathbf{R}) \cap \mathbf{C}(\mathbf{R})\right)$ *to the Cauchy problem (3.40). Moreover, the asymptotics*

$$u(t,x) = t^{-\frac{1}{\alpha}} (\log t)^{-\frac{1}{\alpha}} \left(G(\xi) + O\left(\langle \xi \rangle^{-\alpha-1} \frac{\log\log t}{\log t} \right) \right) \tag{3.42}$$

is true for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}$, *where* $G(\xi) = \overline{\mathcal{F}}_{\eta\to\xi}\left(e^{-|\eta|^{\alpha}}\right)$, $\xi = t^{-\frac{1}{\alpha}} x$.

Remark 3.17. Since the asymptotics of the function $G(\xi)$ is (see Lemma 1.41)

$$G(\xi) = \frac{\sqrt{2}}{\sqrt{\pi}} \Gamma(\alpha+1) |\xi|^{-1-\alpha} \sin\frac{\pi\alpha}{2} + O\left(|\xi|^{-1-2\alpha}\right) \tag{3.43}$$

for $\xi \to \infty$, then we see that the second summand in the right-hand side of (3.42) is a remainder, when simultaneously $t \to \infty$ and $|\xi| \to \infty$. When the parameter $\alpha = 2k$, $k \in \mathbf{N}$, the main term in the asymptotics (3.43) vanishes. Indeed for this case the function $G(\xi)$ decays exponentially, so that formula (3.42) gives only the estimate of the solution as $t \to \infty$, and $\xi = xt^{-\frac{1}{\alpha}} \to \infty$. Some additional decay conditions for the initial data $u_0(x)$ must be fulfilled to attain the asymptotic representation for large t and x (see Theorem 3.19 below).

Proof of Theorem 3.16

Define the Green operator

$$\mathcal{G}(t)\phi = \overline{\mathcal{F}}_{\eta\to x} e^{-|\eta|^{\alpha} t} \hat{\phi}(\eta) = t^{-\frac{1}{\alpha}} \int_{\mathbf{R}} G\left(t^{-\frac{1}{\alpha}}(x-y)\right) \phi(y)dy \tag{3.44}$$

with a kernel

$$G(\xi) = \overline{\mathcal{F}}_{\eta\to\xi}\left(e^{-|\eta|^{\alpha}}\right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{i\xi\eta} e^{-|\eta|^{\alpha}} d\eta.$$

Define our basic norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0} \left(\langle t \rangle^{\frac{1}{\alpha}} \|\phi(t)\|_{\mathbf{L}^{\infty}} + \langle t \rangle^{-1} \|\phi(t)\|_{\mathbf{L}^{\infty,\alpha+1}} \right)$$

and the norm $\|f\|_{\mathbf{Y}} = \|\langle t \rangle f(t)\|_{\mathbf{X}}$. Then the $\mathbf{L}^{1,\gamma}$ norm can be estimated as follows

$$\langle t\rangle^{-\frac{\gamma}{\alpha}} \|\phi(t)\|_{\mathbf{L}^{1,\gamma}} = \langle t\rangle^{-\frac{\gamma}{\alpha}} \int_{|x|\le\langle t\rangle^{\frac{1}{\alpha}}} \langle x\rangle^{\gamma} |\phi(t,x)|\, dx$$
$$+ \langle t\rangle^{-\frac{\gamma}{\alpha}} \int_{|x|>\langle t\rangle^{\frac{1}{\alpha}}} \left|\langle t\rangle^{-\frac{1}{\alpha}} x\right|^{1+\alpha-\gamma} \langle x\rangle^{\gamma} |\phi(t,x)| \left|\langle t\rangle^{-\frac{1}{\alpha}} x\right|^{\gamma-\alpha-1} dx$$
$$\le C\langle t\rangle^{\frac{1}{\alpha}} \|\phi(t)\|_{\mathbf{L}^\infty} + C\langle t\rangle^{-1} \|\phi(t)\|_{\mathbf{L}^{\infty,\alpha+1}} \int_{|y|>1} |y|^{\gamma-\alpha-1} dy \le C\|\phi\|_{\mathbf{X}}, \tag{3.45}$$

where $\gamma\in[0,1]$, $\gamma<\alpha$. Define the function $g(t)$

$$g(t) = 1+\kappa\log(1+t)$$

with some $\kappa>0$. By the estimates of Lemma 1.40 we can see that the function $G_0(t,x) = (1+t)^{-\frac{1}{\alpha}} G\left(x(1+t)^{-\frac{1}{\alpha}}\right)$ is the asymptotic kernel for the Green operator $\mathcal{G}$ in spaces $\mathbf{X},\mathbf{Z}$. We choose the functional $f(\phi)=\int_{\mathbf{R}}\phi(t,x)\,dx$.

Let us show that the triad $(\mathbf{X},\mathbf{Y},\mathcal{G})$ is concordant. In view of the estimate $g^{-1}(\tau)\le C$ and Lemma 3.12 we get

$$\left\|\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)f(\tau)\,d\tau\right\|_{\mathbf{L}^\infty} + \left\|\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)f(\tau)\,d\tau\right\|_{\mathbf{L}^{\infty,\alpha+1}}$$
$$\le C\int_0^t \|f(\tau)\|_{\mathbf{L}^\infty} d\tau + C\int_0^t \langle t-\tau\rangle \|f(\tau)\|_{\mathbf{L}^1} d\tau + C\int_0^t \|f(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} d\tau$$
$$\le Cg^{-1}(t)\,\|\langle t\rangle f\|_{\mathbf{X}}$$

for all $0\le t\le 4$. We now consider $t>4$. Via the condition of the lemma for the function $g(t)$ we have the estimate $\langle t\rangle^{-\frac{1}{2\alpha}}\le Cg^{-1}(t)$ and estimate (3.15); hence by virtue of (3.45) and the third estimate of Lemma 1.40 with $\beta=1+\alpha$, $\gamma\in[0,1]$, $\gamma<\alpha$, we obtain

$$\left\|\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)f(\tau)\,d\tau\right\|_{\mathbf{L}^\infty}$$
$$\le C\int_0^{\sqrt{t}} (t-\tau)^{-\frac{1+\gamma}{\alpha}} \langle\tau\rangle^{\frac{\gamma}{\alpha}-1} d\tau \sup_{\tau>0}\langle\tau\rangle^{1-\frac{\gamma}{\alpha}} \|f(\tau)\|_{\mathbf{L}^{1,\gamma}}$$
$$+C\int_0^{\sqrt{t}} (t-\tau)^{-1-\frac{1}{\alpha}} d\tau \sup_{\tau>0}\|f(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}}$$
$$+Cg^{-1}(t)\int_{\sqrt{t}}^{\frac{t}{2}} (t-\tau)^{-\frac{1+\gamma}{\alpha}} \langle\tau\rangle^{\frac{\gamma}{\alpha}-1} d\tau \sup_{\tau>0}\langle\tau\rangle^{1-\frac{\gamma}{\alpha}} \|f(\tau)\|_{\mathbf{L}^{1,\gamma}}$$
$$+Cg^{-1}(t)\int_{\sqrt{t}}^{\frac{t}{2}} (t-\tau)^{-1-\frac{1}{\alpha}} d\tau \sup_{\tau>0}\|f(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}}$$
$$+Cg^{-1}(t)\int_{\frac{t}{2}}^{t} \tau^{-1-\frac{1}{\alpha}} d\tau \sup_{\tau>0}\langle\tau\rangle^{1+\frac{1}{\alpha}} \|f(\tau)\|_{\mathbf{L}^\infty}$$
$$\le C\|\langle t\rangle f(t)\|_{\mathbf{X}} \left(t^{-\frac{1}{\alpha}-\frac{1}{2\alpha}} + g^{-1}(t)\,t^{-\frac{1}{\alpha}}\right) \le C\|\langle t\rangle f(t)\|_{\mathbf{X}}\, g^{-1}(t)\, t^{-\frac{1}{\alpha}}.$$

Likewise

$$\begin{aligned}
&\left\| |\cdot|^{1+\alpha} \int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{L}^\infty} \\
&\le C \int_0^{\sqrt{t}} d\tau \sup_{\tau>0} \|f(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} \\
&+ C \int_0^{\sqrt{t}} (t-\tau)^{1-\frac{\gamma}{\alpha}} \langle\tau\rangle^{\frac{\gamma}{\alpha}-1} d\tau \sup_{\tau>0} \langle\tau\rangle^{1-\frac{\gamma}{\alpha}} \|f(\tau)\|_{\mathbf{L}^{1,\gamma}} \\
&+ C g^{-1}(t) \int_{\sqrt{t}}^t d\tau \sup_{\tau>0} \|f(\tau)\|_{\mathbf{L}^{\infty,\alpha+1}} \\
&+ C g^{-1}(t) \int_{\sqrt{t}}^t (t-\tau)^{1-\frac{\gamma}{\alpha}} \langle\tau\rangle^{\frac{\gamma}{\alpha}-1} d\tau \sup_{\tau>0} \langle\tau\rangle^{1-\frac{\gamma}{\alpha}} \|f(\tau)\|_{\mathbf{L}^{1,\gamma}} \\
&\le C \|\langle t\rangle f(t)\|_{\mathbf{X}} \left(t^{\frac{1}{2}} + t^{1-\frac{\gamma}{2\alpha}} + t g^{-1}(t) \right) \\
&\le C t g^{-1}(t) \|\langle t\rangle f(t)\|_{\mathbf{X}}
\end{aligned}$$

for all $t > 4$. Thus the triad $(\mathbf{X}, Y, \mathcal{G})$ is concordant. By the proof of Theorem 3.11 we see that the nonlinearity is nonconvective critical, such that conditions (3.2) and (3.4) are fulfilled.

We have for $\mathcal{K}(v) = \mathcal{N}(v) - \frac{v}{\theta} f(\mathcal{N}(v))$

$$\begin{aligned}
&\langle t\rangle^{1+\frac{1}{\alpha}} \|\mathcal{K}(v(t)) - \mathcal{K}(w(t))\|_{\mathbf{L}^\infty} \\
&\le C \left(\|v-w\|_{\mathbf{X}}\right) \left(\|v\|_{\mathbf{X}}^\alpha + \|w\|_{\mathbf{X}}^\alpha\right) \\
&+ \frac{1}{\theta} \left(\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}\right) \sup_{t>0} \langle t\rangle \|\mathcal{N}(v(t)) - \mathcal{N}(w(t))\|_{\mathbf{L}^1} \\
&+ \frac{1}{\theta} \|v-w\|_{\mathbf{X}} \sup_{t>0} \langle t\rangle \left(\|\mathcal{N}(v(t))\|_{\mathbf{L}^1} + \|\mathcal{N}(w(t))\|_{\mathbf{L}^1}\right) \\
&\le C \|v-w\|_{\mathbf{X}} \left(\|v\|_{\mathbf{X}}^\alpha + \|w\|_{\mathbf{X}}^\alpha\right) \left(1 + \frac{1}{\theta}\left(\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}\right)\right)
\end{aligned}$$

and in the same manner

$$\begin{aligned}
&\|\mathcal{K}(v(t)) - \mathcal{K}(w(t))\|_{\mathbf{L}^{\infty,\alpha+1}} \\
&\le C \|v-w\|_{\mathbf{X}} \left(\|v\|_{\mathbf{X}}^\alpha + \|w\|_{\mathbf{X}}^\alpha\right) \left(1 + \frac{1}{\theta}\left(\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}\right)\right)
\end{aligned}$$

for all $t > 0$. This yields the estimate

$$\begin{aligned}
&\|\mathcal{K}(v) - \mathcal{K}(w)\|_{\mathbf{Y}} \\
&\le C \|v-w\|_{\mathbf{X}} \left(\|v\|_{\mathbf{X}}^{\frac{\rho}{n}} + \|w\|_{\mathbf{X}}^{\frac{\rho}{n}}\right) \left(1 + \frac{1}{\theta}\|v\|_{\mathbf{X}} + \frac{1}{\theta}\|w\|_{\mathbf{X}}\right),
\end{aligned}$$

where $\|w\|_{\mathbf{Y}} = \|\langle t\rangle w\|_{\mathbf{X}}$. Therefore since the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ is concordant we see that condition (3.5) is fulfilled. Now applying Theorem 3.2 we obtain a unique solution $u \in \mathbf{C}\left([0,\infty);\mathbf{L}^{\infty,\alpha+1}(\mathbf{R})\cap\mathbf{C}(\mathbf{R})\right)$ to the Cauchy problem (3.40) satisfying estimates

$$\left\|\left\langle t^{-\frac{1}{\alpha}}(\cdot)\right\rangle^{\alpha+1}\left(u(t)-\theta t^{-\frac{1}{\alpha}}G\left(t^{-\frac{1}{\alpha}}(\cdot)\right)g^{-\frac{1}{\alpha}}\right)\right\|_{\mathbf{L}^{\infty}} \leq C\varepsilon^{1+\alpha}\langle t\rangle^{-\frac{1}{\alpha}}g^{-1-\frac{1}{\alpha}}(t)\log g(t). \tag{3.46}$$

This completes the proof of Theorem 3.16.

3.3.2 Large initial data

When the order of derivative $\alpha \in (0,2]$, we can prove global existence of solutions to the Cauchy problem (3.40) without any restriction on the size of the initial data.

Theorem 3.18. *Let $0<\alpha\leq 2$. Suppose that the initial data $u_0>0$ in $\mathbf{R}$ and $u_0\in\mathbf{L}^{\infty,\alpha+1}(\mathbf{R})\cap\mathbf{C}(\mathbf{R})$. Then there exists a unique solution*

$$u \in \mathbf{C}\left([0,\infty);\mathbf{L}^{\infty,\alpha+1}(\mathbf{R})\cap\mathbf{C}(\mathbf{R})\right)$$

to the Cauchy problem for equation (3.40). Moreover the asymptotics (3.42) is true.

Proof of Theorem 3.18

Using the result of Theorem 3.16 as in the proof of estimates (3.33) and (3.34) we have

$$\|u(t)\|_{\mathbf{L}^{\infty}} \leq C\varepsilon^{-1}\langle t\rangle^{-\frac{1}{\alpha}}(\log(2+t))^{-\frac{1}{\alpha}} \tag{3.47}$$

and

$$\|u(t)\|_{\mathbf{L}^{\infty,1+\alpha}} \leq C\varepsilon^{-1}\langle t\rangle(\log(2+t))^{-\frac{1}{\alpha}} \tag{3.48}$$

for all $t>0$. Applying the method of the proof of estimate (3.36) because of estimates (3.47), (3.48), Gronwall's inequality and Lemma 1.40 we get the estimates

$$\|v(t)\|_{\mathbf{L}^{\infty,1+\alpha}} \leq C\langle t\rangle,\ \ \|v(t)\|_{\mathbf{L}^{\infty}} \leq C\langle t\rangle^{-\frac{1}{\alpha}}$$

for all $t>0$. Then as in the proof of Theorem 3.16 we find the estimate

$$\left\|\left\langle t^{-\frac{1}{\alpha}}(\cdot)\right\rangle^{\alpha+1}\left(u(t)-\theta t^{-\frac{1}{\alpha}}G\left(t^{-\frac{1}{\alpha}}(\cdot)\right)g^{-\frac{1}{\alpha}}(t)\right)\right\|_{\mathbf{L}^{\infty}} \leq Ct^{-\frac{1}{\alpha}}(\log t)^{-1-\frac{1}{\alpha}}\log\log t.$$

This completes the proof of Theorem 3.18.

3.3.3 Nonlinear heat equation

In the case of the nonlinear heat equation $\alpha = 2$ we put the initial time at $t = 1$ for convenience.

Theorem 3.19. *Suppose that the initial data $u_1(x) \geq 0$ in $\mathbf{R}$, and are such that $u_1(x)e^{\frac{x^2}{4}}, (\partial_x u_1(x))e^{\frac{x^2}{4}} \in \mathbf{L}^\infty(\mathbf{R}) \cap \mathbf{C}(\mathbf{R})$. Then there exists a unique solution $u \in \mathbf{C}([1,\infty) \times \mathbf{R})$ to the Cauchy problem (3.40). Moreover there exists a function $A \in \mathbf{L}^\infty$, such that the asymptotics*

$$\begin{aligned} u(t,x) &= t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} A\left(\frac{x}{2t}\right)\left(1 + O\left((\log t)^{-1}\right)\right) \\ &\times \left(1 + \frac{1}{\sqrt{3}}\left(A^2\left(\frac{x}{2t}\right) + O\left((\log t)^{-1}\right)\right)\int_1^t \sqrt{\frac{3t}{3t-2\tau}} e^{-\frac{\tau x^2}{2t(3t-2\tau)}} \frac{d\tau}{\tau}\right)^{-\frac{1}{2}} \end{aligned} \tag{3.49}$$

is true for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$.

Remark 3.20. Note that

$$\int_1^t \sqrt{\frac{3t}{3t-2\tau}} e^{-\frac{\tau}{3t-2\tau}\frac{x^2}{2t}} \frac{d\tau}{\tau} = \log\left(1 + \frac{t^2}{t+x^2}\right) + O(1)$$

for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$. In other words in the short-range region $|x| << t$ the asymptotics (3.49) describes a logarithmic correction comparing with the linear case. In the long-range region $|x| \geq t$ the asymptotics (3.49) has a quasi-linear character.

Proof of Theorem 3.19

As in paper Hayashi and Ozawa [1988] we change the dependent and independent variables $u(t,x) = t^{-\frac{1}{2}} e^{-t\xi^2} v(t,\xi)$, and $\xi = \frac{x}{2t}$. Then from (3.40) we obtain

$$\begin{cases} v_t - \frac{1}{4t^2} v_{\xi\xi} + \frac{1}{t} e^{-2t\xi^2} v^3(t,\xi) = 0, \ t > 1, \ \xi \in \mathbf{R}, \\ v(1,\xi) = v_1(\xi), \ \xi \in \mathbf{R}, \end{cases} \tag{3.50}$$

where $v_1(\xi) = e^{\xi^2} u_1(2\xi)$. Then changing the dependent variable $v = we^{-g}$ we get

$$\begin{aligned} &w_t - \frac{1}{4t^2} w_{\xi\xi} + \frac{1}{t} e^{-2t\xi^2} e^{-2g} w^3 - w g_t \\ &+ \frac{1}{2t^2} g_\xi w_\xi - \frac{1}{4t^2}(g_\xi)^2 w + \frac{1}{4t^2} g_{\xi\xi} w = 0, \end{aligned}$$

We now choose a function g to satisfy the equation

$$\begin{cases} g_t = \frac{1}{t} e^{-2t\xi^2} e^{-2g} w^2 + \frac{1}{2t^2}(g_\xi)^2 + \frac{1}{4t^2} g_{\xi\xi}, \ t > 1, \ \xi \in \mathbf{R}, \\ g(1,\xi) = 0, \ \xi \in \mathbf{R}, \end{cases} \tag{3.51}$$

then we obtain

$$\begin{cases} w_t = \frac{3}{4t^2} (g_\xi)^2 w - \frac{1}{2t^2} g_\xi w_\xi + \frac{1}{4t^2} w_{\xi\xi},\ t > 1,\ \xi \in \mathbf{R}, \\ w(1,\xi) = v_1(\xi),\ \xi \in \mathbf{R}. \end{cases} \tag{3.52}$$

We change $h = e^{2g}$ in (3.51), then

$$\begin{cases} h_t = \frac{2}{t} e^{-2t\xi^2} w^2 + \frac{1}{4t^2} h_{\xi\xi},\ t > 1,\ \xi \in \mathbf{R}, \\ h(1,\xi) = 1,\ \xi \in \mathbf{R}. \end{cases} \tag{3.53}$$

Proposition 3.21. *Let the initial data $v_1, \partial_x v_1 \in \mathbf{L}^\infty$ be small such that*

$$\|v_1\|_{\mathbf{L}^\infty} + \|\partial_x v_1\|_{\mathbf{L}^\infty} \le \varepsilon.$$

Also we suppose that

$$\theta = \inf_{|\xi| \le 1} \int_{\mathbf{R}} e^{-(\xi-\eta)^2} v_1(\eta)\, d\eta \ge C\varepsilon > 0.$$

Then there exists a unique solution $v \in \mathbf{C}([1,\infty) \times \mathbf{R})$ to the Cauchy problem (3.50). Moreover, there exists a function $A \in \mathbf{L}^\infty$, such that the asymptotics

$$w(t,\xi) = A(\xi) + O\left(\frac{\varepsilon^2}{Q(t)}\right) \tag{3.54}$$

and

$$h(t,\xi) = 1 + \frac{2}{\sqrt{3}} \left(A^2(\xi) + O\left(\frac{\varepsilon^3}{Q(t)}\right)\right) \int_1^t \sqrt{\frac{3t}{3t-2\tau}} e^{-\frac{2t\tau}{3t-2\tau}\xi^2} \frac{d\tau}{\tau} \tag{3.55}$$

are true for $t \to \infty$ uniformly with respect to $\xi \in \mathbf{R}$, where $Q(t) = 1 + \varepsilon^2 \log(1+t)$.

Proof. Denote the Green function

$$G(t,\tau,\xi) = \frac{1}{\sqrt{\pi}} \sqrt{\frac{t\tau}{t-\tau}} e^{-\frac{t\tau}{t-\tau}\xi^2},$$

and

$$V(t,\xi) = \int_{\mathbf{R}} G(t,1,\xi-\eta)\, v_1(\eta)\, d\eta,$$

then the Cauchy problems (3.51) and (3.52) can be written in the form of the integral equation

$$g(t,\xi) = \int_1^t \frac{d\tau}{\tau} \int_{\mathbf{R}} G(t,\tau,\xi-\eta) \left(e^{-2\tau\eta^2} h^{-1}(\tau,\eta)\, w^2(\tau,\eta)\right) d\eta \tag{3.56}$$

and

$$w(t,\xi) = V(t,\xi) + \int_1^t \frac{d\tau}{4\tau^2} \int_{\mathbf{R}} G(t,\tau,\xi-\eta) F(\tau,\eta)\, d\eta, \tag{3.57}$$

where

$$F(\tau,\eta) = 3 (g_\eta(\tau,\eta))^2 w(\tau,\eta) - 2 g_\eta(\tau,\eta) w_\eta(\tau,\eta).$$

We prove the following estimates

$$\|w_\xi(t)\|_{\mathbf{L}^\infty} + \|g_\xi(t)\|_{\mathbf{L}^\infty} < \varepsilon + \frac{\varepsilon^2 \sqrt{t}}{Q(t)} \tag{3.58}$$

and

$$\|w(t)\|_{\mathbf{L}^\infty} < C\varepsilon, \ \left\| e^{-2t\xi^2} h^{-1}(t,\xi) \right\|_{\mathbf{L}^\infty} < \frac{1}{Q(t)} \tag{3.59}$$

for all $t \geq 1$. Note that estimates (3.58) and (3.59) are valid for $t = 1$. On the contrary we suppose that (3.58) and (3.59) are violated at some time $t = T$, then by the continuity with respect to $t \geq 1$ we have estimates

$$\|w_\xi(t)\|_{\mathbf{L}^\infty} + \|g_\xi(t)\|_{\mathbf{L}^\infty} \leq \varepsilon + \frac{C\varepsilon^2 \sqrt{t}}{Q(t)}, \tag{3.60}$$

and

$$\|w(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon, \ \left\| e^{-2t\xi^2} h^{-1}(t,\xi) \right\|_{\mathbf{L}^\infty} \leq \frac{C}{Q(t)} \tag{3.61}$$

for all $t \in [1,T]$. First we estimate the derivative $\partial_\xi V(t,\xi)$

$$\|\partial_\xi V(t)\|_{\mathbf{L}^\infty} = \frac{\sqrt{t}}{\sqrt{\pi(t-1)}} \left\| \partial_\xi \int_{\mathbf{R}} e^{-\frac{t}{t-1}\eta^2} v_1(\xi-\eta)\, d\eta \right\|_{\mathbf{L}^\infty}$$
$$\leq \|v_1'\|_{\mathbf{L}^\infty} < \varepsilon.$$

To prove estimate (3.58) we differentiate equations (3.56) and (3.57) with respect to ξ

$$\begin{aligned} &|g_\xi(t,\xi)| + |w_\xi(t,\xi)| \leq |V_\xi(t,\xi)| \\ &+ \int_1^t d\tau \int_{\mathbf{R}} |\partial_\xi G(t,\tau,\xi-\eta)| \left(\frac{1}{\tau} e^{-2\tau\eta^2} h^{-1}(\tau,\eta) w^2(\tau,\eta) \right. \\ &\left. + \frac{1}{2\tau^2} g_\eta(\tau,\eta) |w_\eta(\tau,\eta)| + \frac{3}{4\tau^2} |w(\tau,\eta)| (g_\eta(\tau,\eta))^2 \right) d\eta. \end{aligned} \tag{3.62}$$

Via (3.60) and (3.61) we obtain

$$\begin{aligned} &\left\| \frac{1}{\tau} e^{-2\tau\eta^2} h^{-1}(\tau,\eta) w^2(\tau,\eta) \right. \\ &\left. + \frac{1}{2\tau^2} g_\eta(\tau,\eta) |w_\eta(\tau,\eta)| + \frac{3}{4\tau^2} |w(\tau,\eta)| (g_\eta(\tau,\eta))^2 \right\|_{\mathbf{L}^\infty} \\ &\leq \frac{C\varepsilon^2}{\tau Q(\tau)} + \frac{1}{\tau^2} \left(\varepsilon + \frac{C\varepsilon^2 \sqrt{t}}{Q(t)} \right)^2 \leq \frac{C\varepsilon^2}{\tau Q(\tau)} + C\varepsilon^2 \tau^{-2} \leq \frac{C\varepsilon^2}{\tau Q(\tau)}; \end{aligned}$$

therefore,

$$\left\| \int_{\mathbf{R}} |\partial_\xi G(t,\tau,\xi-\eta)| \left(\frac{1}{\tau} e^{-2\tau\eta^2} h^{-1}(\tau,\eta) w^2(\tau,\eta) \right.\right.$$
$$\left.\left. + \frac{1}{2\tau^2} g_\eta(\tau,\eta) |w_\eta(\tau,\eta)| + \frac{3}{4\tau^2} |w(\tau,\eta)| (g_\eta(\tau,\eta))^2 \right) d\eta \right\|_{\mathbf{L}^\infty}$$
$$\leq \frac{C\varepsilon^2}{\sqrt{\tau} Q(\tau)}.$$

Then from (3.62) we have

$$\|w_\xi(t)\|_{\mathbf{L}^\infty} + \|g_\xi(t)\|_{\mathbf{L}^\infty} < \varepsilon + C\varepsilon^2 \int_1^t \frac{d\tau}{\sqrt{\tau} Q(\tau)}$$
$$= \varepsilon + C\frac{\varepsilon^2\sqrt{t}}{Q(t)} + \varepsilon^2 \int_1^t \frac{Q'(\tau)\sqrt{\tau} d\tau}{Q(\tau)^2} < \varepsilon + \frac{\varepsilon^2\sqrt{t}}{Q(t)}$$

for all $t \in [1,T]$.

Now we consider the function $V(t)$

$$V(t,\xi) = \frac{\sqrt{t}}{\sqrt{\pi(t-1)}} \int_{\mathbf{R}} e^{-\frac{t}{t-1}(\xi-\eta)^2} v_1(\eta) d\eta$$
$$= \frac{1}{\sqrt{\pi}} \int_{\mathbf{R}} e^{-(\xi-\eta)^2} v_1(\eta) d\eta$$
$$+ \frac{1}{\sqrt{\pi}} \left(\sqrt{\frac{t}{t-1}} - 1 \right) \int_{\mathbf{R}} e^{-\frac{t}{t-1}(\xi-\eta)^2} v_1(\eta) d\eta$$
$$+ \frac{1}{\sqrt{\pi}} \int_{\mathbf{R}} e^{-(\xi-\eta)^2} \left(e^{-\frac{1}{t-1}(\xi-\eta)^2} - 1 \right) v_1(\eta) d\eta$$
$$= \frac{1}{\sqrt{\pi}} \int_{\mathbf{R}} e^{-(\xi-\eta)^2} v_1(\eta) d\eta + O\left(t^{-1} \|v_1\|_{\mathbf{L}^\infty}\right)$$

for $t > 2$, and

$$V(t,\xi) = \frac{\sqrt{t}}{\sqrt{\pi(t-1)}} \int_{\mathbf{R}} e^{-\frac{t}{t-1}(\xi-\eta)^2} v_1(\eta) d\eta = O(\|v_1\|_{\mathbf{L}^\infty})$$

for $t \in [1,2]$. In view of (3.60) we have

$$\left| \int_{\frac{t}{2}}^t \frac{d\tau}{\tau^2} \int_{\mathbf{R}} d\eta G(t,\tau,\xi-\eta) F(\tau,\eta) \right|$$
$$\leq C\varepsilon^2 \int_{\frac{t}{2}}^t \frac{d\tau}{\tau^2} + C\varepsilon^4 \int_{\frac{t}{2}}^t \frac{d\tau}{\tau\left(1+\varepsilon^2 \log(1+\tau)\right)^2} \leq \frac{C\varepsilon^2}{Q(t)}.$$

Denoting

$$G(\infty,\tau,\xi)=\frac{1}{\sqrt{\pi}}\sqrt{\tau}e^{-\tau\xi^2},$$

and

$$B(\xi)=\int_1^\infty\frac{d\tau}{4\tau^2}\int_{\mathbf{R}}d\eta G(\infty,\tau,\xi-\eta)F(\tau,\eta),$$

we get

$$\begin{aligned}&\int_1^{\frac{t}{2}}\frac{d\tau}{4\tau^2}\int_{\mathbf{R}}d\eta G(t,\tau,\xi-\eta)F(\tau,\eta)\\&=B(\xi)-\int_{\frac{t}{2}}^\infty\frac{d\tau}{4\tau^2}\int_{\mathbf{R}}d\eta G(\infty,\tau,\xi-\eta)F(\tau,\eta)\\&+\int_1^{\frac{t}{2}}\frac{d\tau}{4\tau^2}\int_{\mathbf{R}}d\eta\left(G(t,\tau,\xi-\eta)-G(\infty,\tau,\xi-\eta)\right)F(\tau,\eta)\\&=B(\xi)+O\left(\frac{\varepsilon^2}{Q(t)}\right),\end{aligned}$$

since

$$\begin{aligned}&\int_{\frac{t}{2}}^\infty\frac{d\tau}{\tau^2}\int_{\mathbf{R}}d\eta G(\infty,\tau,\xi-\eta)F(\tau,\eta)\\&\le C\varepsilon^2\int_{\frac{t}{2}}^\infty\frac{d\tau}{\tau^2}+C\varepsilon^4\int_{\frac{t}{2}}^\infty\frac{d\tau}{\tau\left(1+\varepsilon^2\log(1+\tau)\right)^2}\le\frac{C\varepsilon^2}{Q(t)}\end{aligned}$$

and

$$\begin{aligned}&\int_1^{\frac{t}{2}}\frac{d\tau}{\tau^2}\int_{\mathbf{R}}d\eta\left(G(t,\tau,\xi-\eta)-G(\infty,\tau,\xi-\eta)\right)F(\tau,\eta)\\&=\frac{1}{\sqrt{\pi}}\int_1^{\frac{t}{2}}\tau^{-\frac{3}{2}}d\tau\int_{\mathbf{R}}d\eta\left(\sqrt{\frac{t}{t-\tau}}-1\right)e^{-\frac{t\tau}{t-\tau}(\xi-\eta)^2}F(\tau,\eta)\\&+\frac{1}{\sqrt{\pi}}\int_1^{\frac{t}{2}}\tau^{-\frac{3}{2}}d\tau\int_{\mathbf{R}}d\eta\left(e^{-\frac{\tau^2}{t-\tau}(\xi-\eta)^2}-1\right)e^{-\tau(\xi-\eta)^2}F(\tau,\eta)\\&\le\frac{C\varepsilon^2}{t}\int_1^{\frac{t}{2}}\frac{d\tau}{\tau}+\frac{C\varepsilon^4}{t}\int_1^{\frac{t}{2}}\frac{d\tau}{\left(1+\varepsilon^2\log(1+\tau)\right)^2}\le\frac{C\varepsilon^2}{Q(t)}.\end{aligned}$$

Note also that

$$B(\xi)\le C\varepsilon^2$$

for all $\xi\in\mathbf{R}$. Denote a function

$$A(\xi)=\int_{\mathbf{R}}e^{-(\xi-\eta)^2}v_1(\eta)\,d\eta+B(\xi),$$

then we see that $A(\xi)\ge C\varepsilon>0$ for $|\xi|<1$ and

$$w(t,\xi) = V(t,\xi) + \int_1^t \frac{d\tau}{4\tau^2}\int_{\mathbf{R}} d\eta G(t,\tau,\xi-\eta) F(\tau,\eta)$$
$$= A(\xi) + O\left(\frac{\varepsilon^2}{Q(t)}\right). \tag{3.63}$$

Now we write the Cauchy problem (3.53) for $h(t,\xi)$ in the form of an integral equation

$$h(t,\xi) = 1 + \frac{2}{\sqrt{3}} A^2(\xi)\int_1^t \sqrt{\frac{3t}{3t-2\tau}} e^{-\frac{2t\tau}{3t-2\tau}\xi^2}\frac{d\tau}{\tau} + 2R_1,$$

where

$$R_1 = \int_1^t \frac{d\tau}{\tau}\int_{\mathbf{R}} d\eta G(t,\tau,\xi-\eta) e^{-2\tau\eta^2} w^2(\tau,\eta)$$
$$- \frac{1}{\sqrt{3}} A^2(\xi)\int_1^t \sqrt{\frac{3t}{3t-2\tau}} e^{-\frac{2t\tau}{3t-2\tau}\xi^2}\frac{d\tau}{\tau}.$$

Using the identity

$$\frac{t\tau}{t-\tau}(\xi-\eta)^2 + 2\tau\eta^2 = \frac{\tau(3t-2\tau)}{t-\tau}\left(\eta - \frac{\xi}{3-2\frac{\tau}{t}}\right)^2 + \frac{2\tau t}{3t-2\tau}\xi^2;$$

we represent the integral R_1 in the following manner:

$$R_1 = \frac{1}{\sqrt{3}}\int_1^t \sqrt{\frac{t}{3t-2\tau}} e^{-\frac{2\tau t}{3t-2\tau}\xi^2}\left(w^2(\tau,\xi) - A^2(\xi)\right)\frac{d\tau}{\tau}$$
$$+ \frac{1}{\sqrt{3}}\int_1^t \sqrt{\frac{t}{3t-2\tau}} e^{-\frac{2\tau t}{3t-2\tau}\xi^2}\left(w^2\left(\tau,\frac{\xi t}{3t-2\tau}\right) - w^2(\tau,\xi)\right)\frac{d\tau}{\tau}$$
$$+ \frac{1}{\sqrt{\pi}}\int_1^t \frac{d\tau}{\sqrt{\tau}}\sqrt{\frac{t}{t-\tau}} e^{-\frac{2\tau t}{3t-2\tau}\xi^2}$$
$$\times \int_{\mathbf{R}} e^{-\frac{\tau(3t-2\tau)}{t-\tau}\left(\eta-\frac{\xi t}{3t-2\tau}\right)^2}\left(w^2(\tau,\eta) - w^2\left(\tau,\frac{\xi t}{3t-2\tau}\right)\right) d\eta.$$

Note that in view of (3.60), (3.61) and (3.63) we have the estimate

$$|R_1| \le C\varepsilon^3 \int_1^t e^{-\frac{2}{3}\tau\xi^2}\frac{d\tau}{\tau Q(\tau)}$$
$$= C\varepsilon^3\left(e^{-\frac{2}{3}t\xi^2}\log t\frac{1}{Q(t)} - \int_1^t \log\tau d e^{-\frac{2}{3}\tau\xi^2}\frac{1}{Q(\tau)}\right)$$
$$= C\varepsilon^3\left(e^{-\frac{2}{3}t\xi^2}\frac{\log t}{Q(t)} - \int_1^t \log\tau\left(-\frac{2}{3}\xi^2 e^{-\frac{2}{3}\tau\xi^2}\frac{1}{Q(\tau)} - e^{-\frac{2}{3}\tau\xi^2}\frac{Q'(\tau)}{Q(\tau)^2}\right)d\tau\right)$$
$$\le \frac{C\varepsilon^3}{Q(t)}\int_1^t e^{-\frac{2}{3}\tau\xi^2}\frac{d\tau}{\tau}$$

for $t \to \infty$, uniformly with respect to $\xi \in \mathbf{R}$. Thus we obtain

$$h(t,\xi) = 1 + \frac{2}{\sqrt{3}} \left(A^2(\xi) + O\left(\frac{\varepsilon^3}{Q(t)} \right) \right) \int_1^t \sqrt{\frac{3t}{3t-2\tau}} e^{-\frac{2t\tau}{3t-2\tau}\xi^2} \frac{d\tau}{\tau}. \quad (3.64)$$

By virtue of (3.64) the estimate (3.59) follows

$$\begin{aligned} \left| e^{-2t\xi^2} h^{-1}(t,\xi) \right| &\leq C e^{-2t\xi^2} \left(1 + A^2(\xi) \int_1^t e^{-\frac{2}{3}\tau\xi^2} \frac{d\tau}{\tau} \right)^{-1} \\ &\leq C e^{-2t\xi^2} \left(1 + \varepsilon^2 \log\left(1 + \frac{t}{\langle t\xi^2 \rangle} \right) \right)^{-1} < \frac{C}{Q(t)} \end{aligned}$$

for all $t \in [1,T]$. The contradiction obtained proves estimates (3.58), (3.59) for all $t \geq 1$. Formulas (3.63) and (3.64) yield the asymptotics (3.54) and (3.55). Proposition 3.21 is proved.

We take sufficiently small $\varepsilon > 0$ and consider the following two auxiliary Cauchy problems

$$\begin{cases} U_t - \frac{1}{4t^2} U_{\xi\xi} + \frac{1}{t} e^{-2t\xi^2} U^3(t,\xi) = 0, \ \xi \in \mathbf{R}, \ t > 1, \\ U(1,\xi) = \varepsilon e^{\xi^2} u_1(2\xi), \xi \in \mathbf{R} \end{cases} \quad (3.65)$$

and

$$\begin{cases} V_t - \frac{1}{4t^2} V_{\xi\xi} + \frac{1}{t} \varepsilon^2 e^{-2t\xi^2} V^3(t,\xi) = 0, \ \xi \in \mathbf{R}, \ t > 1, \\ V(1,\xi) = e^{\xi^2} u_1(2\xi), \xi \in \mathbf{R}. \end{cases} \quad (3.66)$$

Note that problem (3.66) can be reduced to problem (3.65) by changing $V = \varepsilon^{-1} U$. In addition the problem (3.65) has a sufficiently small initial data. Therefore for the functions U and V the large time asymptotics are known by virtue of Proposition 3.21. Then by Lemma 3.14 we get

$$U(t,\xi) \leq v(t,\xi) \leq V(t,\xi)$$

for all $t > 0$ and $\xi \in \mathbf{R}$. In particular, using the estimates (3.58) and (3.59) of Proposition 3.21 we have for $v = wh^{-\frac{1}{2}}$

$$\frac{\varepsilon C}{\sqrt{Q(t)}} \leq \left\| e^{-\frac{1}{2}t\xi^2} v(t,\xi) \right\|_{\mathbf{L}^\infty} \leq \frac{C}{\sqrt{Q(t)}} \quad (3.67)$$

for all $t > 1$, $\xi \in \mathbf{R}$, where $Q(t) = 1 + \varepsilon^2 \log(1+t)$. We rewrite (3.50) in the form of the integral equation

$$v(t,\xi) = V(t,\xi) + \int_1^t \frac{d\tau}{\tau} \int_{\mathbf{R}} G(t,\tau,\xi-\eta) e^{-2\tau\eta^2} v^3(\tau,\eta) d\eta$$

and estimate the derivative

$$\|v_\xi(t,\xi)\|_{\mathbf{L}^\infty} \le C + \int_1^t \frac{d\tau}{\tau} \|\partial_\xi G(t,\tau,\xi)\|_{\mathbf{L}^1} \left\| e^{-t\xi^2} v(t,\xi) \right\|_{\mathbf{L}^\infty}^3$$
$$\le C + \int_1^t \frac{d\tau}{\sqrt{\tau}} \sqrt{\frac{t}{t-\tau}} Q^{-\frac{3}{2}}(\tau) \le C\sqrt{t} Q^{-\frac{3}{2}}(t)$$

for all $t \ge 1$. Now applying the maximum principle to equation (3.51) we get for the function $m(t) = \max_{\xi \in \mathbf{R}} |g_\xi(t,\xi)|$

$$\frac{d}{dt} m \le \frac{C}{\sqrt{t}} Q^{-1}(t);$$

hence integrating with respect to time we obtain

$$\|g_\xi(t)\|_{\mathbf{L}^\infty} \le \frac{C\sqrt{t}}{Q(t)}$$

for all $t \ge 1$. After that we apply the maximum principle to equation (3.52), we get for $n(t) = \max_{\xi \in \mathbf{R}} |w(t,\xi)|$

$$\frac{d}{dt} n \le \frac{C}{t} Q^{-2}(t)\, n.$$

Then integrating with respect to time we obtain

$$\|w(t)\|_{\mathbf{L}^\infty} \le C$$

for all $t \ge 1$. We apply the maximum principle to equation (3.53) and get for $k(t) = \max_{\xi \in \mathbf{R}} h(t,\xi)$

$$\frac{d}{dt} k \le \frac{C}{t} e^{-t\xi^2};$$

integrating with respect to time yields

$$\|h(t)\|_{\mathbf{L}^\infty} \le C \int_1^t \frac{d\tau}{\tau} e^{-\tau\xi^2} \le CQ(t)$$

for all $t \ge 1$. Now by the identity

$$w_\xi = v_\xi \sqrt{h} - w g_\xi$$

we have the estimate

$$\|w_\xi(t)\|_{\mathbf{L}^\infty} \le \frac{C\sqrt{t}}{Q(t)}$$

for all $t \ge 1$. Consider two equations

$$h_t = \frac{2}{t} e^{-2t\xi^2} v^2 h + \frac{1}{4t^2} h_{\xi\xi},$$

and

$$H_t = \frac{2}{t} e^{-2t\xi^2} U^2 H + \frac{1}{4t^2} H_{\xi\xi},$$

where $v(t,\xi) \geq U(t,\xi)$ for all $\xi \in \mathbf{R}$, $t > 0$. Define

$$W(t,\xi) = U(t,\xi)\sqrt{H(t,\xi)},$$

which gives us a decomposition $U(t,\xi) = \frac{W(t,\xi)}{\sqrt{H(t,\xi)}}$ similar to $v(t,\xi) = \frac{w(t,\xi)}{\sqrt{h(t,\xi)}}$. By a standard comparison principle for the heat equation we get

$$h(t,\xi) \geq H(t,\xi) \geq \frac{1}{2}\left(1 + A^2(\xi)\log\left(1 + \frac{t}{\langle t\xi^2\rangle}\right)\right)$$

for all $\xi \in \mathbf{R}$, $t > 0$. Therefore we obtain the estimate

$$\begin{aligned} w(t,\xi) &\geq v(t,\xi)\sqrt{h(t,\xi)} \geq U(t,\xi)\sqrt{H(t,\xi)} \\ &= W(t,\xi) \geq \frac{1}{2}\lim_{t\to\infty} W(t,\xi) > 0. \end{aligned}$$

As in (3.63) we state that there exists a function $A(\xi) \in \mathbf{L}^\infty$ such that

$$w(t,\xi) = A(\xi) + O\left(Q^{-1}(t)\right)$$

and

$$\lim_{t\to\infty} w(t,\xi) = A(\xi) > 0.$$

Using these estimates we have the asymptotics as in Proposition 3.21. Theorem 3.19 is proved.

3.4 Complex Landau-Ginzburg equation

In this section we consider the Cauchy problem for the complex Landau-Ginzburg equation

$$\begin{cases} \partial_t u - \alpha \Delta u + \beta |u|^{\frac{2}{n}} u = 0, & x \in \mathbf{R}^n,\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{3.68}$$

where $\alpha, \beta \in \mathbf{C}$. We are interested in the dissipative case $\operatorname{Re}\alpha > 0$. Suppose that $\operatorname{Re}\delta(\alpha,\beta) \geq 0$, $\theta = \left|\int_{\mathbf{R}^n} u_0(x)\,dx\right| \neq 0$, where

$$\delta(\alpha,\beta) = \frac{\beta|\alpha|^{n-1} n^{\frac{n}{2}}}{((n+1)|\alpha|^2 + \alpha^2)^{\frac{n}{2}}}.$$

Our purpose is to study the global existence of small solutions and the large time asymptotics of solutions to the Cauchy problem (3.68). Below we show that the nonlinearity in equation (3.68) is critical from the point of view of the large time asymptotic behavior of solutions. The nonlinearity of equation

(3.68) does not have enough regularity to get smooth solutions in the higher order Sobolev spaces. Here we are working in the Lebesgue spaces and so by using smoothing properties of the linear evolution group we consider the problem under the assumptions of rather small regularity on the initial data $u_0 \in \mathbf{L}^r(\mathbf{R}^n)$, $r > 1$. Also to obtain the estimates of the remainder terms in the large time asymptotic formulas we have to assume that the initial data satisfy the decay condition at infinity, such that $u_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n)$ with some $a \in (0,1)$.

We define $\theta = \left|\int_{\mathbf{R}^n} u_0(x)\,dx\right|$, $\eta = \frac{\theta^{\frac{2}{n}}}{2\pi n} \operatorname{Re}\delta(\alpha,\beta)$,

$$\mu = \frac{\theta^{\frac{4}{n}}}{(4\pi)^2} (\operatorname{Im}\delta(\alpha,\beta))^2 \operatorname{Re}\left(\left(1+\frac{1}{n}\right)\nu_1 - \frac{1}{n}\nu_2\right),$$

$$\omega = \frac{\theta^2}{4\pi} \operatorname{Im}\delta(\alpha,\beta),$$

$$\nu_l = \log\frac{1}{1-\kappa_l} + \sum_{j=2}^{m} \frac{1}{j-1}\left((1-\kappa_l)^{-j+1} - 1\right), \tag{3.69}$$

for $n = 2m$, and

$$\nu_l = 2\log\frac{2}{1+\sqrt{1-\kappa_l}} + \sum_{j=1}^{m} \frac{2}{2j-1}\left((1-\kappa_l)^{\frac{1}{2}-j} - 1\right)$$

for $n = 2m+1$, $l = 1,2$,

$$\kappa_1 = \frac{(\overline{\alpha}+\alpha)^2}{((n+1)\overline{\alpha}+\alpha)^2},$$

$$\kappa_2 = \frac{2|\alpha|^2 + (n+1)\overline{\alpha}^2 + (1-n)\alpha^2}{|(n+1)\alpha + \overline{\alpha}|^2}.$$

The sums like $\sum_{j=2}^{m}$ in the case of $m = 1$ we assume to be absent.

We prove the following result.

Theorem 3.22. *Assume that the initial data* $u_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^r(\mathbf{R}^n)$, *where* $r > 1$, $a \in (0,1)$, *have sufficiently small norm* $\epsilon = \|u_0\|_{\mathbf{L}^{1,a}} + \|u_0\|_{\mathbf{L}^r}$ *and are such that* $\theta = \left|\int_{\mathbf{R}^n} u_0(x)\,dx\right| \geq C\varepsilon$. *Then there exists a unique solution* $u(t,x) \in \mathbf{C}\left([0,\infty);\mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^r(\mathbf{R}^n)\right) \cap \mathbf{C}((0,\infty);\mathbf{L}^\infty(\mathbf{R}^n))$ *of the Cauchy problem (3.68), satisfying the following time decay estimates for all* $t > 0$:

$$\|u(t)\|_\infty \leq C\varepsilon t^{-\frac{n}{2}} (1 + \eta\log\langle t\rangle)^{-\frac{n}{2}},$$

if $\eta > 0$,

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon t^{-\frac{n}{2}} (1 + \mu\log\langle t\rangle)^{-\frac{n}{4}},$$

if $\eta = 0$, $\mu > 0$, *finally*

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\varepsilon t^{-\frac{n}{2}} \left(1 + \kappa \log \langle t \rangle\right)^{-\frac{n}{6}},$$

if $\eta = 0$, $\mu = 0$, $\kappa > 0$, *where* κ *is a positive constant defined explicitly below in (3.96). Furthermore the following asymptotic formulas of the solution are valid*

$$u(t,x) = \frac{\theta}{(4\pi\alpha t)^{\frac{n}{2}} (1 + \eta \log t)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4\alpha t} - \frac{i\omega}{\eta} \log\log t + i \arg \widehat{u}_0(0)\right) + O\left(\frac{\varepsilon^{5/2}}{(t\log t)^{\frac{n}{2}} \log t}\right),$$

if $\eta > 0$,

$$u(t,x) = \frac{\theta}{(4\pi\alpha t)^{\frac{n}{2}} (1 + \mu \log t)^{\frac{n}{4}}} \exp\left(-\frac{|x|^2}{4\alpha t} - \frac{2i\omega}{\sqrt{\mu}} \sqrt{\log t} + i \arg \widehat{u}_0(0)\right) + O\left(\frac{\varepsilon^3}{t^{\frac{n}{2}} (\log t)^{\frac{n}{4}} (\log t)^{\frac{1}{2}}}\right),$$

if $\eta = 0$, $\mu > 0$, *and finally*

$$u(t,x) = \frac{\theta}{(4\pi\alpha t)^{\frac{n}{2}} (1 + \kappa \log t)^{\frac{n}{6}}} \exp\left(-\frac{|x|^2}{4\alpha t} - \frac{3i\omega}{2\sqrt[3]{\kappa}} \sqrt[3]{\log^2 t} + i \arg \widehat{u}_0(0)\right) + O\left(\frac{\varepsilon^3}{t^{\frac{n}{2}} (\log t)^{\frac{n}{6}} (\log t)^{\frac{1}{3}}}\right),$$

if $\eta = 0$, $\mu = 0$, $\kappa > 0$.

Remark 3.23. In order to check the condition $\mu > 0$, *that is to examine the inequality* $\operatorname{Re}\left(\left(1 + \frac{1}{n}\right)\nu_1 - \frac{1}{n}\nu_2\right) > 0$ *for the values of* α *such that* $\operatorname{Re}\alpha > 0$, *we substitute* $\alpha = e^{i\phi}$, $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *into the functions* κ_1 *and* κ_2. *Then*

$$\kappa_1 = \left(\frac{1 + e^{-2i\phi}}{1 + (n+1)e^{-2i\phi}}\right)^2, \quad \kappa_2 = \frac{2 + (n+1)e^{-2i\phi} + (1-n)e^{2i\phi}}{|n+1+e^{-2i\phi}|^2}.$$

Hence

$$f_n(\phi) = \operatorname{Re}\left(\left(1 + \frac{1}{n}\right)\nu_1 - \frac{1}{n}\nu_2\right) = \log \frac{|1-\kappa_2|^{\frac{1}{n}}}{|1-\kappa_1|^{1+\frac{1}{n}}} - \sum_{j=2}^{m} \frac{1}{j-1} + \sum_{j=2}^{m} \frac{1}{j-1} \operatorname{Re}\left(\left(1 + \frac{1}{n}\right)(1-\kappa_1)^{1-j} - \frac{1}{n}(1-\kappa_2)^{1-j}\right),$$

if $n = 2m$ *and*

$$f_n(\phi) = \mathrm{Re}\left(\left(1+\frac{1}{n}\right)\nu_1 - \frac{1}{n}\nu_2\right) = 2\log\frac{2\left|1+\sqrt{1-\kappa_2}\right|^{\frac{1}{n}}}{\left|1+\sqrt{1-\kappa_1}\right|^{1+\frac{1}{n}}} - \sum_{j=1}^{m}\frac{2}{2j-1}$$
$$+\sum_{j=1}^{m}\frac{2}{2j-1}\mathrm{Re}\left(\left(1+\frac{1}{n}\right)(1-\kappa_1)^{\frac{1}{2}-j} - \frac{1}{n}(1-\kappa_2)^{\frac{1}{2}-j}\right)$$

for the case $n = 2m+1$. We have plotted the functions $f_n(\phi)$ by using the Maple program, taking the dimensions $n = 0, 1, 2, ..., 10$. The qualitative behavior of the function $f_n(\phi)$ appears to be the same for any spacial dimension n: the functions $f_n(\phi)$ are odd, $f_n(\phi_n) = 0 = f_n\left(\frac{\pi}{2}\right)$. Moreover the function $f_n(\phi)$ is positive in $(0, \phi_n)$ and negative in $\left(\phi_n, \frac{\pi}{2}\right)$, where the roots ϕ_n are close to 1.

3.4.1 Preliminaries

The Green operator $\mathcal{G}$ of the problem (3.68) is given by

$$\mathcal{G}(t)\phi = \overline{\mathcal{F}}_{\xi\to x}e^{-\alpha\xi^2 t}\hat{\phi}(\xi) = \int_{\mathbf{R}^n} G(t, x-y)\,\phi(y)dy,$$

and $G(t,x) = (4\pi\alpha t)^{-\frac{n}{2}}e^{-|x|^2/4\alpha t}$. Denote $\vartheta = (2\pi)^{\frac{n}{2}}\hat{\phi}(0)$.

By Lemma 1.28 we get

Lemma 3.24. *The following estimates are true, provided that the right-hand sides are bounded:*

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^{p,b}} \le C\{t\}^{\frac{n}{2r}\left(1-\frac{1}{p}\right)}\langle t\rangle^{\frac{b}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)}\left(\|\phi\|_{\mathbf{L}^{1,b}} + \|\phi\|_{\mathbf{L}^{r,b}}\right)$$

and

$$\left\||\cdot|^b\left(\mathcal{G}(t)\phi - \vartheta G(t)\right)\right\|_{\mathbf{L}^p} \le C\left(\{t\}^{\frac{n}{2}\left(1-\frac{1}{r}\right)\left(1-\frac{1}{p}\right)} + \{t\}^{\frac{b}{2}}\right)\langle t\rangle^{\frac{b-a}{2}}t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}\|\phi\|_{\mathbf{L}^{1,a}}$$

for all $t > 0$, where $1 \le p \le \infty$, $b \in [0, a]$, $0 < a < 1$, and $r > 1$.

Denote $g(t) = 1 + \zeta\log\langle t\rangle$ with some $\zeta > 0$ and

$$\|f\|_{\mathbf{F}} = \sup_{t>0}\sup_{1\le s\le p}\{t\}^{-\rho}t^{1-\frac{\lambda}{2}+\frac{n}{2}\left(1-\frac{1}{s}\right)}\left\||\cdot|^\lambda f(t)\right\|_{\mathbf{L}^s},$$

where $1 \le p \le \infty$, $\rho \ge 0$, $\lambda > 0$. Then using Lemma 3.24 we have

Lemma 3.25. *Let the function $f(t,x)$ have the zero mean value $\hat{f}(t,0) = 0$ and the norm $\|f\|_{\mathbf{F}}$ be bounded. Then the following inequality is valid*

$$\left\||\cdot|^b\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)\,f(\tau)\,d\tau\right\|_{\mathbf{L}^p} \le Cg^{-1}(t)\{t\}^\rho t^{\frac{b}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)}\|f\|_{\mathbf{F}},$$

for all $t > 0$, $0 \le b \le \lambda$.

3.4.2 Proof of Theorem 3.22 in the case of $\mathrm{Re}\,\delta(\alpha,\beta)>0$

To apply Theorem 3.2 we choose the space

$$\mathbf{Z}=\left\{\phi\in\mathbf{L}^{r}\left(\mathbf{R}^{n}\right)\cap\mathbf{L}^{1,a}\left(\mathbf{R}^{n}\right)\right\}$$

with $a\in(0,1)$ and $r>1$ and the space

$$\mathbf{X}=\left\{\phi\in\mathbf{C}\left((0,\infty);\mathbf{L}^{1,\lambda}\left(\mathbf{R}^{n}\right)\cap\mathbf{W}_{\infty}^{0,\lambda}\left(\mathbf{R}^{n}\right)\right):\|\phi\|_{\mathbf{X}}<\infty\right\}$$

where the norm

$$\|\phi\|_{\mathbf{X}}=\sup_{t>0}\sup_{1\leq p\leq\infty}\sup_{b\in[0,\lambda]}\{t\}^{\rho\left(1-\frac{1}{p}\right)}t^{\frac{n}{2}\left(1-\frac{1}{p}\right)}\langle t\rangle^{-\frac{b}{2}}g\left(t\right)\|r\left(t\right)\|_{\mathbf{L}^{p,b}}.$$

Denote the Green operator $\mathcal{G}\left(t\right)$

$$\mathcal{G}\left(t\right)\phi=\int_{\mathbf{R}^{n}}G\left(t,x-y\right)\phi\left(y\right)dy,$$

where the kernel $G\left(t,x\right)$ is

$$G\left(t,x\right)=\left(4\pi\alpha t\right)^{-\frac{n}{2}}e^{-\frac{|x|^{2}}{4\alpha t}}.$$

Using Lemma 3.24 we clearly see that according to Definition 2.1 the function

$$G_{0}\left(t,x\right)=\left(4\pi\left(t+1\right)\right)^{-\frac{n}{2}}e^{-\frac{|x|^{2}}{4(t+1)}}$$

is the asymptotic kernel for operator $\mathcal{G}$ in spaces $\mathbf{X},\mathbf{Z}$ with continuous linear functional

$$f(\phi)=\int_{\mathbf{R}^{n}}\phi dx$$

and $\gamma=\frac{a}{2}$. Define $\|\phi\|_{\mathbf{Y}}=\|\langle t\rangle\,\phi\left(t\right)\|_{\mathbf{X}}$. Using Lemma 3.25 we easily see that the triad $(\mathbf{X},\mathbf{Y},\mathcal{G})$ is concordant.

By an easy computation we obtain

$$\begin{aligned}&\theta^{1+\frac{2}{n}}\,\mathrm{Re}\int_{\mathbf{R}^{n}}\beta\left|G_{0}\right|^{\frac{2}{n}}G_{0}\left(t,x\right)dx\\&=\frac{\theta^{1+\frac{2}{n}}}{4\pi t}\,\mathrm{Re}\left(\frac{\beta}{\left(4\pi\alpha t\right)^{\frac{n}{2}}\left|\alpha\right|}\int_{\mathbf{R}^{n}}e^{-\frac{|x|^{2}}{4tn}\left(\frac{1}{\alpha}+\frac{1}{\overline{\alpha}}\right)-\frac{|x|^{2}}{4t\alpha}}dx\right)\\&=\frac{\theta^{1+\frac{2}{n}}}{4\pi(t+1)}\,\mathrm{Re}\left(\frac{\beta\left|\alpha\right|^{n-1}n^{\frac{n}{2}}}{\left(\left(n+1\right)\left|\alpha\right|^{2}+\alpha^{2}\right)^{\frac{n}{2}}}\right)=\frac{\theta^{1+\frac{2}{n}}}{4\pi(t+1)}\,\mathrm{Re}\,\delta\left(\alpha,\beta\right).\end{aligned}$$

Therefore we have

$$\int_0^t \operatorname{Re} f\left(\mathcal{N}\left(\theta G_0(\tau)\right)\right) d\tau = \int_0^t \frac{\theta^{1+\frac{2}{n}}}{4\pi(\tau+1)} \operatorname{Re} \delta(\alpha,\beta)\, d\tau$$
$$= \eta \theta^{\frac{2}{n}} \log(1+t),$$

where $\eta = \frac{\theta}{2\pi n} \operatorname{Re} \delta(\alpha, \beta)$. Thus nonlinearity $\mathcal{N} = |u|^{\frac{2}{n}} u$ is critical. Also it is evident that

$$e^z \mathcal{N}\left(ue^{-z}\right) = e^z \left|ue^{-z}\right|^{\sigma} ue^{-z} = e^{-\sigma z} \mathcal{N}(u);$$

hence conditions (3.2), (3.3) and (3.6) are fulfilled. Since

$$\log(2+t) \left\|\mathcal{N}(v(t)) - \mathcal{N}(w(t))\right\|_{\mathbf{L}^1}$$
$$\leq C \log(2+t) \left(\|v(t)\|_{\mathbf{L}^\infty}^{\frac{2}{n}} + \|w(t)\|_{\mathbf{L}^\infty}^{\frac{2}{n}}\right) \|v(t) - w(t)\|_{\mathbf{L}^1}$$
$$\leq C \{t\}^{-\frac{1}{r}} \langle t \rangle^{-1} \left\|\log(2+t)(v-w)\right\|_{\mathbf{X}} \left(\|v\|_{\mathbf{X}}^{\frac{2}{n}} + \|w\|_{\mathbf{X}}^{\frac{2}{n}}\right),$$

condition (3.4) is true. Also we have

$$\|\mathcal{K}(v) - \mathcal{K}(w)\|_{\mathbf{Y}} \leq \|\mathcal{N}(v) - \mathcal{N}(w)\|_{\mathbf{Y}}$$
$$+ \frac{1}{\theta}\left(\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}\right) \sup_{t>0} \{t\}^{\frac{1}{r}} \langle t \rangle \left\|\mathcal{N}(v(t)) - \mathcal{N}(w(t))\right\|_{\mathbf{L}^1}$$
$$+ \frac{1}{\theta} \|v - w\|_{\mathbf{X}} \sup_{t>0} \{t\}^{\frac{1}{r}} \langle t \rangle \left(\|\mathcal{N}(v(t))\|_{\mathbf{L}^1} + \|\mathcal{N}(w(t))\|_{\mathbf{L}^1}\right)$$
$$\leq C \|v - w\|_{\mathbf{X}} \left(\|v\|_{\mathbf{X}}^{\frac{2}{n}} + \|w\|_{\mathbf{X}}^{\frac{2}{n}}\right) \left(1 + \frac{1}{\theta}\|v\|_{\mathbf{X}} + \frac{1}{\theta}\|w\|_{\mathbf{X}}\right),$$

where $\|\phi\|_{\mathbf{Y}} = \|\langle t \rangle \phi\|_{\mathbf{X}}$ and

$$\mathcal{K}(v(\tau)) = \mathcal{N}(v(\tau)) - \frac{v(\tau)}{\theta} f\left(\mathcal{N}(v(\tau))\right).$$

Since the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ is concordant we see that condition (3.5) is fulfilled. Now applying Theorem 3.2 we easily get the results of Theorem 3.22 in the case of $\eta > 0$.

3.4.3 Proof of Theorem 3.22 in the case of $\eta = 0, \mu > 0$

In this subsection we consider the case $\eta = 0, \mu > 0$. We need to modify the proof of Theorem 3.2. Note that system(3.7) for problem (3.68) has the form

$$\begin{cases} v_t - \alpha \Delta v + \beta e^{-\frac{2}{n}\varphi} \left(|v|^{\frac{2}{n}} - \frac{1}{\theta} \int_{\mathbf{R}^n} |v|^{\frac{2}{n}} v dx\right) v = 0, \\ \varphi' = \frac{1}{\theta} e^{-\frac{2}{n}\varphi} \left(\operatorname{Re} \int_{\mathbf{R}^n} \beta |v|^{\frac{2}{n}} v dx\right), \\ v(0,x) = v_0(x), \quad \varphi(0) = 0, \end{cases} \tag{3.70}$$

where $v_0(x) = u(0,x)\exp\left(-i\arg\int_{\mathbf{R}^n} u(0,x)\,dx\right)$. Multiplying the second equation of system (3.70) by the factor $g(t) = e^{\frac{4}{n}\varphi(t)}$ and then integrating with respect to time $t > 0$, we obtain

$$g(t) = 1 + \frac{4}{n\theta}\int_0^t \sqrt{g(\tau)}\,\mathrm{Re}\int_{\mathbf{R}^n} \beta\,|v|^{\frac{2}{n}}\,v(\tau,x)\,dxd\tau.$$

Therefore we have the system of integral equations

$$\begin{cases} v(t) = \mathcal{G}(t)\,v_0 - \beta\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)\,f_1(\tau)\,d\tau \\ g(t) = 1 + \frac{4}{n\theta}\int_0^t \sqrt{g(\tau)}\,\mathrm{Re}\int_{\mathbf{R}^n}\beta\,|v|^{\frac{2}{n}}\,v(\tau,x)\,dxd\tau, \end{cases} \tag{3.71}$$

where

$$f_1(\tau) = \sqrt{g(\tau)}\left(|v|^{\frac{2}{n}}\,v(\tau) - \frac{v(\tau)}{\theta}\int_{\mathbf{R}^n}|v|^{\frac{2}{n}}\,v(\tau,x)\,dx\right).$$

We find a solution v of system (3.71) in the neighborhood of the second approximation of the perturbation theory $\Phi(t) + \Psi(t)$, where $\Phi(t) = \mathcal{G}(t)\,v_0$ and

$$\Psi(t) = -\beta\int_0^t \mathcal{G}(t-\tau)\left(|\Phi(\tau)|^{\frac{2}{n}}\,\Phi(\tau) - \frac{\Phi(\tau)}{\theta}\int_{\mathbf{R}^n}|\Phi(\tau)|^{\frac{2}{n}}\,\Phi(\tau)\,dx\right)\frac{d\tau}{g^{\frac{1}{2}}(\tau)}.$$

We put $r = v - \Phi - \Psi$, then we get

$$\begin{cases} r(t) = -\beta\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)\,f(\tau)\,d\tau, \\ g(t) = 1 + \frac{4}{n\theta}\int_0^t \sqrt{g(\tau)}\,\mathrm{Re}\int_{\mathbf{R}^n}\beta\,|v|^{\frac{2}{n}}\,v(\tau,x)\,dxd\tau, \end{cases} \tag{3.72}$$

where

$$f(\tau) = \sqrt{g(\tau)}\left(|v|^{\frac{2}{n}}\,v(\tau) - |\Phi(\tau)|^{\frac{2}{n}}\,\Phi(\tau)\right.$$
$$\left. - \frac{v(\tau)}{\theta}\int_{\mathbf{R}^n}|v|^{\frac{2}{n}}\,v(\tau,x)\,dx + \frac{\Phi(\tau)}{\theta}\int_{\mathbf{R}^n}|\Phi(\tau)|^{\frac{n}{2}}\,\Phi(\tau)\,dx\right).$$

We define the mappings $\mathcal{M}(r,g)$ and $\mathcal{R}(r,g)$ by

$$\mathcal{M}(r,g) = -\beta\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)\,f(\tau)\,d\tau,$$
$$\mathcal{R}(r,g) = 1 + \frac{4}{n\theta}\int_0^t \sqrt{g(\tau)}\,\mathrm{Re}\int_{\mathbf{R}^n}\beta\,|v|^{\frac{2}{n}}\,v(\tau,x)\,dxd\tau.$$

To show the existence of solutions to (3.72), we prove that the transformation $(\mathcal{M}(r,g), \mathcal{R}(r,g))$ is the contraction mapping in the following set

$$\mathbf{X} = \{ r \in \mathbf{C} \left((0,\infty) ; \mathbf{W}_1^{0,\lambda} (\mathbf{R}^n) \cap \mathbf{W}_\infty^{0,\lambda} (\mathbf{R}^n) \right), g \in \mathbf{C} (0,\infty) :$$
$$\sup_{t>0} \sup_{1\le p\le\infty} \sup_{b\in[0,\lambda]} \{t\}^{\rho\left(1-\frac{1}{p}\right)} t^{\frac{n}{2}\left(1-\frac{1}{p}\right)} \langle t\rangle^{-\frac{b}{2}} g(t) \| \langle\cdot\rangle^b r(t) \|_{\mathbf{L}^p} \le C\varepsilon^{q+\frac{2}{n}};$$
$$\frac{1}{2}(1+\mu\log\langle t\rangle) \le g(t) \le 2(1+\mu\log\langle t\rangle),$$
$$|g'(t)| \le C\varepsilon^{\frac{4}{n}} t^{-1} \text{ for all } t>0 \},$$

where $\rho = \frac{1}{2}\left(1-\frac{1}{r}\right) > 0, \lambda \in (0,a)$, and

$$\mu = \frac{\theta^{\frac{4}{n}} (\operatorname{Im}\delta)^2}{(4\pi)^2} \operatorname{Re}\left(\left(1+\frac{1}{n}\right)\nu_1 - \frac{1}{n}\nu_2\right) > 0.$$

When $(r,g) \in \mathbf{X}$ we get

$$\sup_{t>0} \sup_{1\le p\le\infty} \{t\}^{-\rho\left(1-\frac{1}{p}\right)} t^{1-\frac{b}{2}+\frac{n}{2}\left(1-\frac{1}{p}\right)} \left\| |\cdot|^b f(t) \right\|_{\mathbf{L}^p} \le C\varepsilon^q,$$

where $b \in [0,\lambda]$. We define Ψ as a solution to the Cauchy problem

$$\begin{aligned} \partial_t\Psi - \alpha\Delta\Psi + \beta e^{-2\varphi}\left(|\Phi|^{\frac{2}{n}} - \frac{1}{\theta}\int_{\mathbf{R}^n} |\Phi|^{\frac{2}{n}} \Phi dx\right)\Phi = 0, \\ \Psi(0,x) = 0, \end{aligned}$$

where $\Phi(t) = \mathcal{G}(t) v_0.$.

We use the following result.

Lemma 3.26. *We assume that* $v_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^r(\mathbf{R}^n)$, *the norm* $\|v_0\|_{\mathbf{L}^r} + \|v_0\|_{\mathbf{L}^{1,a}} = \varepsilon$ *is sufficiently small,* $\hat{v}_0(0) = |\hat{u}_0(0)| = \theta(2\pi)^{-\frac{n}{2}} \ge C\varepsilon > 0$, $\eta = 0$, $\mu > 0$. *Let the function* $v(t,x)$ *satisfy the estimates*

$$\|v(t)\|_{\mathbf{L}^p} \le \varepsilon\{t\}^\rho t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

and

$$\|v(t) - \Phi(t) - \Psi(t)\|_{\mathbf{L}^p} \le \varepsilon^q g^{-1}(t) t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

for all $t>0$, $q \le p \le \infty$, *where* $\rho > 0$. *We also assume that the function* $g(t)$ *is such that*

$$\frac{1}{2}(1+\mu\log\langle t\rangle) \le g(t) \le 2(1+\mu\log\langle t\rangle)$$

and $|g'(t)| \le C\varepsilon^{\frac{4}{n}} t^{-1}$ *for all* $t>0$.

Then the following inequalities are valid

$$\frac{1}{\theta}\left|\sqrt{g(t)} \operatorname{Re}\int_{\mathbf{R}^n} \beta|v|^{\frac{2}{n}} v(t,x)\,dx\right| \le C\varepsilon^{\frac{2}{n}} t^{-1}\{t\}^\rho \tag{3.73}$$

and

$$\frac{9}{10}(1+\mu\log\langle t\rangle) \le 1+\frac{4}{n\theta}\int_0^t d\tau\sqrt{g(\tau)}\,\mathrm{Re}\int_{\mathbf{R}^n}\beta|v|^{\frac{2}{n}}v(\tau,x)\,dx$$
$$\le \frac{11}{10}(1+\mu\log\langle t\rangle) \tag{3.74}$$

for all $t>0$.

Applying Lemma 3.25 and Lemma 3.26 in the same way as in the proof of Theorem 3.2 we can show that the mapping $(\mathcal{M}(r,g),\mathcal{R}(r,g))$ is the contraction mapping from the set $\mathbf{X}$ into itself and as above we get the asymptotic formula for the solution for the case $\eta=0$, $\mu>0$.

Now we turn to the proof of Lemma 3.26.

Proof. We need only to consider the case $t\ge 1$. First let us prove the representation

$$\Psi(t)=\frac{\theta^q\delta}{4\pi\sqrt{g(t)}}\Gamma(t)+R(t), \tag{3.75}$$

where

$$\Gamma(t)=-(4\pi\alpha t)^{-\frac{n}{2}}\int_0^1\left(\frac{1}{(1-\sigma z)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4\alpha t(1-\sigma z)}}-e^{-\frac{|x|^2}{4\alpha t}}\right)\frac{dz}{z}$$

with $\sigma=\frac{\alpha+\overline{\alpha}}{\alpha+(n+1)\overline{\alpha}}$. We prove that the remainder term R can be estimated as follows

$$\left\||\cdot|^b R(t)\right\|_{\mathbf{L}^p}\le C\varepsilon^q t^{\frac{b}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)}g^{-1}(t)$$

for all $t>0$, $0\le b\le\lambda$, $1\le p\le\infty$, where $0<\lambda<\min(a,\rho)$. We have

$$\Psi=-\beta\theta^q\int_0^t g^{-\frac{1}{2}}(\tau)\,\mathcal{G}(t-\tau)\left(|\Phi|^{\frac{2}{n}}\Phi(\tau)-\frac{\Phi(\tau)}{\theta}\int_{\mathbf{R}^n}|\Phi|^{\frac{2}{n}}\Phi(\tau,x)\,dx\right)d\tau$$
$$=-\beta\theta^q\int_0^t g^{-\frac{1}{2}}(\tau)\,\mathcal{G}(t-\tau)\left(|G|^{\frac{2}{n}}G(\tau)-G(\tau)\int_{\mathbf{R}^n}|G|^{\frac{2}{n}}G(\tau,x)\,dx\right)d\tau$$
$$+R_1(t),$$

where

$$R_1(t)=\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)f(\tau)\,d\tau$$

and

$$f(\tau)=-\beta\sqrt{g(\tau)}\left(|\Phi|^{\frac{2}{n}}\Phi(\tau)-\frac{\Phi(\tau)}{\theta}\int_{\mathbf{R}^n}|\Phi|^{\frac{2}{n}}\Phi(\tau,x)\,dx\right.$$
$$\left.-\theta^q|G|^{\frac{2}{n}}G(\tau)+G(\tau)\theta^q\int_{\mathbf{R}^n}|G|^{\frac{2}{n}}G(\tau,x)\,dx\right).$$

Via Lemma 3.24 we see that $f(\tau)$ satisfies the estimate

$$\left\| |\cdot|^{\lambda} f(\tau) \right\|_{\mathbf{L}^p} \leq C\varepsilon^q \tau^{\frac{\lambda}{2}-1-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

for all $\tau > 0$ and $1 \leq p \leq \infty$, where $0 < \lambda < \min(a, \rho)$. Therefore by virtue of Lemma 3.25 we get

$$\left\| |\cdot|^{b} R_1(t) \right\|_{\mathbf{L}^p} \leq C\varepsilon^q t^{\frac{b}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)} g^{-1}(t)$$

for all $t > 0$, $0 \leq b \leq \lambda$, $1 \leq p \leq \infty$. Then integrating by parts with respect to time t we obtain

$$\begin{aligned} \Psi(t) = &-\frac{\beta\theta^q}{g^{\frac{1}{2}}(t)} \int_0^t \mathcal{G}(t-\tau) \left(|G|^{\frac{2}{n}} G(\tau) - G(\tau) \int_{\mathbf{R}^n} |G|^{\frac{2}{n}} G(\tau, x)\, dx \right) d\tau \\ &+ R(t), \end{aligned}$$

where we denote $R(t) = R_1(t) + R_2(t)$ and

$$\begin{aligned} R_2(t) = C\theta^q \int_0^t \frac{d\tau g'(\tau)}{g^{\frac{3}{2}}(\tau)} \int_0^{\tau} \mathcal{G}(t-\zeta) \\ \times \left(|G|^{\frac{2}{n}} G(\zeta) - G(\zeta) \int_{\mathbf{R}^n} |G|^{\frac{2}{n}} G(\zeta, x)\, dx \right) d\zeta. \end{aligned}$$

Then using the conditions on the function $g(t)$, estimates of Lemma 3.24, we obtain

$$\begin{aligned} &\left\| |\cdot|^{b} R_2(t) \right\|_{\mathbf{L}^p} \leq C\varepsilon^q \int_0^{t/2} g^{-\frac{3}{2}}(\tau)\, \tau^{-1} d\tau \int_0^{\tau} (t-\zeta)^{\frac{b-a}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)} \\ &\times \left\| |x|^{a} \left(|G|^{\frac{2}{n}} G(\zeta) - G(\zeta) \int_{\mathbf{R}^n} |G|^{\frac{2}{n}} G(\zeta, x)\, dx \right) \right\|_{\mathbf{L}^1} d\zeta \\ &+ C\varepsilon^q \int_{t/2}^{t} g^{-\frac{3}{2}}(\tau)\, \tau^{-1} d\tau \int_0^{\tau/2} (t-\zeta)^{\frac{b-a}{2}} \\ &\times \left\| |x|^{a} \left(|G|^{\frac{2}{n}} G(\zeta) - G(\zeta) \int_{\mathbf{R}^n} |G|^{\frac{2}{n}} G(\zeta, x)\, dx \right) \right\|_{\mathbf{L}^1} d\zeta \\ &+ C\varepsilon^q \int_{t/2}^{t} g^{-\frac{3}{2}}(\tau)\, \tau^{-1} d\tau \int_{\tau/2}^{\tau} (t-\zeta)^{\frac{b-a}{2}} \\ &\times \left\| |x|^{a} \left(|G|^{\frac{2}{n}} G(\zeta) - G(\zeta) \int_{\mathbf{R}^n} |G|^{\frac{2}{n}} G(\zeta, x)\, dx \right) \right\|_{\mathbf{L}^p} d\zeta; \end{aligned}$$

hence

$$\left\| |\cdot|^b R_2(t) \right\|_{\mathbf{L}^p} \leq C\varepsilon^q \int_0^{\sqrt{t}} (t-\tau)^{\frac{b-a}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)} \tau^{\frac{a}{2}-1} d\tau$$
$$+ C\varepsilon^q g^{-\frac{3}{2}}(t) \int_{\sqrt{t}}^{t/2} (t-\tau)^{\frac{b-a}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)} \tau^{\frac{a}{2}-1} d\tau$$
$$+ C\varepsilon^q g^{-\frac{3}{2}}(t) \int_{t/2}^{t} (t-\tau)^{\frac{b-a}{2}} \tau^{\frac{a}{2}-1-\frac{n}{2}\left(1-\frac{1}{p}\right)} d\tau$$
$$\leq C\varepsilon^q t^{\frac{b}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)} \left(t^{-\frac{a}{4}} + g^{-\frac{3}{2}}(t)\right) \leq C\varepsilon^q g^{-\frac{3}{2}}(t)\, t^{\frac{b}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$, $0 \leq b \leq \lambda$, $1 \leq p \leq \infty$. We have by a direct calculation

$$|G|^{\frac{2}{n}} G - G \int_{\mathbf{R}^n} |G|^{\frac{2}{n}} G(\tau, x)\, dx$$
$$= \frac{1}{(4\pi\tau)^{\frac{n}{2}+1} \alpha^{\frac{n}{2}} |\alpha|} \left(\exp\left(-\frac{|x|^2}{4\tau n} \left(\frac{n+1}{\alpha} + \frac{1}{\overline{\alpha}} \right) \right) \right.$$
$$\left. - \frac{n^{\frac{n}{2}}}{\left((n+1) + \frac{\alpha}{\overline{\alpha}}\right)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4\alpha\tau} \right) \right).$$

Using the formula

$$\mathcal{G}(t-\tau)\, e^{-b|x|^2} = (1 + 4\alpha b (t-\tau))^{-\frac{n}{2}} \exp\left(-\frac{b|x|^2}{1 + 4\alpha b (t-\tau)} \right)$$

we then obtain

$$\beta \mathcal{G}(t-\tau) \left(|G|^{\frac{2}{n}} G - G \int_{\mathbf{R}^n} |G|^{\frac{2}{n}} G(\tau, x)\, dx \right)$$
$$= \frac{\beta}{(4\pi\tau)^{\frac{n}{2}+1} \alpha^{\frac{n}{2}} |\alpha|} \left(\mathcal{G}(t-\tau) \exp\left(-\frac{|x|^2}{4\tau n} \left(\frac{n+1}{\alpha} + \frac{1}{\overline{\alpha}} \right) \right) \right.$$
$$\left. - \frac{n^{\frac{n}{2}}}{\left(n+1+\frac{\alpha}{\overline{\alpha}}\right)^{\frac{n}{2}}} \mathcal{G}(t-\tau) \exp\left(-\frac{|x|^2}{4\alpha\tau} \right) \right)$$
$$= \frac{\delta}{(4\pi)^{\frac{n}{2}+1} \tau \alpha^{\frac{n}{2}}} \left(\frac{1}{(t-\sigma\tau)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha(t-\sigma\tau)}} - \frac{1}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t}} \right),$$

where $\sigma = \frac{\alpha+\overline{\alpha}}{\alpha+(n+1)\overline{\alpha}}$. This implies

$$\Psi(t) = -\frac{\theta^q \delta}{(4\pi)^{\frac{n}{2}+1} \alpha^{\frac{n}{2}} \sqrt{g(t)}} \int_0^t \left(\frac{1}{(t-\sigma\tau)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha(t-\sigma\tau)}} - \frac{1}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t}} \right) \frac{d\tau}{\tau}$$
$$+ R(t),$$

so (3.75) follows. Denote $r = v - \Phi - \Psi$, then by (3.75) and the Taylor formula we get

$$\begin{aligned}
&\left\| |v|^{\frac{2}{n}} v - |\Phi|^{\frac{2}{n}} \Phi - \left(1+\frac{1}{n}\right) |\Phi|^{\frac{2}{n}} \Psi - \frac{1}{n} \Phi \overline{\Phi}^{-1} |\Phi|^{\frac{2}{n}} \overline{\Psi} \right\|_{\mathbf{L}^1} \\
&\leq C \left\| \Phi \overline{\Phi}^{-2} |\Phi|^{\frac{2}{n}} \overline{\Psi}^2 \right\|_{\mathbf{L}^1} + C \left\| \Phi^{-1} |\Phi|^{\frac{2}{n}} \Psi^2 \right\|_{\mathbf{L}^1} + C \left\| \overline{\Phi}^{-1} |\Phi|^{\frac{2}{n}} \Psi \overline{\Psi} \right\|_{\mathbf{L}^1} \\
&+ C \left\| |\Phi|^{\frac{2}{n}} r \right\|_{\mathbf{L}^1} + C \left\| \Phi \overline{\Phi}^{-1} |\Phi|^{\frac{2}{n}} \overline{r} \right\|_{\mathbf{L}^1} \\
&\leq C \varepsilon^{q+\frac{2}{n}} t^{-1} g^{-1}(t) . \qquad (3.76)
\end{aligned}$$

We next prove (3.73) and (3.74). By (3.76), Lemma 3.24 and the fact that $\eta = 0$, that is

$$\operatorname{Re} \int_{\mathbf{R}^n} \beta |\Phi|^{\frac{2}{n}} \Phi(\tau, x) \, dx = O\left(\varepsilon^{q+\frac{2}{n}} t^{-1} g^{-1}(t)\right),$$

we have

$$\begin{aligned}
\operatorname{Re} \int_{\mathbf{R}^n} \beta |v|^{\frac{2}{n}} v(\tau, x) \, dx &= \left(1+\frac{1}{n}\right) \operatorname{Re} \int_{\mathbf{R}^n} \beta |\Phi|^{\frac{2}{n}} \Psi(\tau, x) \, dx \\
+ \frac{1}{n} \operatorname{Re} \int_{\mathbf{R}^n} \beta \Phi \overline{\Phi}^{-1} |\Phi|^{\frac{2}{n}} \overline{\Psi}(\tau, x) \, dx &+ O\left(\varepsilon^{q+\frac{2}{n}} t^{-1} g^{-1}(t)\right). \qquad (3.77)
\end{aligned}$$

We compute the right-hand side of (3.77). By Lemma 3.24 we have

$$\|\Phi(t) - \theta G(t)\|_{\mathbf{L}^q} \leq C \varepsilon t^{-\frac{1}{q} - \frac{a}{2}},$$

and by virtue of (3.75) we get

$$\begin{aligned}
&\int_{\mathbf{R}^n} |\Phi|^{\frac{2}{n}} \Psi(t, x) \, dx \\
&= -\frac{\theta^{q+\frac{2}{n}} \delta}{(4\pi)^2 (4\pi\alpha t)^{\frac{n}{2}} |\alpha| t} g^{-\frac{1}{2}}(t) \int_0^1 \frac{dz}{z} \int_{\mathbf{R}^n} \left(\frac{1}{(1-\sigma z)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\sigma z)} - \frac{|x|^2}{4tn}\left(\frac{1}{\alpha} + \frac{1}{\overline{\alpha}}\right)} \right. \\
&\left. - e^{-\frac{|x|^2}{4tn}\left(\frac{n+1}{\alpha} + \frac{1}{\overline{\alpha}}\right)} \right) dx + O\left(\varepsilon^{q+\frac{2}{n}} t^{-1} g^{-1}(t)\right).
\end{aligned}$$

We have by a simple calculation

$$\begin{aligned}
&\frac{1}{(4\pi\alpha t)^{\frac{n}{2}}} \left(\frac{1}{(1-\sigma z)^{\frac{n}{2}}} \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4\alpha t(1-\sigma z)} - \frac{|x|^2}{4tn}\left(\frac{1}{\alpha} + \frac{1}{\overline{\alpha}}\right)} dx - \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4tn}\left(\frac{n+1}{\alpha} + \frac{1}{\overline{\alpha}}\right)} dx \right) \\
&= \frac{\delta |\alpha|}{\beta} \left(\frac{1}{(1-\sigma^2 z)^{\frac{n}{2}}} - 1 \right).
\end{aligned}$$

Thus we get

$$\int_{\mathbf{R}^n} |\Phi|^{\frac{2}{n}} \Psi(t, x) \, dx = -\frac{\theta^{q+\frac{2}{n}} \delta^2}{(4\pi)^2 \beta t} g^{-\frac{1}{2}}(t) \nu_1 + O\left(\varepsilon^{q+\frac{2}{n}} t^{-1} g^{-1}(t)\right), \qquad (3.78)$$

where

$$\nu_1 = \int_0^1 \left(\frac{1}{(1-\kappa_1 z)^{\frac{n}{2}}} - 1 \right) \frac{dz}{z} = \int_0^{\kappa_1} \left(\frac{1}{(1-t)^{\frac{n}{2}}} - 1 \right) \frac{dt}{t}$$
$$= \int_{1-\kappa_1}^1 \left(\frac{1}{y^{\frac{n}{2}}} - 1 \right) \frac{dy}{1-y}.$$

Accordingly for $n = 2m$ we have

$$\nu_1 = \int_{1-\kappa_1}^1 \left(\frac{1}{y^m} - 1 \right) \frac{dy}{1-y} = \sum_{j=1}^m \int_{1-\kappa_1}^1 y^{-j} dy$$
$$= \log \frac{1}{1-\kappa_1} + \sum_{j=2}^m \frac{1}{j-1} \left((1-\kappa_1)^{1-j} - 1 \right),$$

and for the case $n = 2m + 1$

$$\nu_1 = \int_{1-\kappa_1}^1 \left(\frac{1}{y^{m+\frac{1}{2}}} - 1 \right) \frac{dy}{1-y}$$
$$= \sum_{j=1}^m \int_{1-\kappa_1}^1 y^{-j-\frac{1}{2}} dy + \int_{1-\kappa_1}^1 \frac{1}{\sqrt{y}\left(1+\sqrt{y}\right)} dy$$
$$= 2 \log \frac{2}{1+\sqrt{1-\kappa_1}} + \sum_{j=1}^m \frac{1}{j-\frac{1}{2}} \left((1-\kappa_1)^{\frac{1}{2}-j} - 1 \right),$$

where $\kappa_1 = \sigma^2 = \frac{(\overline{\alpha}+\alpha)^2}{((n+1)\overline{\alpha}+\alpha)^2}$. We next compute the term

$$\int_{\mathbf{R}^n} \Phi \overline{\Phi}^{-1} |\Phi|^{\frac{2}{n}} \overline{\Psi} (\tau, x)\, dx.$$

We have

$$\int_{\mathbf{R}^n} \Phi \overline{\Phi}^{-1} |\Phi|^{\frac{2}{n}} \overline{\Psi} (t, x)\, dx$$
$$= -\frac{\theta^{1+\frac{4}{n}} \overline{\delta}}{(4\pi)^{\frac{n}{2}+2} t\, |\alpha|\, (\alpha t)^{\frac{n}{2}}} g^{-\frac{1}{2}}(t) \int_{\mathbf{R}^n} dx \int_0^1 \frac{dz}{z} \left(\frac{1}{(1-\overline{\sigma} z)^{\frac{n}{2}}} \right.$$
$$\times \exp \left(-\frac{|x|^2}{4\overline{\alpha} t (1-\overline{\sigma} z)} - \frac{|x|^2}{4tn} \left(\frac{1+n}{\alpha} + \frac{1-n}{\overline{\alpha}} \right) \right)$$
$$\left. - \exp \left(-\frac{|x|^2}{4tn} \left(\frac{1+n}{\alpha} + \frac{1}{\overline{\alpha}} \right) \right) \right) + O\left(\varepsilon^{q+\frac{2}{n}} t^{-1} g^{-1}(t) \right)$$
$$= -\frac{\theta^{1+\frac{4}{n}} |\delta|^2}{(4\pi)^2 \beta t} g^{-\frac{1}{2}}(t)\, \nu_2 + O\left(\varepsilon^{q+\frac{2}{n}} t^{-1} g^{-1}(t) \right), \tag{3.79}$$

where $\nu_2 = \int_0^1 \left(\frac{1}{(1-\kappa_2 z)^{\frac{n}{2}}} - 1 \right) \frac{dz}{z}$. Therefore as above

$$\nu_2 = \log \frac{1}{1-\kappa_2} + \sum_{j=2}^{m} \frac{1}{j-1} \left((1-\kappa_2)^{1-j} - 1 \right)$$

for $n = 2m$, and

$$\nu_2 = 2 \log \frac{2}{1+\sqrt{1-\kappa_2}} + \sum_{j=1}^{m} \frac{1}{j-\frac{1}{2}} \left((1-\kappa_2)^{\frac{1}{2}-j} - 1 \right)$$

for $n = 2m+1$, where

$$\kappa_2 = \frac{(n+1)\,\overline{\alpha} + (1-n)\,\alpha}{(n+1)\,\overline{\alpha} + \alpha} \overline{\sigma} = \frac{2|\alpha|^2 + (n+1)\,\overline{\alpha}^2 + (1-n)\,\alpha^2}{|(n+1)\,\alpha + \overline{\alpha}|^2}.$$

Then by (3.77), (3.78) and (3.79) we have

$$\operatorname{Re} \int_{\mathbf{R}^n} \beta\, |v|^{\frac{2}{n}}\, v(\tau, x)\, dx = \frac{\theta\mu}{4t} g^{-\frac{1}{2}}(t) + O\left(\varepsilon^{q+\frac{2}{n}} t^{-1} g^{-1}(t) \right) \tag{3.80}$$

for all $t \geq 1$. Via (3.80) we get (3.73) and

$$\begin{aligned} &1 + \frac{4}{\theta} \int_0^t \sqrt{g(\tau)}\, \operatorname{Re} \int_{\mathbf{R}^n} \beta\, |v|^{\frac{2}{n}}\, v(\tau, x)\, dx d\tau \\ &= 1 + \mu \log \langle t \rangle + O\left(\varepsilon^{\frac{4}{n}} \sqrt{\log \langle t \rangle} \right) \end{aligned}$$

for all $t \geq 1$; hence, we obtain estimates (3.74), and Lemma 3.26 is proved.

3.4.4 Proof of Theorem 3.22 in the case of $\eta = 0$, $\mu = 0$, $\kappa > 0$

Multiplying the second equation of system (3.70) by the factor $g(t) = e^{\frac{6}{n}\varphi(t)}$ and then integrating with respect to time $t > 0$, we obtain

$$g(t) = 1 + \frac{6}{\theta} \int_0^t g^{\frac{2}{3}}(\tau)\, \operatorname{Re} \int_{\mathbf{R}^n} \beta\, |v|^{\frac{2}{n}}\, v(\tau, x)\, dx d\tau.$$

Therefore, we have the system of integral equations

$$\begin{cases} v(t) = \mathcal{G}(t)\, v_0 - \beta \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau)\, f_2(\tau)\, d\tau, \\ g(t) = 1 + \frac{6}{\theta} \int_0^t g^{\frac{2}{3}}(\tau)\, \operatorname{Re} \int_{\mathbf{R}^n} \beta\, |v|^{\frac{2}{n}}\, v(\tau, x)\, dx d\tau, \end{cases} \tag{3.81}$$

where

$$f_2(\tau) = g^{\frac{2}{3}}(\tau) \left(|v|^2\, v(\tau) - \frac{v(\tau)}{\theta} \int_{\mathbf{R}^n} |v|^{\frac{2}{n}}\, v(\tau, x)\, dx \right).$$

We find a solution v of system (3.81) in the neighborhood of the third approximation of the perturbation theory $\Phi(t)+\Psi(t)+V(t)$, where $\Phi(t)$ is defined as above,

$$\Psi(t) = -\beta\int_0^t g^{-\frac{1}{3}}(\tau)\mathcal{G}(t-\tau) \times\left(|\Phi(\tau)|^{\frac{2}{n}}\Phi(\tau) - \frac{\Phi(\tau)}{\theta}\int_{\mathbf{R}^n}|\Phi(\tau)|^{\frac{2}{n}}\Phi(\tau,x)\,dx\right)d\tau$$

and

$$V(t) = -\beta\int_0^t g^{-\frac{1}{3}}(\tau)\mathcal{G}(t-\tau)\left(\left(1+\frac{1}{n}\right)|\Phi|^{\frac{2}{n}}\Psi(\tau) + \frac{1}{n}\Phi\overline{\Phi}^{-1}|\Phi|^{\frac{2}{n}}\overline{\Psi}(\tau) - \frac{\Phi(\tau)}{\theta}\int_{\mathbf{R}^n}\left(\left(1+\frac{1}{n}\right)|\Phi|^{\frac{2}{n}}\Psi(\tau,x) + \frac{1}{n}\Phi\overline{\Phi}^{-1}|\Phi|^{\frac{2}{n}}\overline{\Psi}(\tau,x)\right)dxd\tau - \frac{\Psi(\tau)}{\theta}\int_{\mathbf{R}^n}|\Phi|^{\frac{2}{n}}\Phi(\tau,x)\,dx\right)d\tau.$$

We put $r = v-\Phi-\Psi-V$, then we get

$$\begin{cases} r(t) = -\beta\int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)f(\tau)\,d\tau, \\ g(t) = 1+\frac{6}{\theta n}\int_0^t g^{\frac{2}{3}}(\tau)\operatorname{Re}\int_{\mathbf{R}^n}\beta|v|^2 v(\tau,x)\,dxd\tau, \end{cases}$$

where

$$f(\tau) = g^{\frac{2}{3}}(\tau)\left(|v|^2 v(\tau) - |\Phi(\tau)|^{\frac{2}{n}}\Phi(\tau) - \left(1+\frac{1}{n}\right)|\Phi|^{\frac{2}{n}}\Psi(\tau) - \frac{1}{n}\Phi\overline{\Phi}^{-1}|\Phi|^{\frac{2}{n}}\overline{\Psi}(\tau) - \frac{v(\tau)}{\theta}\int_{\mathbf{R}^n}|v|^2 v(\tau,x)\,dx + \frac{\Phi(\tau)}{\theta}\int_{\mathbf{R}^n}|\Phi(\tau)|^{\frac{2}{n}}\Phi(\tau)\,dx + \frac{\Phi(\tau)}{\theta}\int_{\mathbf{R}^n}\left(\left(1+\frac{1}{n}\right)|\Phi|^{\frac{2}{n}}\Psi(\tau,x) + \frac{1}{n}\Phi\overline{\Phi}^{-1}|\Phi|^{\frac{2}{n}}\overline{\Psi}(\tau,x)\right)dx + \frac{\Psi(\tau)}{\theta}\int_{\mathbf{R}^n}|\Phi|^{\frac{2}{n}}\Phi(\tau)\,dx\right).$$

We define the mappings $\mathcal{M}(r,g)$ and $\mathcal{R}(r,g)$ by

$$\mathcal{M}(r,g) = -\beta\int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)f(\tau)\,d\tau,$$

$$\mathcal{R}(r,g) = 1+\frac{6}{\theta n}\int_0^t g^{\frac{2}{3}}(\tau)\operatorname{Re}\int_{\mathbf{R}^n}\beta|v|^{\frac{2}{n}}v(\tau,x)\,dxd\tau.$$

We prove that the transformation $(\mathcal{M},\mathcal{R})$ is the contraction mapping in the set

$$\mathbf{X} = \{r \in \mathbf{C}\left((0,\infty);\mathbf{L}^{1,\lambda} \cap \mathbf{W}_{\infty}^{0,\lambda}\right), g \in \mathbf{C}(0,\infty):$$
$$\sup_{t>0}\sup_{1\le p\le\infty}\sup_{b\in[0,\lambda]}\{t\}^{\rho\left(1-\frac{1}{p}\right)} t^{\frac{n}{2}\left(1-\frac{1}{p}\right)}\langle t\rangle^{-\frac{b}{2}} g(t)\left\|\langle\cdot\rangle^{b} r(t)\right\|_{\mathbf{L}^p} \le C\varepsilon^{q+\frac{2}{n}};$$
$$\frac{1}{2}(1+\mu\log\langle t\rangle) \le g(t) \le 2(1+\mu\log\langle t\rangle),$$
$$|g'(t)| \le C\varepsilon^{\frac{4}{n}} t^{-1} \text{ for all } t>0\},$$

where $\rho = \frac{1}{2}\left(1-\frac{1}{r}\right) > 0, \lambda \in (0,a)$ and

$$\mu = \frac{\theta^{\frac{4}{n}}(\operatorname{Im}\delta)^2}{(4\pi)^2}\operatorname{Re}\left(\left(1+\frac{1}{n}\right)\nu_1 - \frac{1}{n}\nu_2\right) > 0.$$

When $(r,g) \in \mathbf{X}$ we get

$$\sup_{t>0}\sup_{1\le p\le\infty}\{t\}^{-\rho\left(1-\frac{1}{p}\right)} t^{1-\frac{b}{2}+\frac{n}{2}\left(1-\frac{1}{p}\right)}\left\||\cdot|^{b} f(t)\right\|_{\mathbf{L}^p} \le C\varepsilon^{q},$$

where $b \in [0,\lambda]$. We use the following lemma . We define Ψ as a solution to the Cauchy problem

$$\Psi_t - \alpha\Psi_{xx} + \beta g^{-\frac{1}{3}}\left(|\Phi|^{\frac{2}{n}} - \frac{1}{\theta}\int_{\mathbf{R}^n}|\Phi|^{\frac{2}{n}}\Phi dx\right)\Phi = 0,$$
$$\Psi(0,x) = 0,$$

where $\Phi = \mathcal{G}v_0$. Let V be a solution to the Cauchy problem

$$V_t - \alpha V_{xx} + \beta g^{-\frac{1}{3}}\left(\left(\left(1+\frac{1}{n}\right)|\Phi|^{\frac{2}{n}}\Psi(\tau) + \frac{1}{n}\Phi\overline{\Phi}^{-1}|\Phi|^{\frac{2}{n}}\overline{\Psi}(\tau)\right.\right.$$
$$- \frac{\Phi(\tau)}{\theta}\int_{\mathbf{R}^n}\left(\left(1+\frac{1}{n}\right)|\Phi|^{\frac{2}{n}}\Psi(\tau,x) + \frac{1}{n}\Phi\overline{\Phi}^{-1}|\Phi|^{\frac{2}{n}}\overline{\Psi}(\tau,x)\right)dx$$
$$\left.-\frac{\Psi(\tau)}{\theta}\int_{\mathbf{R}^n}|\Phi|^{\frac{2}{n}}\Phi(\tau)\,dx\right)d\tau = 0,\ V(0,x) = 0,$$

thus we have

$$\Psi(t) = -\beta\int_0^t g^{-\frac{1}{3}}(\tau)\mathcal{G}(t-\tau)\left(|\Phi|^{\frac{2}{n}}\Phi(\tau) - \frac{\Phi(\tau)}{\theta}\int_{\mathbf{R}^n}|\Phi|^{\frac{2}{n}}\Phi(\tau,x)\,dx\right)d\tau,$$

and

$$V(t) = -\beta\int_0^t g^{-\frac{1}{3}}(\tau)\mathcal{G}(t-\tau)\left(\left(\left(1+\frac{1}{n}\right)|\Phi|^{\frac{2}{n}}\Psi(\tau) + \frac{1}{n}\Phi\overline{\Phi}^{-1}|\Phi|^{\frac{2}{n}}\overline{\Psi}(\tau)\right.\right.$$
$$- \frac{\Phi(\tau)}{\theta}\int_{\mathbf{R}^n}\left(\left(1+\frac{1}{n}\right)|\Phi|^{\frac{2}{n}}\Psi(\tau,x) + \frac{1}{n}\Phi\overline{\Phi}^{-1}|\Phi|^{\frac{2}{n}}\overline{\Psi}(\tau,x)\right)dx$$
$$\left.-\frac{\Psi(\tau)}{\theta}\int_{\mathbf{R}^n}|\Phi|^{\frac{2}{n}}\Phi(\tau,x)\,dx\right)d\tau.$$

Lemma 3.27. *We assume that* $v_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^r(\mathbf{R}^n)$ *, the norm* $\|v_0\|_{\mathbf{L}^r} + \|v_0\|_{\mathbf{L}^{1,a}} = \varepsilon$ *is sufficiently small,* $\hat{v}_0(0) = \theta (2\pi)^{-\frac{n}{2}} \geq C\varepsilon > 0$, $\eta = 0$, $\mu = 0$. *Let the function* $v(t,x)$ *satisfy the estimates*

$$\|v(t)\|_{\mathbf{L}^p} \leq \varepsilon \{t\}^{-\rho} t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

and

$$\|v(t) - \Phi(t) - \Psi(t) - V(t)\|_{\mathbf{L}^p} \leq \varepsilon^{q+\frac{2}{n}} g^{-1}(t) t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$, $q \leq p \leq \infty$, *where* $\rho > 0$. *We also assume that the function* $g(t)$ *is such that*

$$\frac{1}{2}(1 + \kappa \log \langle t \rangle) \leq g(t) \leq 2(1 + \kappa \log \langle t \rangle)$$

and $|g'(t)| \leq C\varepsilon^{\frac{4}{n}} t^{-1}$ *for all* $t > 0$, *where* $\kappa > 0$ *is defined below in (3.96).*

Then the following inequalities are valid

$$\frac{1}{\theta}\left|\sqrt[3]{g^2(t)}\,\mathrm{Re}\int_{\mathbf{R}^n} \beta |v|^{\frac{2}{n}} v(t,x)\,dx\right| \leq C\varepsilon^{\frac{2}{n}} t^{-1}\{t\}^{\rho} \tag{3.82}$$

and

$$\begin{aligned}\frac{9}{10}(1 + \kappa \log \langle t \rangle) &\leq 1 + \frac{6}{n\theta}\int_0^t d\tau \sqrt[3]{g^2(t)}\,\mathrm{Re}\int_{\mathbf{R}^n} \beta |v|^{\frac{2}{n}} v(\tau,x)\,dx \\ &\leq \frac{11}{10}(1 + \kappa \log \langle t \rangle)\end{aligned} \tag{3.83}$$

for all $t > 0$.

As in the proof of Theorem 3.2, by applying Lemma 3.25 and Lemma 3.27 we can show that $(\mathcal{M}(r,g), \mathcal{R}(r,g))$ is the contraction mapping from the set $\mathbf{X}$ into itself and as above we get the asymptotic formula

$$\begin{aligned}\psi'(t) &= -\frac{1}{\theta} g^{-\frac{1}{3}}(t)\,\mathrm{Im}\int_{\mathbf{R}^n} \beta |v|^{\frac{2}{n}} v dx \\ &= -\frac{\omega}{t\sqrt[3]{1 + \kappa \log t}} + O\left(\varepsilon^2 t^{-1} g^{-2/3}(t)\right),\end{aligned}$$

where $\omega = \frac{\theta^{\frac{2}{n}}}{4\pi}\,\mathrm{Im}\,\delta(\alpha,\beta)$. Hence

$$\begin{aligned}\psi(t) &= -\int_2^t \left(\frac{\omega}{\sqrt[3]{1 + \kappa \log \tau}} + O\left(\varepsilon g^{-2/3}(\tau)\right)\right)\frac{d\tau}{\tau} \\ &= -\frac{3\omega}{2\sqrt[3]{\kappa}}\sqrt[3]{\log^2 t} + O\left(\varepsilon \sqrt[3]{\log t}\right).\end{aligned}$$

Therefore via formulas

$$u(t,x) = e^{-\varphi(t) + i\psi(t)} v(t,x) = e^{-\varphi(t) + i\psi(t)}(\Phi + \Psi + V + r)$$

and the estimates for Φ, Ψ, V given in Lemma 3.24 and Lemma 3.26 we obtain the result of Theorem 3.22 in the case $\eta = 0$, $\mu = 0$, $\kappa > 0$.

Now we prove Lemma 3.27.

Proof. Consider the case of $t \geq 1$. Similarly to formula (3.75) (see the proof of Lemma 3.26) we obtain the representation

$$\Psi = \frac{\theta^q \delta}{4\pi} g^{-\frac{1}{3}}(t)\, \Gamma(t) + R_1(t), \tag{3.84}$$

where

$$\Gamma(t) = -\frac{1}{(4\pi\alpha t)^{\frac{n}{2}}} \int_0^1 \left(\frac{1}{(1-\sigma z)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\sigma z)}} - e^{-\frac{|x|^2}{4\alpha t}} \right) \frac{dz}{z};$$

as above, $\sigma = \frac{\alpha+\overline{\alpha}}{\alpha+(n+1)\overline{\alpha}}$, and the remainder term can be estimated as

$$\left\| |\cdot|^b R_1(t) \right\|_{\mathbf{L}^p} \leq C\varepsilon^q t^{\frac{b}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)} g^{-1}(t)$$

for all $t > 0$, $b \in [0, \lambda]$, $1 \leq p \leq q$. Now let us prove the asymptotic formula

$$V(t) = \frac{\theta^{q+\frac{2}{n}} \delta^2}{(4\pi)^2} g^{-\frac{2}{3}}(t)\, \Lambda(t) + R_2(t), \tag{3.85}$$

where R_2 satisfies the estimate

$$\|R_2(t)\|_{\mathbf{L}^q} \leq C\varepsilon^{q+\frac{2}{n}} t^{-\frac{1}{q}} g^{-1}(t)$$

and

$$\begin{aligned}
\Lambda(t)(4\pi\alpha t)^{\frac{n}{2}} &= \frac{n+1}{n} \int_0^1 \frac{dz}{z(1-\sigma^2 z)^{\frac{n}{2}}} \int_0^1 \frac{d\xi}{\xi} \left(\frac{1}{(1-h\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-h\xi)}} \right. \\
&\left. - \frac{1}{(1-\sigma\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\sigma\xi)}} \right) \\
&+ \frac{n+1}{n} \int_0^1 \left(\frac{1}{(1-\sigma^2 z)^{\frac{n}{2}}} - 1 \right) \frac{dz}{z} \int_0^1 \frac{d\xi}{\xi} \left(\frac{1}{(1-\sigma\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\sigma\xi)}} \right. \\
&\left. - e^{-\frac{|x|^2}{4\alpha t}} \right) - \frac{1}{n} \int_0^1 \frac{dz}{z(1-\kappa_2 z)^{\frac{n}{2}}} \int_0^1 \frac{d\xi}{\xi} \\
&\times \left(\frac{1}{(1-\chi\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\chi\xi)}} - \frac{1}{(1-\sigma\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\sigma\xi)}} \right) \\
&- \frac{1}{n} \int_0^1 \left(\frac{1}{(1-\kappa_2 z)^{\frac{n}{2}}} - 1 \right) \frac{dz}{z} \int_0^1 \frac{d\xi}{\xi} \left(\frac{1}{(1-\sigma\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\sigma\xi)}} - e^{-\frac{|x|^2}{4\alpha t}} \right) \\
&- \int_0^1 \frac{dz}{z} \int_0^1 \frac{d\xi}{\xi} \left(\frac{1}{(1-\sigma z\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\sigma z\xi)}} - e^{-\frac{|x|^2}{4\alpha t}} \right),
\end{aligned}$$

with $h = \frac{\sigma - \kappa_3 z}{1 - \sigma^2 z}$, $\chi = \frac{\sigma - \kappa_4 z}{1 - \kappa_2 z}$, $\sigma = \frac{\alpha + \overline{\alpha}}{a + (n+1)\overline{\alpha}}$, $\kappa_3 = \frac{\alpha + (1-n)\overline{\alpha}}{a + (n+1)\overline{\alpha}} \sigma$, $\kappa_4 = \frac{\overline{\alpha} + (1-n)\alpha}{a + (n+1)\overline{\alpha}} \overline{\sigma}$. We have

$$V = -\frac{\beta\delta\theta^q}{4\pi} \int_0^t g^{-\frac{2}{3}}(\tau)\,\mathcal{G}(t-\tau)\Bigg(\Bigg(\left(1+\frac{1}{n}\right)|G|^{\frac{2}{n}}\,\Gamma - \frac{1}{n}G\overline{G}^{-1}|G|^{\frac{2}{n}}\,\overline{\Gamma}$$
$$- G\int_{\mathbf{R}^n}\left(\left(1+\frac{1}{n}\right)|G|^{\frac{2}{n}}\,\Gamma - \frac{1}{n}G\overline{G}^{-1}|G|^{\frac{2}{n}}\,\overline{\Gamma}\right)dx - \Gamma\int_{\mathbf{R}^n}|G|^{\frac{2}{n}}\,G dx\Bigg)d\tau$$
$$+ R_3, \tag{3.86}$$

with

$$R_3(t) = \int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)\,f(\tau)\,d\tau,$$

where

$$f = -\beta g^{\frac{2}{3}}\Bigg(\left(1+\frac{1}{n}\right)|\Phi|^{\frac{2}{n}}\,\Psi + \frac{1}{n}\Phi\overline{\Phi}^{-1}|\Phi|^{\frac{2}{n}}\,\overline{\Psi}$$
$$-\frac{\Phi}{\theta}\int_{\mathbf{R}^n}\left(\left(1+\frac{1}{n}\right)|\Phi|^{\frac{2}{n}}\,\Psi + \frac{1}{n}\Phi\overline{\Phi}^{-1}|\Phi|^{\frac{2}{n}}\,\overline{\Psi}\right)dx - \frac{\Psi}{\theta}\int_{\mathbf{R}^n}|\Phi|^{\frac{2}{n}}\,\Phi dx$$
$$-\frac{\delta\theta^{q+\frac{2}{n}}}{4\pi}g^{-\frac{1}{3}}\Bigg(\left(1+\frac{1}{n}\right)|G|^{\frac{2}{n}}\,\Gamma - \frac{1}{n}G\overline{G}^{-1}|G|^{\frac{2}{n}}\,\overline{\Gamma}$$
$$- G\int_{\mathbf{R}^n}\left(\left(1+\frac{1}{n}\right)|G|^{\frac{2}{n}}\,\Gamma - \frac{1}{n}G\overline{G}^{-1}|G|^{\frac{2}{n}}\,\overline{\Gamma}\right)dx - \Gamma\int_{\mathbf{R}^n}|G|^{\frac{2}{n}}\,G dx\Bigg)\Bigg).$$

As in the proof of Lemma 3.26 by (3.84) and Lemma 3.24, we see that $f(\tau)$ satisfies the estimate

$$\left\| |\cdot|^{\lambda} f(\tau)\right\|_{\mathbf{L}^p} \le C\varepsilon^{q+\frac{2}{n}}\tau^{\frac{\lambda}{2}-1-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

for all $\tau > 0$ and $1 \le p \le q$, where $0 < \lambda < \min(a, \rho)$. Hence via Lemma 3.25 we get

$$\|R_3(t)\|_{\mathbf{L}^q} \le C\varepsilon^{q+\frac{2}{n}}t^{-\frac{1}{q}}g^{-1}(t)$$

for all $t > 0$. As above we have

$$\int_{\mathbf{R}^n}|G|^{\frac{2}{n}}\,G(\tau,x)\,dx = \frac{\delta}{4\pi\beta\tau}$$

and, similarly,

$$\int_{\mathbf{R}^n}\left(\left(1+\frac{1}{n}\right)|G|^{\frac{2}{n}}\,\Gamma - \frac{1}{n}G\overline{G}^{-1}|G|^{\frac{2}{n}}\,\overline{\Gamma}\right)dx$$
$$= -\frac{\delta}{4\pi\beta\tau}\left(\left(1+\frac{1}{n}\right)\nu_1 - \frac{1}{n}\nu_2\right).$$

Then integrating by parts as in the proof of Lemma 3.26 we get from (3.86)

$$V = -\frac{\beta\delta\theta^{q+\frac{2}{n}}}{(4\pi)^2} g^{-\frac{2}{3}}(t) \int_0^t \mathcal{G}(t-\tau) \left(\frac{n+1}{n|\alpha|} \exp\left(-\frac{|x|^2}{4\tau n} \left(\frac{1}{\alpha} + \frac{1}{\overline{\alpha}} \right) \right) \Gamma(\tau) \right.$$

$$- \frac{(\overline{\alpha})^{\frac{n}{2}}}{n\alpha^{\frac{n}{2}}|\alpha|} \exp\left(-\frac{|x|^2}{4\tau n} \left(\frac{1+n}{\alpha} + \frac{1-n}{\overline{\alpha}} \right) \right) \overline{\Gamma}(\tau)$$

$$\left. + \frac{\delta}{\beta} \left(\left(1 + \frac{1}{n} \right) \nu_1 - \frac{1}{n}\nu_2 \right) G(\tau) - \frac{\delta}{\beta}\Gamma(\tau) \right) \frac{d\tau}{\tau} + R_4, \tag{3.87}$$

where R_4 satisfy the estimate

$$\|R_4(t)\|_{\mathbf{L}^q} \leq C\varepsilon^{q+\frac{2}{n}} t^{-\frac{1}{q}} g^{-1}(t)$$

for all $t > 0$. Therefore by (3.87) we obtain

$$\Lambda(t)(4\pi\alpha)^{\frac{n}{2}}$$

$$= \frac{\beta(n+1)}{|\alpha| n\delta} \int_0^t \frac{d\tau}{\tau^{1+\frac{n}{2}}} \int_0^1 \frac{dz}{z} \mathcal{G}(t-\tau) \left(\frac{1}{(1-\sigma z)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha\tau(1-\sigma z)} - \frac{|x|^2(\alpha+\overline{\alpha})}{4\tau n\alpha\overline{\alpha}}} \right.$$

$$\left. - e^{-\frac{|x|^2(\alpha+(n+1)\overline{\alpha})}{4\tau\alpha\overline{\alpha}n}} \right) - \frac{\beta}{|\alpha|\delta n} \int_0^t \frac{d\tau}{\tau^{1+\frac{n}{2}}} \int_0^1 \frac{dz}{z}$$

$$\times \mathcal{G}(t-\tau) \left(\frac{1}{(1-\overline{\sigma}z)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\overline{\alpha}\tau(1-\overline{\sigma}z)} - \frac{|x|^2((1+n)\overline{\alpha}+(1-n)\alpha)}{4\alpha\overline{\alpha}\tau n}} - e^{-\frac{|x|^2(\alpha+(1+n)\overline{\alpha})}{4\tau\alpha\overline{\alpha}}} \right)$$

$$+ \left(\left(1 + \frac{1}{n} \right) \nu_1 - \frac{1}{n}\nu_2 \right) \int_0^t \frac{d\tau}{\tau^{1+\frac{n}{2}}} \mathcal{G}(t-\tau) e^{-\frac{|x|^2}{4\alpha\tau}}$$

$$- \int_0^t \frac{d\tau}{\tau^{1+\frac{n}{2}}} \int_0^1 \frac{dz}{z} \mathcal{G}(t-\tau) \left(\frac{1}{(1-\sigma z)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha\tau(1-\sigma z)}} - e^{-\frac{|x|^2}{4\alpha\tau}} \right).$$

Next using the formula

$$\mathcal{G}(t-\tau) e^{-b|x|^2} = \frac{1}{(1+4\alpha b(t-\tau))^{\frac{n}{2}}} \exp\left(-\frac{b|x|^2}{1+4\alpha b(t-\tau)} \right)$$

we get

$$
\begin{aligned}
&\Lambda(t)\,(4\pi\alpha t)^{\frac{n}{2}} \\
&= \frac{(n+1)}{n}\int_0^1 \frac{dz}{z\,(1-\sigma^2 z)^{\frac{n}{2}}}\int_0^1 \frac{d\xi}{\xi\,(1-h\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-h\xi)}} \\
&-\frac{1}{n}\int_0^1 \frac{dz}{z\,(1-\kappa_2 z)^{\frac{n}{2}}}\int_0^1 \frac{d\xi}{\xi\,(1-\chi\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\chi\xi)}} \\
&-\int_0^1 \frac{dz}{z}\int_0^1 \frac{d\xi}{\xi\,(1-\sigma\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\sigma\xi)}} \\
&+\left(\left(1+\frac{1}{n}\right)\nu_1-\frac{1}{n}\nu_2\right)\int_0^1 \frac{d\xi}{\xi} e^{-\frac{|x|^2}{4\alpha t}} \\
&-\int_0^1 \frac{dz}{z}\int_0^1 \frac{d\xi}{\xi}\left(\frac{1}{(1-\sigma z\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\sigma z\xi)}}-e^{-\frac{|x|^2}{4\alpha t}}\right),
\end{aligned}
$$

thus obtaining (3.85). We put $r = v - \Phi - \Psi - V$, then by virtue of the Taylor formula, estimates (3.84), (3.85) and the conditions of the lemma, we get

$$
\begin{aligned}
&\Big\| |v|^{\frac{2}{n}} v - |\Phi|^{\frac{2}{n}} \Phi - \left(1+\frac{1}{n}\right)|\Phi|^{\frac{2}{n}}(\Psi+V) - \frac{1}{n}\Phi\overline{\Phi}^{-1}|\Phi|^{\frac{2}{n}}\left(\overline{\Psi}+\overline{V}\right) \\
&-\frac{1}{2n}\left(\frac{1}{n}+1\right)\Phi^{-1}|\Phi|^{\frac{2}{n}}\Psi^2-\frac{1}{2n}\left(\frac{1}{n}-1\right)\Phi\overline{\Phi}^{-2}|\Phi|^{\frac{2}{n}}\overline{\Psi}^2 \\
&-\frac{1}{n}\left(\frac{1}{n}+1\right)\overline{\Phi}^{-1}|\Phi|^{\frac{2}{n}}\Psi\overline{\Psi}\Big\|_{\mathbf{L}^1} \\
&\le C\left\||\Phi|^{\frac{2}{n}} r\right\|_{\mathbf{L}^1}+C\left\|\Phi\overline{\Phi}^{-1}|\Phi|^{\frac{2}{n}}\overline{r}\right\|_{\mathbf{L}^1} \\
&+C\left\||\Phi|^{\frac{2}{n}}\Phi^{-2}\Psi^3\right\|_{\mathbf{L}^1}+C\left\||\Phi|^{\frac{2}{n}-2}\Psi^2\overline{\Psi}\right\|_{\mathbf{L}^1} \\
&+C\left\||\Phi|^{\frac{2}{n}}\overline{\Phi}^{-2}\Psi\overline{\Psi}^2\right\|_{\mathbf{L}^1}+C\left\|\Phi|\Phi|^{\frac{2}{n}}\overline{\Phi}^{-3}\overline{\Psi}^3\right\|_{\mathbf{L}^1}\le C\varepsilon t^{-\frac{1}{q}}g^{-1}(t). \qquad (3.88)
\end{aligned}
$$

We next prove (3.82) and (3.83). By a direct calculation we get

$$
\begin{aligned}
&\frac{1}{n}\left(\frac{1}{n}+1\right)\int_{\mathbf{R}^n}\overline{G}^{-1}|G|^{\frac{2}{n}}|\Gamma|^2\,dx \\
&=\frac{n+1}{n^2(4\pi t)^{1+\frac{n}{2}}\alpha^{\frac{n}{2}}|\alpha|}\int_0^1\frac{dz}{z}\int_0^1\frac{d\xi}{\xi}\int_{\mathbf{R}^n}dx\left(e^{-\frac{|x|^2}{4tn}\left(\frac{1}{\alpha}+\frac{1-n}{\overline{\alpha}}\right)-\frac{|x|^2}{4\alpha t}-\frac{|x|^2}{4\overline{\alpha}t}}\right. \\
&+\frac{1}{(1-\sigma z)^{\frac{n}{2}}(1-\overline{\sigma}\xi)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4tn}\left(\frac{1}{\alpha}+\frac{1-n}{\overline{\alpha}}\right)-\frac{|x|^2}{4\alpha t(1-\sigma z)}-\frac{|x|^2}{4\overline{\alpha}t(1-\overline{\sigma}\xi)}} \\
&-\frac{1}{(1-\sigma z)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4tn}\left(\frac{1}{\alpha}+\frac{1-n}{\overline{\alpha}}\right)-\frac{|x|^2}{4\alpha t(1-\sigma z)}-\frac{|x|^2}{4\overline{\alpha}t}} \\
&\left.-\frac{1}{(1-\overline{\sigma}\xi)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4tn}\left(\frac{1}{\alpha}+\frac{1-n}{\overline{\alpha}}\right)-\frac{|x|^2}{4\alpha t}-\frac{|x|^2}{4\overline{\alpha}t(1-\overline{\sigma}\xi)}}\right);
\end{aligned}
$$

thus, by using the identity $\int_{\mathbf{R}^n} e^{-b|x|^2}dx = \left(\frac{\pi}{b}\right)^{\frac{n}{2}}$, we obtain

$$\frac{1}{n}\left(\frac{1}{n}+1\right)\int_{\mathbf{R}^n} \overline{G}^{-1}|G|^{\frac{2}{n}}|\Gamma|^2 dx$$
$$= \frac{(n+1)\,n^{\frac{n}{2}}|\alpha|^{n-1}}{4\pi t n^2\left(\alpha^2+(n+1)|\alpha|^2\right)^{\frac{n}{2}}}\int_0^1 \frac{dz}{z}\int_0^1 \frac{d\xi}{\xi}\left(1-\frac{1}{(1-\sigma^2 z)^{\frac{n}{2}}}\right.$$
$$\left.+\frac{1}{(1-\sigma^2 z-\kappa_2\xi+\sigma\kappa_4 z\xi)^{\frac{n}{2}}}-\frac{1}{(1-\kappa_2\xi)^{\frac{n}{2}}}\right)$$
$$= \frac{(n+1)\,\delta}{4\pi t n^2\beta}\int_0^1 \frac{dz}{z}\left(\frac{\nu\left(\frac{\kappa_2-\sigma\kappa_4 z}{1-\sigma^2 z}\right)}{(1-\sigma^2 z)^{\frac{n}{2}}}-\nu(\kappa_2)\right), \tag{3.89}$$

where $\nu(y) = \int_0^1\left(\frac{1}{(1-y\xi)^{\frac{n}{2}}}-1\right)\frac{d\xi}{\xi}$. In the same manner we have

$$\frac{1}{2n}\left(\frac{1}{n}+1\right)\int_{\mathbf{R}^n} G^{-1}|G|^{\frac{2}{n}}\Gamma^2 dx$$
$$= \frac{(1+n)\,\delta}{8\pi t n^2\beta}\int_0^1 \frac{dz}{z}\left(\frac{\nu\left(\frac{\kappa_5-\sigma\kappa_6 z}{1-\sigma^2 z}\right)}{(1-\sigma^2 z)^{\frac{n}{2}}}-\nu(\kappa_5)\right) \tag{3.90}$$

and

$$\frac{1}{2n}\left(\frac{1}{n}-1\right)\int_{\mathbf{R}^n} G\overline{G}^{-2}|G|^{\frac{2}{n}}\overline{\Gamma}^2 dx$$
$$= \frac{(1-n)\,\delta}{8\pi t n^2\beta}\int_0^1 \frac{dz}{z}\left(\frac{\nu\left(\frac{\kappa_2-\kappa_7 z}{1-\kappa_2 z}\right)}{(1-\kappa_2 z)^{\frac{n}{2}}}-\nu(\kappa_2)\right), \tag{3.91}$$

where $\kappa_5 = \frac{\alpha(1-n)+\overline{\alpha}(1+n)}{\alpha+(1+n)\overline{\alpha}}\sigma$, $\kappa_6 = \frac{\alpha(1-n)+\overline{\alpha}}{\alpha+(1+n)\overline{\alpha}}\sigma^2$, $\kappa_7 = \frac{\alpha(1-2n)+\overline{\alpha}(1+n)}{\alpha+(1+n)\overline{\alpha}}\overline{\sigma}^2$. Then by an easy calculation we obtain

$$
\begin{aligned}
&\int_{\mathbf{R}^n} |G|^{\frac{2}{n}} \Lambda dx \\
&= \frac{1}{4\pi |\alpha| t (4\pi\alpha t)^{\frac{n}{2}}} \int_0^1 \frac{dz}{z} \int_0^1 \frac{d\xi}{\xi} \int_{\mathbf{R}^n} dx e^{-\frac{|x|^2(\alpha+\overline{\alpha})}{4\alpha\overline{\alpha}tn}} \\
&\times \left(\frac{n+1}{n(1-\sigma^2 z)^{\frac{n}{2}}} \left(\frac{1}{(1-h\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-h\xi)}} - \frac{1}{(1-\sigma\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\sigma\xi)}} \right) \right. \\
&+ \frac{n+1}{n} \left(\frac{1}{(1-\sigma^2 z)^{\frac{n}{2}}} - 1 \right) \left(\frac{1}{(1-\sigma\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\sigma\xi)}} - e^{-\frac{|x|^2}{4\alpha t}} \right) \\
&- \frac{1}{n(1-\kappa_2 z)^{\frac{n}{2}}} \left(\frac{1}{(1-\chi\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\chi\xi)}} - \frac{1}{(1-\sigma\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\sigma\xi)}} \right) \\
&- \frac{1}{n} \left(\frac{1}{(1-\kappa_2 z)^{\frac{n}{2}}} - 1 \right) \left(\frac{1}{(1-\sigma\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\sigma\xi)}} - e^{-\frac{|x|^2}{4\alpha t}} \right) \\
&\left. - \left(\frac{1}{(1-\sigma z\xi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\alpha t(1-\sigma z\xi)}} - e^{-\frac{|x|^2}{4\alpha t}} \right) \right);
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_{\mathbf{R}^n} |G|^{\frac{2}{n}} \Lambda dx \\
&= \frac{\delta}{4\pi t\beta} \int_0^1 \frac{dz}{z} \left(\frac{(n+1)\left(\nu(\sigma h) - \nu\left(\sigma^2\right)\right)}{n(1-\sigma^2 z)^{\frac{n}{2}}} - \frac{\nu(\sigma\chi) - \nu\left(\sigma^2\right)}{n(1-\kappa_2 z)^{\frac{n}{2}}} - \nu\left(\sigma^2 z\right) \right) \\
&+ \frac{\delta}{4\pi t n\beta} \nu\left(\sigma^2\right) \left((n+1)\nu\left(\sigma^2\right) - \nu(\kappa_2) \right). \qquad (3.92)
\end{aligned}
$$

In a similar manner we acquire

$$
\begin{aligned}
&\int_{\mathbf{R}^n} G\overline{G}^{-1} |G|^{\frac{2}{n}} \overline{\Lambda} dx \\
&= \frac{\delta}{4\pi t\beta} \int_0^1 \frac{dz}{z} \left(\frac{(n+1)\left(\nu\left(\kappa_8 \overline{h}\right) - \nu(\kappa_2)\right)}{n\left(1-\overline{\sigma}^2 z\right)^{\frac{n}{2}}} - \frac{\left(\nu\left(\kappa_8 \overline{\chi}\right) - \nu(\kappa_2)\right)}{n\left(1-\overline{\kappa_2} z\right)^{\frac{n}{2}}} - \nu(\kappa_2 z) \right) \\
&+ \frac{\delta}{4\pi t n\beta} \nu(\kappa_2) \left((n+1)\nu\left(\overline{\sigma}^2\right) - \nu\left(\overline{\kappa_2}\right) \right), \qquad (3.93)
\end{aligned}
$$

where $\kappa_8 = \frac{\kappa_2}{\overline{\sigma}}$. Since $\eta = 0$ and $\mu = 0$, that is

$$
\operatorname{Re} \int_{\mathbf{R}^n} \beta |\Phi|^{\frac{2}{n}} \Phi(\tau, x) dx = O\left(\varepsilon^q t^{-1} g^{-1}(t)\right)
$$

and

$$\begin{aligned}&\operatorname{Re}\int_{\mathbf{R}^n}\beta\left(\left(1+\frac{1}{n}\right)|\Phi|^{\frac{2}{n}}\Psi(\tau,x)+\frac{1}{n}\Phi\overline{\Phi}^{-1}|\Phi|^{\frac{2}{n}}\overline{\Psi}(\tau,x)\right)dx\\&=O\left(\varepsilon^{q+\frac{2}{n}}t^{-1}g^{-1}(t)\right),\end{aligned}$$

by using the asymptotics $\Phi=\theta G+O\left(\varepsilon t^{-\frac{n}{2}-\lambda}\right)$ and formula (3.88) we obtain

$$\begin{aligned}&\operatorname{Re}\int_{\mathbf{R}^n}\beta|v|^{\frac{2}{n}}vdx\\&=\frac{\theta^{q+\frac{4}{n}}\delta^2}{(4\pi)^2}g^{-\frac{2}{3}}(t)\operatorname{Re}\int_{\mathbf{R}^n}\beta\left(\frac{n+1}{2n^2}G^{-1}|G|^{\frac{2}{n}}\Gamma^2-\frac{n+1}{n^2}\overline{G}^{-1}|G|^{\frac{2}{n}}|\Gamma|^2\right.\\&\left.+\frac{1-n}{2n^2}G\overline{G}^{-2}|G|^{\frac{2}{n}}\overline{\Gamma}^2+\frac{n+1}{n}|G|^{\frac{2}{n}}\Lambda-\frac{1}{n}G\overline{G}^{-1}|G|^{\frac{2}{n}}\overline{\Lambda}\right)dx\\&+O\left(C\varepsilon^q t^{-1}g^{-1}(t)\right).\end{aligned}$$

Substituting formulas (3.89) to (3.93) into the above identity we then gain

$$\begin{aligned}&\operatorname{Re}\int_{\mathbf{R}^n}\beta|v|^{\frac{2}{n}}vdx\\&=\frac{\theta^{q+\frac{4}{n}}i\delta^3}{(4\pi)^3tn}g^{-\frac{2}{3}}(t)\operatorname{Im}\left(\left(\int_0^1\frac{dz}{z}\left(\frac{n+1}{2n}\left(\frac{\nu\left(\frac{\kappa_5-\kappa_6 z}{1-\sigma^2 z}\right)}{(1-\sigma^2 z)^{\frac{n}{2}}}-\nu(\kappa_5)\right)\right.\right.\right.\\&-\frac{n+1}{n}\left(\frac{\nu\left(\frac{\kappa_2-\sigma\kappa_4 z}{1-\sigma^2 z}\right)}{(1-\sigma^2 z)^{\frac{n}{2}}}-\nu(\kappa_2)\right)+\frac{1-n}{2n}\left(\frac{\nu\left(\frac{\kappa_2-\kappa_7 z}{1-\kappa_2 z}\right)}{(1-\kappa_2 z)^{\frac{n}{2}}}-\nu(\kappa_2)\right)\\&+\frac{n+1}{n}\left(\frac{(n+1)\left(\nu(\sigma h)-\nu\left(\sigma^2\right)\right)}{(1-\sigma^2 z)^{\frac{n}{2}}}-\frac{\nu(\sigma\chi)-\nu\left(\sigma^2\right)}{(1-\kappa_2 z)^{\frac{n}{2}}}-n\nu\left(\sigma^2 z\right)\right)\\&\left.-\frac{(n+1)\left(\nu\left(\kappa_8\overline{h}\right)-\nu(\kappa_2)\right)}{n\left(1-\overline{\sigma}^2 z\right)^{\frac{n}{2}}}+\frac{\left(\nu\left(\kappa_8\overline{\chi}\right)-\nu(\kappa_2)\right)}{n\left(1-\overline{\kappa_2}z\right)^{\frac{n}{2}}}+\nu(\kappa_2 z)\right)\\&+\frac{n+1}{n}\nu\left(\sigma^2\right)\left((n+1)\nu\left(\sigma^2\right)-\nu(\kappa_2)\right)\\&\left.-\frac{1}{n}\nu(\kappa_2)\left((n+1)\nu\left(\overline{\sigma}^2\right)-\nu\left(\overline{\kappa_2}\right)\right)\right)+O\left(C\varepsilon^q t^{-1}g^{-1}(t)\right).\end{aligned}\tag{3.94}$$

Via (3.94) we get (3.82) and the following asymptotics

$$\begin{aligned}&1+\frac{6}{\theta}\int_0^t\sqrt[3]{g^2(\tau)}\operatorname{Re}\int_{\mathbf{R}^n}\beta|v|^{\frac{2}{n}}v(\tau)\,dxd\tau\\&=1+\kappa\log t+O\left(\varepsilon^q\sqrt[3]{\log^2\langle t\rangle}\right),\end{aligned}\tag{3.95}$$

where

$$\kappa = \frac{6i\delta^3\theta^{\frac{6}{n}}}{(4\pi)^3 n} \operatorname{Im}\left(\int_0^1 \frac{dz}{z}\left(\frac{n+1}{2n}\left(\frac{\nu\left(\frac{\kappa_5-\kappa_6 z}{1-\sigma^2 z}\right)}{(1-\sigma^2 z)^{\frac{n}{2}}} - \nu(\kappa_5)\right)\right.\right.$$

$$-\frac{n+1}{n}\left(\frac{\nu\left(\frac{\kappa_2-\sigma\kappa_4 z}{1-\sigma^2 z}\right)}{(1-\sigma^2 z)^{\frac{n}{2}}} - \nu(\kappa_2)\right) + \frac{1-n}{2n}\left(\frac{\nu\left(\frac{\kappa_2-\kappa_7 z}{1-\kappa_2 z}\right)}{(1-\kappa_2 z)^{\frac{n}{2}}} - \nu(\kappa_2)\right)$$

$$+\frac{n+1}{n}\left(\frac{(n+1)\left(\nu(\sigma h) - \nu\left(\sigma^2\right)\right)}{(1-\sigma^2 z)^{\frac{n}{2}}} - \frac{\nu(\sigma\chi) - \nu\left(\sigma^2\right)}{(1-\kappa_2 z)^{\frac{n}{2}}} - n\nu\left(\sigma^2 z\right)\right)$$

$$\left.-\frac{(n+1)\left(\nu\left(\kappa_8\overline{h}\right) - \nu(\kappa_2)\right)}{n\left(1-\overline{\sigma}^2 z\right)^{\frac{n}{2}}} + \frac{\left(\nu\left(\kappa_8\overline{\chi}\right) - \nu(\kappa_2)\right)}{n\left(1-\overline{\kappa_2} z\right)^{\frac{n}{2}}} + \nu(\kappa_2 z)\right)$$

$$+\frac{n+1}{n}\nu\left(\sigma^2\right)\left((n+1)\nu\left(\sigma^2\right) - \nu(\kappa_2)\right)$$

$$\left.-\frac{1}{n}\nu(\kappa_2)\left((n+1)\nu\left(\overline{\sigma}^2\right) - \nu\left(\overline{\kappa_2}\right)\right)\right). \qquad (3.96)$$

Note that κ is an odd function of δ. From the condition $\mu = 0$ we have two possibilities for the value of β with different signs. Therefore we can choose a pure imaginary value of δ with a different sign. Thus there exist α, β such that $\mu = 0$, $\nu = 0$ and $\kappa > 0$. Now asymptotics (3.95) yields estimates (3.83). Lemma 3.27 is proved.

3.4.5 Asymptotic expansion

In this subsection we obtain the asymptotic expansion of small solutions to the Cauchy problem for the complex Landau - Ginzburg equation (3.68). For simplicity we consider here the one dimensional case. The case of higher dimensions also can be considered by this method. We suppose that $\operatorname{Re} \frac{\beta}{\sqrt{2|\alpha|^2+\alpha^2}} \geq 0$.

In the previous subsection, we obtained a precise leading term of the large time asymptotics for small solutions to the Cauchy problem (3.68) with complex numbers α, β, such that $\operatorname{Re}\alpha > 0$ and $\operatorname{Re} \frac{\beta}{\sqrt{2|\alpha|^2+\alpha^2}} \geq 0$. Roughly speaking, our purpose in the present subsection is to show that there exist functions $\Gamma_j(\xi)$ such that the asymptotic expansion

$$u(t,x) = t^{-\frac{1}{2}} \sum_{j=0}^{m} \Gamma_j(\xi)\, \varepsilon^{2j+1} (\log t)^{-\frac{2j+1}{2\kappa}} + O\left(\varepsilon^{2m+3} t^{-\frac{1}{2}} (\log t)^{-\frac{2m+3}{2\kappa}}\right)$$

is valid for $t \to \infty$, where $m \in \mathbf{N}$, the value $\varepsilon > 0$ bounds the size of the initial data, $\xi = \frac{x}{\sqrt{t}}$, $\kappa = 1, 2$, or 3 depending on the values of α and β. In the previous subsections we considered the particular case of $m = 0$.

Denote

$$\eta_1 \equiv \frac{\vartheta^2}{2\pi} \operatorname{Re}\delta \quad \text{with} \quad \delta = \frac{\beta}{\sqrt{2|\alpha|^2 + \alpha^2}},$$

$$\eta_2 = \frac{\vartheta^4 (\operatorname{Im}\delta)^2}{(4\pi)^2} \operatorname{Re}(2\nu(\mu_1) - \nu(\mu_2)), \text{ with}$$

$$\nu(x) = 2\log\left(\frac{2}{1+\sqrt{1-x}}\right),$$

$$\mu_1 = \frac{(\alpha+\bar{\alpha})^2}{(\alpha+2\bar{\alpha})^2}, \mu_2 = 2\frac{|\alpha|^2 + \bar{\alpha}^2}{|\alpha+2\bar{\alpha}|^2},$$

and

$$\eta_3 = \frac{6i\delta^3\vartheta^6}{(4\pi)^3} \operatorname{Im}\left(\int_0^1 \frac{dz}{z}\left(\frac{2\nu\left(\frac{\mu_2\left(1-\frac{\sigma}{2}z\right)}{1-\sigma^2 z}\right)}{\sqrt{1-\sigma^2 z}} - 2\nu(\mu_2) - \frac{\nu\left(\frac{\sigma(\sigma-bz)}{1-\sigma^2 z}\right)}{\sqrt{1-\sigma^2 z}}\right.\right.$$
$$\left. +\nu(\sigma^2) - \frac{6\left(\nu(\sigma h) - \nu(\sigma^2)\right)}{\sqrt{1-\sigma^2 z}} + \frac{3\left(\nu(\sigma a) - \nu(\sigma^2)\right)}{\sqrt{1-\mu_2 z}}\right)$$
$$\left. +3\nu(\sigma^2 z) - 3\nu(\sigma^2)\left(2\nu(\sigma^2) - \nu(\mu_2)\right)\right) > 0, \tag{3.97}$$

with

$$\sigma = \frac{\alpha+\overline{\alpha}}{\alpha+2\overline{\alpha}}, \vartheta = \left|\int_{\mathbf{R}^n} \phi(x)\,dx\right| > 0,$$
$$h = \frac{\sigma - bz}{1-\sigma^2 z}, b = \frac{\alpha\sigma}{\alpha+2\overline{\alpha}}, a = \frac{\sigma - \frac{\mu_2}{2}z}{1-\mu_2 z}.$$

Depending on the complex parameters α, β, we denote $\kappa = 1$ if $\eta_1 > 0$, $\kappa = 2$ if $\eta_1 = 0, \eta_2 > 0$, and $\kappa = 3$ if $\eta_1 = \eta_2 = 0$, $\eta_3 > 0$. We write $\alpha = |\alpha| e^{i\gamma}, \beta = |\beta| e^{i\chi}$, $\gamma \equiv \arg\alpha$, $\chi \equiv \arg\beta$, then equation (3.68) is written as

$$\begin{cases} \left(\partial_t - |\alpha| e^{i\gamma}\partial_x^2\right) u + |\beta| e^{i\chi} |u|^2 u = 0, \ x \in \mathbf{R}, t > 0, \\ u(0,x) = \phi(x), \ x \in \mathbf{R}, \end{cases}$$

where $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\frac{\beta}{\sqrt{2|\alpha|^2+\alpha^2}} \geq 0$. The condition $\operatorname{Re}\alpha > 0$ implies $|\gamma| < \frac{\pi}{2}$ and $\operatorname{Re}\frac{\beta}{\sqrt{2|\alpha|^2+\alpha^2}} \geq 0$ means

$$\left|\chi - \frac{1}{2}\arctan\frac{\sin 2\gamma}{2+\cos 2\gamma}\right| \leq \frac{\pi}{2}$$

since

$$\frac{\beta}{\sqrt{2|\alpha|^2+\alpha^2}} = \frac{|\beta|}{|\alpha|}\frac{e^{i\left(\chi - \frac{\varsigma}{2}\right)}}{\sqrt{(2+\cos 2\gamma)^2 + (\sin 2\gamma)^2}},$$

where $\tan\varsigma = \frac{\sin 2\gamma}{2+\cos 2\gamma}$.

Note that the case of $\kappa = 1$ is fulfilled for all $\alpha \in \mathbf{C}$ such that $|\gamma| < \frac{\pi}{2}$ and $\beta \in \mathbf{C}$ and such that

$$\left|\chi - \frac{1}{2}\arctan\frac{\sin 2\gamma}{2+\cos 2\gamma}\right| < \frac{\pi}{2}.$$

If $\beta \in \mathbf{C}$ is such that

$$\chi = \chi_{\pm}(\gamma) \equiv \frac{1}{2}\arctan\frac{\sin 2\gamma}{2+\cos 2\gamma} \pm \frac{\pi}{2},$$

then $\eta_1 = 0$; namely, we see that $\eta_1 = 0$ on the two lines in (γ, χ) plane.

In order to study the case of $\kappa = 2$, which requires the condition $\eta_1 = 0$ and $\eta_2 > 0$, we consider the function

$$\begin{aligned} f(\gamma) &= \operatorname{Re}\left(2\nu(\mu_1) - \nu(\mu_2)\right) \\ &= \log\frac{2\left|\left|2+e^{2i\gamma}\right| + \sqrt{3+2e^{2i\gamma}}\right|\left|2+e^{2i\gamma}\right|}{\left|2+e^{2i\gamma} + \sqrt{3+2e^{2i\gamma}}\right|^2} \end{aligned}$$

for the values of α such that $\gamma = \arg\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Note that

$$f(0) = \log\frac{6}{3+\sqrt{5}} > 0,\ f\left(\pm\frac{\pi}{3}\right) > 0,$$
$$f\left(\pm\frac{2\pi}{5}\right) < 0 \text{ and } f\left(\pm\frac{\pi}{2}\right) = 0.$$

Plotting numerically via the “Maple” program we see that the function $f(\gamma)$ is positive for all $\gamma \in (-\gamma_0, \gamma_0)$ and is negative for all $\gamma \in \left(-\frac{\pi}{2}, -\gamma_0\right) \cup \left(\gamma_0, \frac{\pi}{2}\right)$, where $\gamma_0 \approx 1.146035015$. Thus the case of $\kappa = 2$ is fulfilled on the parts of two lines

$$\chi = \chi_{\pm}(\gamma) \equiv \frac{1}{2}\arctan\frac{\sin 2\gamma}{2+\cos 2\gamma} \pm \frac{\pi}{2}$$

in (γ, χ) plane, where $\gamma \in (-\gamma_0, \gamma_0)$. Note that by numeric computations we have

$$\begin{aligned} &(\gamma_0, \chi_+(\gamma_0)) = (1.146, 1.826),\ \ (-\gamma_0, \chi_+(\gamma_0)) = (-1.146, 1.315), \\ &(\gamma_0, \chi_-(\gamma_0)) = (1.146, -1.315),\ \ (-\gamma_0, \chi_-(\gamma_0)) = (-1.146, -1.826). \end{aligned}$$

Now we consider the case of $\kappa = 3$. When $\eta_1 = 0 = \eta_2$ we have $\gamma = \pm\gamma_0$ and $\chi = \chi_{\pm}(\gamma) \equiv \frac{1}{2}\arctan\frac{\sin 2\gamma}{2+\cos 2\gamma} \pm \frac{\pi}{2}$; therefore, we must consider four points in (γ, χ) plane:

$$(\gamma_0, \chi_+(\gamma_0)),\ (-\gamma_0, \chi_+(-\gamma_0)),\ (\gamma_0, \chi_-(\gamma_0)) \text{ and } (-\gamma_0, \chi_-(-\gamma_0)).$$

In the first two points, we find

$$\delta = \delta\left(\gamma, \chi_{+}\left(\gamma\right)\right) = \frac{|\beta|}{|\alpha|}\frac{i}{\sqrt{\left(2+\cos 2\gamma\right)^{2}+\left(\sin 2\gamma\right)^{2}}} = i\left|\delta\right|$$

for $-\gamma_0 \le \gamma \le \gamma_0$ and in the last two points

$$\delta = \delta\left(\gamma, \chi_{-}\left(\gamma\right)\right) = -\frac{|\beta|}{|\alpha|}\frac{i}{\sqrt{\left(2+\cos 2\gamma\right)^{2}+\left(\sin 2\gamma\right)^{2}}} = -i\left|\delta\right|$$

for $-\gamma_0 \le \gamma \le \gamma_0$. In order to consider the sign of η_3 we need to compute the integral

$$\eta_4\left(\gamma\right) = \operatorname{Im}\left(\int_0^1 \frac{dz}{z}\left(\frac{2\nu\left(\frac{\mu_2\left(1-\frac{\sigma}{2}z\right)}{1-\sigma^2 z}\right)}{\sqrt{1-\sigma^2 z}} - 2\nu\left(\mu_2\right) - \frac{\nu\left(\frac{\sigma(\sigma-bz)}{1-\sigma^2 z}\right)}{\sqrt{1-\sigma^2 z}}\right.\right.$$
$$\left.+\nu\left(\sigma^2\right) - \frac{6\left(\nu\left(\sigma h\right)-\nu\left(\sigma^2\right)\right)}{\sqrt{1-\sigma^2 z}} + \frac{3\left(\nu\left(\sigma a\right)-\nu\left(\sigma^2\right)\right)}{\sqrt{1-\mu_2 z}}\right)$$
$$\left.+3\nu\left(\sigma^2 z\right) - 3\nu\left(\sigma^2\right)\left(2\nu\left(\sigma^2\right)-\nu\left(\mu_2\right)\right)\right)$$

which depends on γ. Numerically we see that $\eta_4\left(\gamma\right) > 0$ for $0 < \gamma < \frac{\pi}{2}$ and $\eta_4\left(\gamma\right) < 0$ for $-\frac{\pi}{2} < \gamma < 0$ and $\eta_4\left(0\right) = 0$. Therefore,

$$\eta_3\left(\gamma_0, \chi_{+}\left(\gamma_0\right)\right) = \frac{6\left|\delta\left(\gamma_0, \chi_{+}\left(\gamma_0\right)\right)\right|^3 \vartheta^6}{\left(4\pi\right)^3}\eta_4\left(\gamma_0\right) > 0,$$
$$\eta_3\left(-\gamma_0, \chi_{+}\left(-\gamma_0\right)\right) = \frac{6\left|\delta\left(-\gamma_0, \chi_{+}\left(-\gamma_0\right)\right)\right|^3 \vartheta^6}{\left(4\pi\right)^3}\eta_4\left(-\gamma_0\right) < 0$$
$$\eta_3\left(\gamma_0, \chi_{-}\left(\gamma_0\right)\right) = -\frac{6\left|\delta\left(\gamma_0, \chi_{-}\left(\gamma_0\right)\right)\right|^3 \vartheta^6}{\left(4\pi\right)^3}\eta_4\left(\gamma_0\right) < 0,$$
$$\eta_3\left(-\gamma_0, \chi_{-}\left(-\gamma_0\right)\right) = -\frac{6\left|\delta\left(-\gamma_0, \chi_{-}\left(-\gamma_0\right)\right)\right|^3 \vartheta^6}{\left(4\pi\right)^3}\eta_4\left(-\gamma_0\right) > 0,$$

which implies that in the third case of $\kappa = 3$ we consider the range of values α, β such that

$$\left(\gamma, \chi\right) = \left(\gamma_0, \chi_{+}\left(\gamma_0\right)\right), \text{ or } \left(\gamma, \chi\right) = \left(-\gamma_0, \chi_{-}\left(-\gamma_0\right)\right).$$

Define $\Gamma_0\left(\xi\right) = G\left(1, \xi\right)$ with the Green function $G\left(t, x\right) = \frac{1}{\sqrt{4\pi\alpha t}}e^{-|x|^2/4\alpha t}$ and the functions $\Gamma_j\left(\xi\right)$ we define by the recurrent relations

$$\Gamma_j\left(\xi\right) = \int_0^1 \frac{dz}{z}\int_{\mathbf{R}^n} G(1-z, \xi - y\sqrt{z})h_{j-1}\left(y\right)dy, \ j = 1, 2, ..., \tag{3.98}$$

and

$$h_n = \sum_{j+k+l=n} \Gamma_j \Gamma_k \overline{\Gamma_l} - \sum_{j+k+l+m=n} \Gamma_j \int_{\mathbf{R}^n} \Gamma_k \Gamma_l \overline{\Gamma_m} d\xi, \; n = 0, 1, 2, ...$$

Below we show that $\Gamma_j(\xi)$ are well-defined and $\|\Gamma_j\|_{\mathbf{L}^{p,a}} < \infty$ for all $j = 0, 1, 2, ...,$ where $a \in (0,1)$, $1 \le p \le \infty$.

We prove the following result.

Theorem 3.28. *Let $\kappa = 1, 2$ or 3. Assume that the initial data $\phi \in \mathbf{L}^{1,a}(\mathbf{R}) \cap \mathbf{L}^r(\mathbf{R})$, $r > 1$, $a \in (0,1)$ have a sufficiently small norm $\|\phi\|_{\mathbf{L}^r} + \|\phi\|_{\mathbf{L}^{1,a}} = \varepsilon$ and the mean value $\vartheta = \left|\int_{\mathbf{R}} \phi(x)\, dx\right| \ge C\varepsilon$. Then there exists a unique solution $u \in \mathbf{C}\left([0,\infty); \mathbf{L}^{1,a}(\mathbf{R}) \cap \mathbf{L}^r(\mathbf{R})\right) \cap \mathbf{C}\left((0,\infty); \mathbf{L}^\infty(\mathbf{R})\right)$ of the Cauchy problem (3.68), which can be represented as the uniformly convergent series*

$$u = e^{i\psi} g^{-\frac{1}{2\kappa}} \sum_{j=0}^{\infty} v_j. \tag{3.99}$$

Formula (3.99) gives us the asymptotic expansion of solution $u(t,x)$ for large values of time t, since the functions $v_j(t,x)$ have asymptotics

$$v_j(t,x) = \frac{\vartheta^{2j+1}}{\sqrt{t}} \Gamma_j\left(\frac{x}{\sqrt{t}}\right) g^{-\frac{1+2j}{2\kappa}} + O\left(\varepsilon^{2j+3} t^{-\frac{1}{2}} g^{-\frac{3+2j}{2\kappa}}\right),$$

where $g(t)$ has the asymptotic representation

$$g(t) = 1 + \eta_1 \log t + O\left(\varepsilon^2 \log(1 + \eta_1 \log t)\right)$$

if $\kappa = 1$,

$$g(t) = 1 + \eta_\kappa \log t + O\left(\varepsilon^2 (1 + \eta_\kappa \log t)^{\frac{1}{\kappa}}\right)$$

if $\kappa = 2, 3$. In addition ψ has the asymptotic behavior for large time $t \to \infty$

$$\psi(t) = -\frac{\omega}{\eta_1} \log\log t + \arg\widehat{\phi}(0) + O\left(\varepsilon^2\right)$$

if $\kappa = 1$,

$$\psi(t) = -\frac{2\omega}{\sqrt{\eta_2}} \sqrt{\log t} - \frac{\omega^2}{\eta_2} \log\log t + \arg\widehat{\phi}(0) + O\left(\varepsilon^2\right)$$

if $\kappa = 2$, and

$$\psi(t) = -\frac{3\omega}{2\sqrt[3]{\eta_3}} \log^{\frac{2}{3}} t - \frac{3\omega^2}{\sqrt{\eta_3}} \log^{\frac{1}{3}} t - \frac{\omega^3}{\eta_3} \log\log t + \arg\widehat{\phi}(0) + O\left(\varepsilon^2\right)$$

if $\kappa = 3$, where $\omega = \frac{\vartheta^2}{4\pi} \operatorname{Im} \delta$.

Before proving Theorem 3.28 we prepare some preliminary estimates. In Lemma 3.29 we prove that the functions $\Gamma_j(\xi)$ (see formulas (3.98)) belong to the space $\mathbf{L}^{p,a}$ for any $1 \le p \le \infty$, $a \in (0,1)$. The estimates and asymptotic formulas of the functions v_j from the series (3.99) are obtained in Lemma 3.30. Finally in Lemma 3.31 we give the estimates of some integrals which are necessary to find the large time asymptotics of the function $g(t)$.

Preliminaries

Denote $\xi = \frac{x}{\sqrt{t}}$, $\Gamma_0(\xi) = G(1,\xi)$, and the functions $\Gamma_j(\xi)$ we define by the recurrent relations (3.98)

Lemma 3.29. *The estimates* $\|\Gamma_m\|_{\mathbf{L}^{p,a}} \le C$ *are valid for all* $1 \le p \le \infty$, $a \in (0,1)$.

Proof. For the case of $m = 0$ we have

$$\|\Gamma_0\|_{\mathbf{L}^{p,a}} \le C \left\| \langle \xi \rangle^a e^{-C\xi^2} \right\|_{\mathbf{L}^p} \le C;$$

by induction we then get

$$\|h_{m-1}\|_{\mathbf{L}^{1,a}} \le C.$$

Since the mean value $\int_{\mathbf{R}} h_{m-1}(y)\, dy = 0$ we write for $b \in [0,a]$, $a > 0$

$$\begin{aligned}
&|\xi|^b \int_{\mathbf{R}} G(1-z, \xi - y\sqrt{z}) h_{m-1}(y)\, dy \\
&= \frac{1}{\sqrt{4\pi\alpha(1-z)}} \int_{\mathbf{R}} \frac{|\xi|^b}{|y|^a} \left(e^{-\frac{(\xi - y\sqrt{z})^2}{4\alpha(1-z)}} - e^{-\frac{\xi^2}{4\alpha(1-z)}} \right) |y|^a h_{m-1}(y)\, dy. \quad (3.100)
\end{aligned}$$

Changing $\zeta = \frac{\xi}{\sqrt{1-z}}$ and $\eta = \frac{y\sqrt{z}}{\sqrt{1-z}}$ we obtain as in the proof of Lemma 3.24

$$\begin{aligned}
&\sup_{y \in \mathbf{R}} \left\| \frac{|\xi|^b}{|y|^a} \left(e^{-\frac{(\xi - y\sqrt{z})^2}{4\alpha(1-z)}} - e^{-\frac{\xi^2}{4\alpha(1-z)}} \right) \right\|_{\mathbf{L}^p} \\
&= z^{\frac{a}{2}} (1-z)^{\frac{1}{2p} + \frac{b-a}{2}} \sup_{\eta \in \mathbf{R}} \left(\int_{\mathbf{R}} \frac{|\zeta|^{pb}}{|\eta|^{pa}} \left| e^{-\frac{(\zeta-\eta)^2}{4\alpha}} - e^{-\frac{\zeta^2}{4\alpha}} \right|^p d\zeta \right)^{\frac{1}{p}} \\
&\le C z^{\frac{a}{2}} (1-z)^{\frac{1}{2p} + \frac{b-a}{2}} \sup_{\eta \in \mathbf{R}} \left(\int_{\mathbf{R}} \left(e^{-C(\zeta-\eta)^2} + e^{-C\zeta^2} \right) d\zeta \right)^{\frac{1}{p}} \\
&\le C z^{\frac{a}{2}} (1-z)^{\frac{1}{2p} + \frac{b-a}{2}} \qquad (3.101)
\end{aligned}$$

for all $z \in (0,1)$, where $p \in [1,\infty)$. (The case of $p = \infty$ is considered in the same way). The substitution of (3.101) into (3.100) yields

$$\left\| |\xi|^b \, \Gamma_m(\xi) \right\|_{\mathbf{L}^p} = \left\| |\xi|^b \int_0^1 \frac{dz}{z} \int_{\mathbf{R}} G(1-z, \xi - y\sqrt{z}) h_{m-1}(y)\, dy \right\|_{\mathbf{L}^p}$$
$$\le C \left\| |\cdot|^a h_{m-1} \right\|_{\mathbf{L}^1} \int_0^1 \frac{dz}{z\sqrt{1-z}} \sup_{y \in \mathbf{R}} \left\| \frac{|\xi|^b}{|y|^a} \left(e^{-\frac{(\xi - y\sqrt{z})^2}{4\alpha(1-z)}} - e^{-\frac{\xi^2}{4\alpha(1-z)}} \right) \right\|_{\mathbf{L}^p}$$
$$\le C \int_0^1 z^{\frac{a}{2}-1} (1-z)^{\frac{1}{2p} - \frac{1}{2} + \frac{b-a}{2}} dz \le C$$

for all $0 \le b \le a$, $1 \le p \le \infty$. Lemma 3.29 is proved.

Consider the Green operator

$$\mathcal{G}(t)\varphi = \int_{\mathbf{R}} G(t, x-y)\varphi(y)\, dy,$$

with kernel $G(t,x) = (4\pi\alpha t)^{-\frac{1}{2}} e^{-x^2/4\alpha t}$. Denote $\vartheta = \sqrt{2\pi}\hat{\varphi}(0)$.

We denote $v_0(t) = \mathcal{G}(t)\phi_0$ and define v_m by the recurrent relations

$$v_m(t) = \int_0^t \mathcal{G}(t-\tau) g^{-\frac{1}{\kappa}} f_{m-1}(\tau)\, d\tau, \tag{3.102}$$

for $m = 1, 2, 3, ...$, where $\kappa = 1, 2, 3$ and

$$f_n = \sum_{j+k+l=n} v_j v_k \overline{v_l} - \frac{1}{\vartheta} \sum_{j+k+l+m=n} v_j \int_{\mathbf{R}} v_k v_l \overline{v_m} dx,$$

for $n = 0, 1, 2, ...$, where $\vartheta = \int_{\mathbf{R}} \phi_0(x)\, dx$. Denote $g(t) = 1 + \zeta \log \langle t \rangle$ with some $\zeta > 0$.

Lemma 3.30. *Assume that* $\phi_0 \in \mathbf{L}^{1,a}(\mathbf{R}) \cap \mathbf{L}^r(\mathbf{R})$, $r > 1$, $a \in (0,1)$, *have a sufficiently small norm* $\|\phi_0\|_{\mathbf{L}^r} + \|\phi_0\|_{\mathbf{L}^{1,a}} = \varepsilon$ *and* $\hat{\phi}_0(0) = \vartheta(2\pi)^{-\frac{1}{2}} \ge C\varepsilon$. *Then the estimate is true*

$$\left\| |\cdot|^b v_m(t) \right\|_{\mathbf{L}^p} \le C\varepsilon^{2m+1} g^{-\frac{m}{\kappa}}(t) \{t\}^{\frac{1}{2r}\left(1-\frac{1}{p}\right)} t^{\frac{b}{2} - \frac{1}{2}\left(1-\frac{1}{p}\right)} \tag{3.103}$$

for all $t > 0$, *where* $1 \le p \le \infty$, $b \in [0,a]$. *Moreover, the asymptotics*

$$v_m(t) = \vartheta^{2m+1} t^{-\frac{1}{2}} g^{-\frac{m}{\kappa}}(t)\, \Gamma_m(\xi) + O\left(\varepsilon^{2m+1} t^{-\frac{1}{2}} g^{-\frac{m}{\kappa}-1}(t)\right) \tag{3.104}$$

is valid for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}$ *where* $\xi = \frac{x}{\sqrt{t}}$.

Proof. Via Lemma 3.24 we have

$$\|v_0(t)\|_{\mathbf{L}^{p,b}} \le C\varepsilon \{t\}^{\frac{1}{2r}\left(1-\frac{1}{p}\right)} t^{\frac{b}{2} - \frac{1}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$, where $1 \le p \le \infty$, $b \in [0,a]$. Then by induction we get

$$\left\| |\cdot|^b g^{\frac{m-1}{\kappa}}(t) f_{m-1}(t) \right\|_{\mathbf{L}^p} \le C\varepsilon^{2m+1} \{t\}^{\frac{1}{r}} t^{\frac{b}{2}-1-\frac{1}{2}\left(1-\frac{1}{p}\right)}$$

and functions f_{m-1} satisfy the condition $\widehat{f_{m-1}}(t,0) = 0$, therefore by virtue of Lemma 3.25 we obtain

$$\left\| |\cdot|^b v_m(t) \right\|_{\mathbf{L}^p} \le \left\| |\cdot|^b \int_0^t \mathcal{G}(t-\tau) g^{-\frac{m}{\kappa}} \left(g^{\frac{m-1}{\kappa}} f_{m-1} \right) d\tau \right\|_{\mathbf{L}^p}$$
$$\le C\varepsilon^{2m+1} g^{-\frac{m}{\kappa}}(t) \{t\}^{\frac{1}{r}} t^{\frac{b}{2}-\frac{1}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$, where $b \in [0,a]$. Thus estimate (3.103) is true for all $m \ge 0$.

Now we prove asymptotics (3.104). Since $G(t,x) = t^{-\frac{1}{2}} \Gamma_0(\xi)$, by applying Lemma 3.24 we find

$$\left\| v_0(t) - \vartheta t^{-\frac{1}{2}} \Gamma_0(\xi) \right\|_{\mathbf{L}^p} \le C t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{a}{2}} \left(\|\phi_0\|_{\mathbf{L}^r} + \|\phi_0\|_{\mathbf{L}^{1,a}} \right)$$

for any $1 \le p \le \infty$. Hence asymptotics (3.104) with $m = 0$ is valid. Turning to $m \ge 1$ and integrating by parts with respect to time τ we get

$$v_m(t) = -\beta g^{-\frac{m}{\kappa}}(t) \int_0^t \mathcal{G}(t-\tau) \left(g^{\frac{m-1}{\kappa}} f_{m-1} \right)(\tau)\, d\tau + R(t),$$

where

$$R(t) = C \int_0^t \frac{d\tau g'(\tau)}{g^{\frac{m}{\kappa}+1}(\tau)} \int_0^\tau \mathcal{G}(t-z) \left(g^{\frac{m-1}{\kappa}} f_{m-1} \right)(z)\, dz.$$

By using the conditions on the function $g(t)$, we have

$$\|R(t)\|_{\mathbf{L}^p} \le C \int_0^{t/2} \frac{d\tau}{\tau g^{\frac{m}{\kappa}+1}(\tau)} \int_0^\tau (t-z)^{-\frac{a}{2}-\frac{1}{2}\left(1-\frac{1}{p}\right)} \left\| |\cdot|^a f_{m-1} \right\|_{\mathbf{L}^1} dz$$
$$+ C \int_{t/2}^t \frac{d\tau}{\tau g^{\frac{m}{\kappa}+1}(\tau)} \int_0^{\tau/2} (t-z)^{-\frac{a}{2}-\frac{1}{2}\left(1-\frac{1}{p}\right)} \left\| |\cdot|^a f_{m-1} \right\|_{\mathbf{L}^1} dz$$
$$+ C \int_{t/2}^t \frac{d\tau}{\tau g^{\frac{m}{\kappa}+1}(\tau)} \int_{\tau/2}^\tau (t-z)^{-\frac{a}{2}} \left\| |\cdot|^a f_{m-1} \right\|_{\mathbf{L}^p} dz;$$

hence

$$\|R(t)\|_{\mathbf{L}^p} \le C\varepsilon^{2m+1} \int_0^{\sqrt{t}} (t-\tau)^{-\frac{a}{2}-\frac{1}{2}\left(1-\frac{1}{p}\right)} \tau^{\frac{a}{2}-1} d\tau$$
$$+ C\varepsilon^{2m+1} g^{-\frac{m}{\kappa}-1}(t)\, t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \int_{\sqrt{t}}^t (t-\tau)^{-\frac{a}{2}} \tau^{\frac{a}{2}-1} d\tau$$
$$\le C\varepsilon^{2m+1} t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \left(t^{-\frac{a}{4}} + g^{-\frac{m}{\kappa}-1}(t) \right)$$
$$\le C\varepsilon^{2m+1} g^{-\frac{m}{\kappa}-1}(t)\, t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}.$$

Then we write the representation

$$v_m = -\beta g^{-\frac{m}{\kappa}} \int_0^t \frac{d\tau}{\tau^{\frac{3}{2}}} \int_{\mathbf{R}} G(t-\tau, x-y)\, h_{m-1}\left(\frac{y}{\sqrt{\tau}}\right) dy + R_1(t), \quad (3.105)$$

where

$$R_1(t) = R(t) - \beta g^{-\frac{m}{\kappa}}(t) \int_0^t d\tau \int_{\mathbf{R}} G(t-\tau, x-y)\, g^{-1}(\tau)\, M(\tau, y)\, dy$$

and

$$M(\tau, y) = g^{1+\frac{m-1}{\kappa}}(\tau)\, f_{m-1}(\tau, y) - \vartheta^{2m+1} \tau^{-\frac{3}{2}} g(\tau)\, h_{m-1}\left(\frac{y}{\sqrt{\tau}}\right).$$

By induction we see that $M(\tau)$ satisfies the estimate

$$\left\| |\cdot|^b M(\tau) \right\|_{\mathbf{L}^p} \le C\varepsilon^{2m+1} \tau^{\frac{b}{2}-1-\frac{1}{2}\left(1-\frac{1}{p}\right)}$$

for all $\tau > 0$ and $1 \le p \le \infty$, where $0 \le b \le a$. Therefore by virtue of Lemma 3.25 we get

$$\|R_1(t)\|_{\mathbf{L}^p} \le C\varepsilon^{2m+1} g^{-\frac{m}{\kappa}-1}(t)\, t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$ and $1 \le p \le \infty$. Now changing the variables of integration $y = y'\sqrt{\tau}$ and $\tau = zt$ in the integral (3.105) and taking into account the identity

$$\sqrt{t} G\left(t(1-z), \sqrt{t}\left(\xi - y'\sqrt{z}\right)\right) = G\left(1-z, \xi - y'\sqrt{z}\right),$$

we get asymptotics (3.104). Lemma 3.30 is then proved.

In the next lemma we sum the results of Lemmas 3.25 through 3.26. Denote $r_m = v(t) - u_m(t)$, where $u_m = \sum_{j=0}^{m-1} v_j$.

Lemma 3.31. *Assume that* $\phi \in \mathbf{L}^{1,a}(\mathbf{R}) \cap \mathbf{L}^r(\mathbf{R})$, $r > 1$, $a \in (0,1)$, *the norm* $\|\phi\|_{\mathbf{L}^r} + \|\phi\|_{\mathbf{L}^{1,a}} = \varepsilon$ *is sufficiently small,* $\hat{\phi}(0) = \vartheta(2\pi)^{-\frac{1}{2}} \ge C\varepsilon$. *Let the function* $r_m(t,x)$ *satisfy the estimates*

$$\|r_m(t)\|_{\mathbf{L}^p} \le \varepsilon^{2m+1} g^{-\frac{m}{\kappa}}(t)\, \{t\}^{\rho}\, t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)},\ m \ge \kappa$$

for all $t > 0$, $3 \le p \le \infty$, *where* $\rho > 0$. *Also assume that the function* $g(t)$ *is such that*

$$\frac{1}{2}\left(1 + \zeta \log\langle t\rangle\right) \le g(t) \le 2\left(1 + \zeta \log\langle t\rangle\right)$$

with some $\zeta > 0$ *and* $|g'(t)| \le C\varepsilon^2 t^{-1}$ *for all* $t > 0$.

Then the following inequalities are valid

$$\frac{1}{\vartheta}\left| g^{1-\frac{1}{\kappa}}(t)\, \mathrm{Re} \int_{\mathbf{R}} \beta |v|^2 v(t,x)\, dx \right| \le C\varepsilon^2 t^{-1} \{t\}^{\rho}$$

and

$$\frac{9}{10}\left(1+\eta_\kappa \log\langle t\rangle\right) \le 1+\frac{2\kappa}{\vartheta}\int_0^t d\tau g^{1-\frac{1}{\kappa}}(\tau)\operatorname{Re}\int_{\mathbf{R}}\beta|v|^2 v(\tau,x)\,dx$$
$$\le \frac{11}{10}\left(1+\eta_\kappa \log\langle t\rangle\right)$$

for all $t>0$, where η_κ are defined above.

Proof of Theorem 3.28

As in the proof of Theorem3.2 we change the dependent variable $u(t,x) = e^{-\varphi(t)+i\psi(t)}v(t,x)$, then we get system (3.70) from the Landau - Ginzburg equation (3.68). Multiplying the second equation of system (3.70) by the factor $g(t) = e^{2\kappa\varphi(t)}$, ($\kappa = 1, 2$ or 3 is defined after (3.97)), and then integrating with respect to time $t > 0$, we obtain

$$\begin{cases} v(t) = \mathcal{G}(t)\,\phi_0 - \beta\int_0^t g^{-\frac{1}{\kappa}}(\tau)\,\mathcal{G}(t-\tau)F_1 d\tau, \\ g(t) = 1+\frac{2\kappa}{\vartheta}\int_0^t g^{1-\frac{1}{\kappa}}(\tau)\operatorname{Re}\int_{\mathbf{R}}\beta|v|^2 v(\tau,x)\,dxd\tau. \end{cases}$$

We find a solution v of the above system of equations in the neighborhood of

$$u_m(t) = \sum_{j=0}^{m-1} v_j(t),$$

where $v_0(t) = \mathcal{G}(t)\,\phi_0$ and v_j are defined by the recurrent relations (3.102). We set $r = v - u_m$, $m \ge \kappa$, then we acquire

$$\begin{cases} r(t) = -\beta\int_0^t \mathcal{G}(t-\tau)g^{-\frac{1}{\kappa}}(\tau)\,F_m(\tau)\,d\tau, \\ g(t) = 1+\frac{2\kappa}{\vartheta}\int_0^t g^{1-\frac{1}{\kappa}}(\tau)\operatorname{Re}\int_{\mathbf{R}}\beta|v|^2 v(\tau,x)\,dxd\tau, \end{cases}$$

where

$$F_m = F_1 - \sum_{j=0}^{m-2} f_j$$

for $m \ge 2$. We define the mappings $\mathcal{M}(r,g)$ and $\mathcal{R}(r,g)$ by

$$\begin{cases} \mathcal{M}(r,g) = -\beta\int_0^t \mathcal{G}(t-\tau)g^{-\frac{1}{k}}(\tau)\,F_m(\tau)\,d\tau, \\ \mathcal{R}(r,g) = 1+\frac{2\kappa}{\vartheta}\int_0^t g^{1-\frac{1}{\kappa}}(\tau)\operatorname{Re}\int_{\mathbf{R}}\beta|v|^2 v(\tau,x)\,dxd\tau. \end{cases} \tag{3.106}$$

We prove that $(\mathcal{M},\mathcal{R})$ is the contraction mapping in the set

$$\mathbf{X} = \left\{r \in \mathbf{C}\left((0,\infty);\mathbf{L}^{1,\lambda}(\mathbf{R})\cap\mathbf{L}^{\infty,\lambda}(\mathbf{R})\right), g \in \mathbf{C}(0,\infty):\right.$$
$$\sup_{t>0}\sup_{1\le p\le\infty}\sup_{b\in[0,\lambda]}\{t\}^{-\rho}t^{-\frac{b}{2}+\frac{1}{2}\left(1-\frac{1}{p}\right)}g^{\frac{m}{\kappa}}(t)\left\||\cdot|^b r(t)\right\|_{\mathbf{L}^p} \le C\varepsilon^{2m+1},$$
$$\frac{1}{2}\left(1+\eta_\kappa\log\langle t\rangle\right) \le g(t) \le 2\left(1+\eta_\kappa\log\langle t\rangle\right), \text{ for all } t>0,$$

where $\rho = \frac{1}{2}\left(1-\frac{1}{r}\right) > 0,\ \lambda \in (0,a)$. The values η_κ were defined after (3.97). First we prove that the mapping $(\mathcal{M},\mathcal{R})$ transforms the set $\mathbf{X}$ into itself. When $(r,g) \in \mathbf{X}$ by virtue of Lemma 3.30 we get

$$\left\| |\cdot|^b v \right\|_{\mathbf{L}^p} \le C\varepsilon \{t\}^{\rho\left(1-\frac{1}{p}\right)} t^{\frac{b}{2}-\frac{1}{2}\left(1-\frac{1}{p}\right)};$$

we then have by a direct calculation

$$\left\| |\cdot|^b \left(|v|^2 v - |u_{m-1}|^2 u_{m-1} \right) \right\|_{\mathbf{L}^p} \le C\varepsilon^{2m+1} g^{-\frac{m-1}{\kappa}}(t) \{t\}^{\rho} t^{\frac{b}{2}-1-\frac{1}{2}\left(1-\frac{1}{p}\right)},$$

so

$$\sup_{t>0} \sup_{1\le p\le\infty} \{t\}^{-\rho} t^{1-\frac{b}{2}+\frac{1}{2}\left(1-\frac{1}{p}\right)} \left\| |\cdot|^b g^{\frac{m-1}{\kappa}}(t) F_m(t) \right\|_{\mathbf{L}^p} \le C\varepsilon^{2m+1}$$

where $b \in [0,\lambda]$. We also see that $\widehat{F_m}(t,0) = 0$ since $\widehat{v_j}(t,0) = 0$ if $j \ne 0$. Therefore, by applying Lemma 3.25 we get the estimates

$$\sup_{t>0} \sup_{1\le p\le\infty} g^{\frac{m}{\kappa}}(t) \{t\}^{-\rho} t^{-\frac{b}{2}+\frac{1}{2}\left(1-\frac{1}{p}\right)} \left\| |\cdot|^b \mathcal{M}(r,g)(t) \right\|_{\mathbf{L}^p} \le C\varepsilon^{2m+1}.$$

Furthermore Lemma 3.31 yields

$$\frac{1}{2}\left(1+\eta_\kappa \log\langle t\rangle\right) \le \mathcal{R}(r,g)(t) \le 2\left(1+\eta_\kappa \log\langle t\rangle\right)$$

for all $t > 0$, where $\eta_\kappa > 0$. Thus the transformations $\mathcal{M}(r,g)$ and $\mathcal{R}(r,g)$ are the contraction mappings from the set $\mathbf{X}$ into itself. We have by the $\mathbf{L}^2(\mathbf{R}) - \mathbf{L}^2(\mathbf{R})$ estimates of the heat kernel and by the fact that $(r_j, g_j) \in \mathbf{X}$, for $j = 1,2$

$$\begin{aligned}
&\sup_{t>0} \langle t\rangle^{-\epsilon} \left\| \mathcal{M}(r_1,g_1)(t) - \mathcal{M}(r_2,g_2)(t) \right\|_{\mathbf{L}^2} \\
&\le C\varepsilon^2 \sup_{t>0} \langle t\rangle^{-\epsilon} \int_0^t \frac{d\tau}{\tau^{1-\epsilon}} \left(\sup_{t>0} \langle t\rangle^{-\epsilon} \left\| r_1(t) - r_2(t) \right\|_{\mathbf{L}^2} \right. \\
&\left. + \sup_{t>0} \langle t\rangle^{-\epsilon} \left| g_1(t) - g_2(t) \right| \right) \\
&\le \varepsilon \left(\sup_{t>0} \langle t\rangle^{-\epsilon} \left\| r_1(t) - r_2(t) \right\|_{\mathbf{L}^2} + \sup_{t>0} \langle t\rangle^{-\epsilon} \left| g_1(t) - g_2(t) \right| \right)
\end{aligned}$$

with some small $\epsilon > 0$ and

$$\begin{aligned}
&\sup_{t>0} \langle t\rangle^{-\epsilon} \left| \mathcal{R}(r_1,g_1)(t) - \mathcal{R}(r_2,g_2)(t) \right| \\
&\le \sup_{t>0} C\varepsilon^2 \langle t\rangle^{-\epsilon} \int_0^t \frac{d\tau}{\tau^{1-\epsilon}} \left(\sup_{t>0} \langle t\rangle^{-\epsilon} \left\| r_1(t) - r_2(t) \right\|_{\mathbf{L}^2} \right. \\
&\left. + \sup_{t>0} \langle t\rangle^{-\epsilon} \left| g_1(t) - g_2(t) \right| \right) \\
&\le \varepsilon \left(\sup_{t>0} \langle t\rangle^{-\epsilon} \left\| r_1(t) - r_2(t) \right\|_{\mathbf{L}^2} + \sup_{t>0} \langle t\rangle^{-\epsilon} \left| g_1(t) - g_2(t) \right| \right).
\end{aligned}$$

Thus we see that $(\mathcal{M},\mathcal{R})$ is the contraction mapping. Therefore there exists a unique solution (r,g) of the system of integral equations (3.106) in the set $\mathbf{X}$.

Now in the case of $\kappa = 1$ from Lemma 3.31 we have $g(t) = \eta_1 \log t + O(\varepsilon \log\log t)$ as $t \to \infty$ and by (3.70) we obtain

$$\begin{aligned}
\psi'(t) &= -\frac{1}{\vartheta} g^{-1}(t) \operatorname{Im} \int_{\mathbf{R}} \beta |v|^2 v dx \\
&= -\vartheta^2 g^{-1}(t) \operatorname{Im} \int_{\mathbf{R}} \beta |G|^2 G dx + O\left(\frac{\varepsilon^2}{\langle t\rangle} g^{-2}(t)\right) \\
&= -\frac{\omega}{(1+\eta_1 \log\langle t\rangle)\langle t\rangle} + O\left(\frac{\varepsilon^2}{\langle t\rangle} g^{-2}(t)\right),
\end{aligned}$$

where $\omega = \frac{\vartheta^2}{4\pi} \operatorname{Im} \delta(\alpha,\beta)$. Hence we get

$$\begin{aligned}
\psi(t) &= \psi(0) - \int_0^t \left(\frac{\omega}{1+\eta_1 \log\langle \tau\rangle} + O\left(\varepsilon^2 g^{-2}(\tau)\right)\right)\frac{d\tau}{\langle\tau\rangle} \\
&= \arg \widehat{u}_0(0) - \frac{\omega}{\eta_1} \log(1+\eta_1 \log\langle t\rangle) + O\left(\varepsilon^2\right).
\end{aligned}$$

Therefore via formulas

$$u(t,x) = e^{-\varphi(t)+i\psi(t)} v(t,x) = e^{-\varphi(t)+i\psi(t)}(u_m + r),$$

estimates of v_j given in Lemma 3.30 and the estimate of the remainder r, we obtain the asymptotics (3.99) of the solution to the Cauchy problem (3.68). Theorem 3.28 is proved for the case of $\kappa = 1$.

In the case of $\kappa = 2$ as above we get

$$\begin{aligned}
\psi'(t) &= -\frac{1}{\vartheta} g^{-\frac{1}{2}}(t) \operatorname{Im} \beta \int_{\mathbf{R}} |v|^2 v dx \\
&= -\frac{1}{\vartheta} g^{-\frac{1}{2}}(t) \operatorname{Im} \beta \left(\int_{\mathbf{R}} |v_0|^2 v_0 + 2v_1 |v_0|^2 + v_0^2 \overline{v_1} dx\right) + O\left(\frac{\varepsilon^6}{\langle t\rangle} g^{-\frac{3}{2}}(t)\right) \\
&= -\frac{\omega}{\sqrt{1+\eta_2 \log\langle t\rangle}(\langle t\rangle)} - \frac{\omega^2}{(1+\eta_2 \log\langle t\rangle)(\langle t\rangle)} + O\left(\frac{\varepsilon^6}{\langle t\rangle} g^{-\frac{3}{2}}(t)\right),
\end{aligned}$$

where we also denote $\omega = \frac{\vartheta^2}{4\pi} \operatorname{Im} \delta(\alpha,\beta)$. Hence we get

$$\begin{aligned}
\psi(t) &= \psi(0) - \int_0^t \left(\frac{\omega}{\sqrt{1+\eta_2 \log\langle\tau\rangle}} + \frac{\omega^2}{(1+\eta_2 \log\langle\tau\rangle)}\right. \\
&\quad \left. + O\left(\varepsilon^6 g^{-\frac{3}{2}}(\tau)\right)\right)\frac{d\tau}{\langle\tau\rangle} \\
&= \psi(0) - 2\omega\eta_2^{-1/2}\sqrt{\log t} - \omega^2 \eta_2^{-1}\sqrt{\log t} + O(\varepsilon^2).
\end{aligned}$$

Thus Theorem 3.28 is proved in the case of $\kappa = 2$.

In the case of $\kappa = 3$ we have

$$\begin{aligned}
\psi'(t) &= -\frac{1}{\vartheta} g^{-\frac{1}{3}}(t) \operatorname{Im} \beta \int_{\mathbf{R}} |v|^2 v dx \\
&= -\frac{1}{\vartheta} g^{-\frac{1}{3}}(t) \operatorname{Im} \beta \left(\int_{\mathbf{R}} |v_0|^2 v_0 + 2v_1 |v_0|^2 + v_0^2 \overline{v_1} + 2v_2 |v_0|^2 \right. \\
&\left. + v_0^2 \overline{v_2} + 4v_0 |v_1|^2 + 2v_1^2 \overline{v_0} dx \right) + O\left(\frac{\varepsilon^8}{\langle t \rangle} g^{-\frac{4}{3}}(t) \right) \\
&= -\frac{\omega}{\sqrt[3]{1 + \eta_3 \log \langle t \rangle} \langle t \rangle} - \frac{\omega^2}{(1 + \eta_3 \log \langle t \rangle)^{\frac{2}{3}} \langle t \rangle} \\
&- \frac{\omega^3}{(1 + \eta_3 \log \langle t \rangle)(\langle t \rangle)} + O\left(\frac{\varepsilon^8}{\langle t \rangle} g^{-\frac{4}{3}}(t) \right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\psi(t) &= \psi(0) - \int_0^t \left(\frac{\omega}{\sqrt[3]{1 + \eta_3 \log \langle \tau \rangle}} + \frac{\omega^2}{(1 + \eta_3 \log \langle t \rangle)^{\frac{2}{3}}} \right. \\
&\left. + \frac{\omega^3}{(1 + \eta_3 \log \langle t \rangle)} \frac{d\tau}{\langle \tau \rangle} \right) + \int_0^t O\left(\frac{\varepsilon^8}{\langle t \rangle} g^{-\frac{4}{3}}(t) \right) dt \\
&= \psi(0) - \frac{3\omega}{2\eta_3^{\frac{1}{3}}} \log^{\frac{2}{3}} t - \frac{3\omega^2}{\eta_3^{\frac{2}{3}}} \log^{\frac{1}{3}} t - \frac{3\omega^3}{\eta_3} \log \log t + O\left(\varepsilon^2\right).
\end{aligned}$$

Thus we obtain the result of Theorem 3.28 in the case of $\kappa = 3$. Theorem 3.28 is proved.

3.5 Damped wave equation

3.5.1 Small initial data

This section is devoted to the study of the nonlinear damped wave equation

$$\begin{cases} \mathcal{L}u + \lambda \mathcal{N}(u) = 0, \ x \in \mathbf{R}^n, \ t > 0 \\ u(0, x) = \varepsilon u_0(x), \ \partial_t u(0, x) = \varepsilon u_1(x), \ x \in \mathbf{R}^n, \end{cases} \tag{3.107}$$

where $\mathcal{L} = \partial_t^2 + \partial_t - \Delta$, $\varepsilon > 0$, the critical nonlinearity $\mathcal{N}(u)$ is defined by

$$\mathcal{N}(u) = u^{1 + \frac{2}{n}}.$$

Denote

$$\theta = \int_{\mathbf{R}^n} (u_0(x) + u_1(x)) dx, \ \kappa = \frac{\lambda}{4\pi} (\varepsilon \theta)^{\frac{2}{n}} \left(\frac{n}{n+2} \right)^{\frac{n}{2}}.$$

Define $g(t) = 1 + \kappa \log \langle t \rangle$ and let

$$G(t,x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$$

be the heat kernel. The result of this section is the following.

Theorem 3.32. *Let the initial data u_0, u_1 be such that*

$$u_0 \in \mathbf{H}^{\delta,0}(\mathbf{R}^n) \cap \mathbf{H}^{0,\delta}(\mathbf{R}^n),\ u_1 \in \mathbf{H}^{\delta-1,0}(\mathbf{R}^n) \cap \mathbf{H}^{-1,\delta}(\mathbf{R}^n),$$

where $\delta > \frac{n}{2}$. Also we assume

$$\lambda\theta^{\frac{2}{n}} > 0,\ \int_{\mathbf{R}^n} u_0(x)\,dx > 0.$$

Then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \le \varepsilon_0$ the Cauchy problem (3.107) has a unique global solution $u \in \mathbf{C}\left([0,\infty); \mathbf{H}^{\delta,0}(\mathbf{R}^n)\right)$ satisfying the following asymptotic property

$$\left\| u(t) - \varepsilon\theta G(t,x) e^{-\varphi(t)} \right\|_{\mathbf{L}^\infty} \le C\varepsilon^{1+\frac{2}{n}} g^{-1-\frac{n}{2}}(t) \langle t\rangle^{-\frac{n}{2}}$$

where the function $\varphi(t)$ satisfy the estimate

$$\left| e^{\frac{2}{n}\varphi(t)} - g(t) \right| \le C\varepsilon^{\frac{2}{n}} \log g(t)$$

for all $t > 0$.

Remark 3.33. The nonlinearity $u^{1+\frac{2}{n}}$ can be replaced by $|u|^{\frac{2}{n}} u$ or $|u|^{1+\frac{2}{n}}$ if we assume $\lambda > 0$ or $\lambda\theta > 0$ instead of $\lambda\theta^{\frac{2}{n}} > 0$, respectively. In these cases $\kappa = \frac{\lambda}{4\pi}(\varepsilon|\theta|)^{\frac{2}{n}} \left(\frac{n}{n+2}\right)^{\frac{n}{2}}$ for $|u|^{\frac{2}{n}} u$ and $\kappa = \frac{\lambda}{4\pi}\varepsilon^{\frac{2}{n}} |\theta|^{1+\frac{2}{n}} \theta^{-1} \left(\frac{n}{n+2}\right)^{\frac{n}{2}}$. We note that our conditions always keep $\kappa > 0$.

Preliminary Lemmas

The solution of the linear Cauchy problem

$$\begin{cases} \mathcal{L}u = f,\ x \in \mathbf{R}^n,\ t > 0, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x),\ x \in \mathbf{R}^n \end{cases} \tag{3.108}$$

where $\mathcal{L} = \partial_t^2 + \partial_t - \Delta$, $\varepsilon > 0$; we write by the Duhamel formula

$$u(t) = \widetilde{\mathcal{G}}(t) u_0 + \mathcal{G}(t) u_1 + \int_0^t \mathcal{G}(t-\tau) f(\tau)\,d\tau, \tag{3.109}$$

where

$$\widetilde{\mathcal{G}}(t) = (\partial_t + 1)\mathcal{G}(t) = e^{-\frac{t}{2}} \overline{\mathcal{F}}_{\xi\to x} L_0(t,\xi) \mathcal{F}_{x\to\xi},$$
$$\mathcal{G}(t) = e^{-\frac{t}{2}} \overline{\mathcal{F}}_{\xi\to x} L_1(t,\xi) \mathcal{F}_{x\to\xi},$$

with

$$L_0(t,\xi) = \cos\left(t\sqrt{|\xi|^2 - \frac{1}{4}}\right) + \frac{1}{2}L_1(t,\xi),$$

$$L_1(t,\xi) = \frac{\sin\left(t\sqrt{|\xi|^2 - \frac{1}{4}}\right)}{\sqrt{|\xi|^2 - \frac{1}{4}}}.$$

Also we define the operators

$$\mathcal{G}'_j(t) = \overline{\mathcal{F}}_{\xi\to x}\frac{\partial}{\partial t}\left(e^{-\frac{t}{2}}L_j(t,\xi)\right)\mathcal{F}_{x\to\xi}$$

$j = 0,1$. Note that the symbols $L_0(t,\xi)$ and $L_1(t,\xi)$ are smooth and bounded: $L_j(t,\xi) \in \mathbf{C}^\infty(\mathbf{R}^n)$, $j = 0,1$. Moreover the symbol $L_1(t,\xi)$ decays as $\frac{1}{|\xi|}$ for $|\xi| \to \infty$ which means the gain of regularity concerning the initial datum u_1. Using Lemma 1.35 we get the following result

Lemma 3.34. *The estimates*

$$\left\|(-\Delta)^\alpha \widetilde{\mathcal{G}}(t)\phi\right\|_{\mathbf{L}^2} \le C\left\|(-\Delta)^\alpha\phi\right\|_{\mathbf{L}^2},$$

$$\left\|(-\Delta)^\alpha \mathcal{G}(t)\phi\right\|_{\mathbf{L}^2} \le C\left\|(-\Delta)^\alpha\langle\Delta\rangle^{-\frac{1}{2}}\phi\right\|_{\mathbf{L}^2},$$

$$\left\|(-\Delta)^\alpha\langle\Delta\rangle^{-\frac{1}{2}}\mathcal{G}'_j(t)\phi\right\|_{\mathbf{L}^2} \le C\langle t\rangle^{-1}\left\|(-\Delta)^\alpha\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2},$$

and

$$\left\||\cdot|^\alpha \widetilde{\mathcal{G}}(t)\phi\right\|_{\mathbf{L}^2} \le C\left\||\cdot|^\alpha\phi\right\|_{\mathbf{L}^2} + C\langle t\rangle^{\frac{\alpha}{2}}\|\phi\|_{\mathbf{L}^2},$$

$$\left\||\cdot|^\alpha \mathcal{G}(t)\phi\right\|_{\mathbf{L}^2} \le C\left\||\cdot|^\alpha\langle\Delta\rangle^{-\frac{1}{2}}\phi\right\|_{\mathbf{L}^2} + C\langle t\rangle^{\frac{\alpha}{2}}\left\|\langle\Delta\rangle^{-\frac{1}{2}}\phi\right\|_{\mathbf{L}^2}$$

are true for all $t \ge 0$, where $\alpha \ge 0$, provided that the right-hand sides are finite.

The following lemma says that the asymptotic behavior of solutions to the linear Cauchy problem (3.108) is similar to that of the heat equation. (For the proof see Lemma 1.37.)

Lemma 3.35. *Let $\mathcal{G}_0 = \widetilde{\mathcal{G}}$ and $\mathcal{G}_1 = \mathcal{G}$. The estimates*

$$\begin{aligned}
&\left\|(-\Delta)^{\frac{\alpha}{2}}\left(\mathcal{G}_j(t)\phi - G(t,x)\widehat{\phi}(0)\right)\right\|_{\mathbf{L}^2} \\
&\le Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}}\left\|(-\Delta)^\alpha\langle\Delta\rangle^{-\frac{j-1}{2}}\phi\right\|_{\mathbf{L}^2} \\
&+ Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}}\left\|\langle\cdot\rangle^\delta\langle\Delta\rangle^{-\frac{j-1}{2}}\phi\right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}}\left\|\langle\Delta\rangle^{-\frac{j-1}{2}}\phi\right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}} \\
&+ Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}}\left\|\langle\cdot\rangle^\delta\langle\Delta\rangle^{-\frac{j-1}{2}}\phi\right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}}\left\|\langle\Delta\rangle^{-\frac{j-1}{2}}\phi\right\|_{\mathbf{L}^2}^{1-\frac{n+2\gamma}{2\delta}},
\end{aligned}$$

$$\begin{aligned}
&\left\| (-\Delta)^{\frac{\alpha}{2}} \langle \Delta \rangle^{-\frac{1}{2}} \left(\mathcal{G}_j'(t)\phi - \Delta G(t,x)\widehat{\phi}(0) \right) \right\|_{\mathbf{L}^2} \\
&\leq Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1} \left\| (-\Delta)^{\alpha} \langle \Delta \rangle^{-\frac{j-1}{2}} \phi \right\|_{\mathbf{L}^2} \\
&+ Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1} \left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{j-1}{2}} \phi \right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}} \left\| \langle \Delta \rangle^{-\frac{j-1}{2}} \phi \right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}} \\
&+ Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1} \left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{j-1}{2}} \phi \right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}} \left\| \langle \Delta \rangle^{-\frac{j-1}{2}} \phi \right\|_{\mathbf{L}^2}^{1-\frac{n+2\gamma}{2\delta}}
\end{aligned}$$

and

$$\begin{aligned}
&\left\| |\cdot|^{\delta} \left(\mathcal{G}_j(t)\phi - G(t)\widehat{\phi}(0) \right) \right\|_{\mathbf{L}^2} \\
&\leq C \left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{j-1}{2}} \phi \right\|_{\mathbf{L}^2} \\
&+ Ct^{\frac{\delta-\gamma}{2}-\frac{n}{4}} \left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{j-1}{2}} \phi \right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}} \left\| \langle \Delta \rangle^{-\frac{j-1}{2}} \phi \right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}} \\
&+ Ct^{\frac{\delta-\gamma}{2}-\frac{n}{4}} \left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{j-1}{2}} \phi \right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}} \left\| \langle \Delta \rangle^{-\frac{j-1}{2}} \phi \right\|_{\mathbf{L}^2}^{1-\frac{n+2\gamma}{2\delta}}
\end{aligned}$$

are true for all $t \geq 1$, $j = 0, 1$, *where* $\delta > \frac{n}{2}$, $\alpha \geq 0$ *and* $0 < \gamma < \min\left(1, \delta - \frac{n}{2}\right)$, *provided that the right-hand sides are finite.*

We let

$$g(t) = 1 + \kappa \log \langle t \rangle$$

with some $\kappa > 0$, and we define two norms

$$\begin{aligned}
\|\phi\|_{\mathbf{X}} &= \sup_{t>0} \sup_{0 \leq \alpha \leq \delta} \langle t \rangle^{\frac{n}{4}+\frac{\alpha}{2}} \left\| (-\Delta)^{\frac{\alpha}{2}} \phi(t) \right\|_{\mathbf{L}^2} \\
&+ \sup_{t>0} \sup_{0 \leq \alpha \leq \delta} \langle t \rangle^{\frac{n}{4}+\frac{\alpha}{2}+\frac{1}{2}} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle \Delta \rangle^{-\frac{1}{2}} \partial_t \phi(t) \right\|_{\mathbf{L}^2} \\
&+ \sup_{t>0} \langle t \rangle^{\frac{n}{4}-\frac{\delta}{2}} \left\| |\cdot|^{\delta} \phi(t) \right\|_{\mathbf{L}^2}
\end{aligned}$$

and

$$\begin{aligned}
\|\phi\|_{\mathbf{Y}} &= \sup_{t>0} \sup_{0 \leq \alpha \leq \delta} \langle t \rangle^{1+\frac{n}{4}+\frac{\alpha}{2}} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle \Delta \rangle^{-\frac{1}{2}} \phi(t) \right\|_{\mathbf{L}^2} \\
&+ \sup_{t>0} \langle t \rangle^{1+\frac{n}{4}-\frac{\delta}{2}} \left\| |\cdot|^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} \phi(t) \right\|_{\mathbf{L}^2}.
\end{aligned}$$

Now we prove that the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ is concordant.

Lemma 3.36. *Let the function* $f(t,x)$ *have a zero mean value* $\widehat{f}(t,0) = 0$. *Then the following inequality*

$$\left\| g(t) \int_0^t g^{-1}(\tau) \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{X}} \leq C \|f\|_{\mathbf{Y}}$$

is valid, provided that the right-hand side is finite.

Proof. By Lemma 3.34 we get

$$\begin{aligned}
&\left\| (-\Delta)^{\frac{\alpha}{2}} \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau)\, f(\tau)\, d\tau \right\|_{\mathbf{L}^2} \\
&\leq C \int_0^t g^{-1}(\tau) \left\| (-\Delta)^{\frac{\alpha}{2}} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2} d\tau \leq C \|f\|_{\mathbf{Y}},
\end{aligned}$$

$$\begin{aligned}
&\left\| (-\Delta)^{\frac{\alpha}{2}} \langle \Delta \rangle^{-\frac{1}{2}} \partial_t \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau)\, f(\tau)\, d\tau \right\|_{\mathbf{L}^2} \\
&= \left\| (-\Delta)^{\frac{\alpha}{2}} \langle \Delta \rangle^{-\frac{1}{2}} \int_0^t g^{-1}(\tau)\, \mathcal{G}'(t-\tau)\, f(\tau)\, d\tau \right\|_{\mathbf{L}^2} \\
&\leq C \int_0^t g^{-1}(\tau) \left\| (-\Delta)^{\frac{\alpha}{2}} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2} d\tau \leq C \|f\|_{\mathbf{Y}}
\end{aligned}$$

and

$$\begin{aligned}
&\left\| |\cdot|^{\alpha} \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau)\, f(\tau)\, d\tau \right\|_{\mathbf{L}^2} \\
&\leq C \int_0^t g^{-1}(\tau) \langle t-\tau \rangle^{\frac{\alpha}{2}} \left\| \langle \cdot \rangle^{\alpha} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2} d\tau \leq C \|f\|_{\mathbf{Y}}
\end{aligned}$$

for all $t \in [0,1]$, where $\alpha \in [0,\delta]$. We now consider $t \geq 1$. In view of Lemma 3.35 we obtain

$$\begin{aligned}
&\left\| (-\Delta)^{\frac{\alpha}{2}} \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau)\, f(\tau)\, d\tau \right\|_{\mathbf{L}^2} \\
&\leq \int_0^{\frac{t}{2}} g^{-1}(\tau) \left\| (-\Delta)^{\frac{\alpha}{2}} \mathcal{G}(t-\tau)\, f(\tau) \right\|_{\mathbf{L}^2} d\tau \\
&+ \int_{\frac{t}{2}}^t g^{-1}(\tau) \left\| (-\Delta)^{\frac{\alpha}{2}} \mathcal{G}(t-\tau)\, f(\tau) \right\|_{\mathbf{L}^2} d\tau \\
&\leq C \int_0^{\frac{t}{2}} g^{-1}(\tau)\, (t-\tau)^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2} d\tau \\
&+ C \int_0^{\frac{t}{2}} g^{-1}(\tau)\, (t-\tau)^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}} \\
&\times \Bigg(\left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}} \left\| \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}} \\
&+ \left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}} \left\| \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{1-\frac{n+2\gamma}{2\delta}} \Bigg) d\tau \\
&+ C \int_{\frac{t}{2}}^t g^{-1}(\tau) \left\| (-\Delta)^{\frac{\alpha}{2}} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2} d\tau;
\end{aligned}$$

hence,

$$\left\|(-\Delta)^{\frac{\alpha}{2}} \int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau) f(\tau)\,d\tau\right\|_{\mathbf{L}^2}$$

$$\leq C\int_0^{\frac{t}{2}} g^{-1}(\tau)(t-\tau)^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}}\tau^{\frac{\gamma}{2}-1}d\tau$$

$$\times \sup_{t>0}\langle t\rangle^{1+\frac{n}{4}}\Big(\langle t\rangle^{\frac{\alpha}{2}}\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\Delta\rangle^{-\frac{1}{2}} f(t)\right\|_{\mathbf{L}^2}$$

$$+\left\|\langle\Delta\rangle^{-\frac{1}{2}} f(t)\right\|_{\mathbf{L}^2}+\langle t\rangle^{-\frac{\delta}{2}}\left\||\cdot|^{\delta}\langle\Delta\rangle^{-\frac{1}{2}} f(t)\right\|_{\mathbf{L}^2}\Big)$$

$$+C\int_{\frac{t}{2}}^{t} g^{-1}(\tau)\tau^{-1-\frac{n}{4}-\frac{\alpha}{2}}d\tau\sup_{t>0}\langle t\rangle^{1+\frac{n}{4}+\frac{\alpha}{2}}\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\Delta\rangle^{-\frac{1}{2}} f(t)\right\|_{\mathbf{L}^2}$$

$$\leq Ct^{-\frac{n}{4}-\frac{\alpha}{2}}g^{-1}(t)\,\|f\|_{\mathbf{Y}}\,.$$

Similarly by virtue of the second estimate of Lemma 3.35 we have

$$\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\Delta\rangle^{-\frac{1}{2}}\partial_t\int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau) f(\tau)\,d\tau\right\|_{\mathbf{L}^2}$$

$$\leq C\int_0^{\frac{t}{2}} g^{-1}(\tau)(t-\tau)^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1}\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\Delta\rangle^{-\frac{1}{2}} f(\tau)\right\|_{\mathbf{L}^2}d\tau$$

$$+C\int_0^{\frac{t}{2}} g^{-1}(\tau)(t-\tau)^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1}$$

$$\times\Bigg(\left\|\langle\cdot\rangle^{\delta}\langle\Delta\rangle^{-\frac{1}{2}} f(\tau)\right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}}\left\|\langle\Delta\rangle^{-\frac{1}{2}} f(\tau)\right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}}$$

$$+\left\|\langle\cdot\rangle^{\delta}\langle\Delta\rangle^{-\frac{1}{2}} f(\tau)\right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}}\left\|\langle\Delta\rangle^{-\frac{1}{2}} f(\tau)\right\|_{\mathbf{L}^2}^{1-\frac{n+2\gamma}{2\delta}}\Bigg)d\tau$$

$$+C\int_{\frac{t}{2}}^{t} g^{-1}(\tau)\langle t-\tau\rangle^{-1}\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\Delta\rangle^{-\frac{1}{2}} f(\tau)\right\|_{\mathbf{L}^2}d\tau;$$

thus

$$\left\| (-\Delta)^{\frac{\alpha}{2}} \partial_t \int_0^t g^{-1}(\tau) \mathcal{G}(t-\tau) f(\tau) \, d\tau \right\|_{\mathbf{L}^2}$$
$$\leq C \int_0^{\frac{t}{2}} g^{-1}(\tau)(t-\tau)^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1} \tau^{\frac{\gamma}{2}-1} d\tau$$
$$\times \sup_{t>0} \langle t \rangle^{1+\frac{n}{4}} \left(\langle t \rangle^{\frac{\alpha}{2}} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle \Delta \rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} \right.$$
$$\left. + \left\| \langle \Delta \rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} + \langle t \rangle^{-\frac{\delta}{2}} \left\| |\cdot|^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} \right)$$
$$+ C \int_{\frac{t}{2}}^t g^{-1}(\tau) \langle t-\tau \rangle^{-1} \tau^{-1-\frac{n}{4}-\frac{\alpha}{2}} d\tau$$
$$\times \sup_{t>0} \langle t \rangle^{1+\frac{n}{4}+\frac{\alpha}{2}} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle \Delta \rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2}$$
$$\leq C t^{-\frac{n}{4}-\frac{\alpha+1}{2}} g^{-1}(t) \|f\|_{\mathbf{Y}}.$$

Finally for all $t \geq 1$ applying the third estimate of Lemma 3.35 we get

$$\left\| |\cdot|^{\delta} \int_0^t g^{-1}(\tau) \mathcal{G}(t-\tau) f(\tau) \, d\tau \right\|_{\mathbf{L}^2}$$
$$\leq C \int_0^{\frac{t}{2}} g^{-1}(\tau) \left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2} d\tau$$
$$+ C \int_0^{\frac{t}{2}} g^{-1}(\tau)(t-\tau)^{\frac{\delta-\gamma}{2}-\frac{n}{4}}$$
$$\times \left(\left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}} \left\| \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}} \right.$$
$$\left. + \left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}} \left\| \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{1-\frac{n+2\gamma}{2\delta}} \right) d\tau$$
$$+ C \int_{\frac{t}{2}}^t g^{-1}(\tau) \left(\left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2} \right.$$
$$\left. + \langle t-\tau \rangle^{\frac{\delta}{2}} \left\| \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2} \right) d\tau.$$

Therefore,

$$\left\| |\cdot|^{\delta} \int_0^t g^{-1}(\tau) \mathcal{G}(t-\tau) f(\tau) \, d\tau \right\|_{\mathbf{L}^2}$$
$$\leq \left(\int_0^t g^{-1}(\tau) \tau^{\frac{\delta}{2}-\frac{n}{4}-1} d\tau + \int_0^{\frac{t}{2}} g^{-1}(\tau)(t-\tau)^{\frac{\delta-\gamma}{2}-\frac{n}{4}} \tau^{\frac{\gamma}{2}-1} d\tau \right)$$
$$\times \sup_{t>0} \langle t \rangle^{1+\frac{n}{4}} \left(\left\| \langle \Delta \rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} + \langle t \rangle^{-\frac{\delta}{2}} \left\| |\cdot|^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} \right)$$
$$\leq C \langle t \rangle^{\frac{\delta}{2}-\frac{n}{4}} g^{-1}(t) \|f\|_{\mathbf{Y}}.$$

This completes the proof of Lemma 3.36.

Consider the following Cauchy problem

$$\begin{cases} \frac{d}{dt}\left(h'(t)\left(e^{t}-\beta\right)\right)=\frac{2\lambda}{n\varepsilon\theta}e^{t}\int_{\mathbf{R}^{n}}\mathcal{N}\left(v\left(t,x\right)\right)dx \\ \quad +\frac{n+2}{2h(t)}\left(h'(t)\right)^{2}\left(e^{t}-\beta\right)-\beta h'(t), \\ \quad h(0)=1, h'(0)=0. \end{cases} \tag{3.110}$$

Denote

$$g(t)=1+\kappa\log\langle t\rangle,\ \kappa=\frac{\lambda}{2n\pi}\left(\varepsilon\theta\right)^{\frac{2}{n}}\left(\frac{n}{n+2}\right)^{\frac{n}{2}}>0,$$

and define

$$v_{0}(t)=\varepsilon\sum_{j=0}^{1}\mathcal{G}_{j}(t)\,u_{j}.$$

Lemma 3.37. *Suppose that*

$$\|v\|_{\mathbf{X}}\leq C\varepsilon,\ \ \|v(t)-v_{0}(t)\|_{\mathbf{L}^{p}}\leq C\varepsilon^{1+\frac{2}{n}}g^{-1}(t)\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

for all $t>0$, $1\leq p\leq\infty$, *then there exists a unique solution* $h(t)\in\mathbf{C}^{1}((0,\infty))$ *of the Cauchy problem (3.110) such that*

$$|h(t)-g(t)|\leq C\varepsilon^{\frac{2}{n}}\log g(t),\ \ |h'(t)|\leq C\varepsilon^{\frac{2}{n}}\langle t\rangle^{-1} \tag{3.111}$$

for all $t>0$.

Proof. Integration of (3.110) with respect to time yields

$$h'(t)=\frac{2\lambda}{n\varepsilon\theta\left(e^{t}-\beta\right)}\int_{0}^{t}d\tau e^{\tau}\int_{\mathbf{R}^{n}}\mathcal{N}\left(v\left(\tau,x\right)\right)dx$$
$$+\frac{n+2}{2\left(e^{t}-\beta\right)}\int_{0}^{t}d\tau\frac{\left(e^{\tau}-\beta\right)}{h(\tau)}\left(h'(\tau)\right)^{2}+\frac{\beta\left(1-h(t)\right)}{e^{t}-\beta},\ h(0)=1. \tag{3.112}$$

Integration by parts gives us

$$\int_{0}^{t}d\tau e^{\tau}\int_{\mathbf{R}^{n}}\mathcal{N}\left(v\left(\tau,x\right)\right)dx=e^{t}\int_{\mathbf{R}^{n}}\mathcal{N}\left(v\left(t,x\right)\right)dx-\int_{\mathbf{R}^{n}}\mathcal{N}\left(v\left(0,x\right)\right)dx$$
$$-\int_{0}^{t}d\tau e^{\tau}\int_{\mathbf{R}^{n}}\partial_{\tau}\mathcal{N}\left(v\left(\tau,x\right)\right)dx. \tag{3.113}$$

Therefore by virtue of (3.112) and (3.113) we have

$$\begin{cases} h'(t)=\frac{2\lambda}{n\varepsilon\theta}\int_{\mathbf{R}^{n}}\mathcal{N}\left(v\left(t,x\right)\right)dx+Q(t), \\ \quad h(0)=1, \end{cases} \tag{3.114}$$

where

$$
\begin{aligned}
Q(t) &= \frac{2\lambda}{n\varepsilon\theta\left(e^{t}-\beta\right)}\left(\beta\int_{\mathbf{R}^{n}}\mathcal{N}(v(t,x))\,dx-\int_{\mathbf{R}^{n}}\mathcal{N}(v(0,x))\,dx\right.\\
&\left.-\int_{0}^{t}d\tau e^{\tau}\int_{\mathbf{R}^{n}}\partial_{\tau}\mathcal{N}(v(\tau,x))\,dx\right)\\
&+\frac{n+2}{2\left(e^{t}-\beta\right)}\int_{0}^{t}d\tau\frac{\left(e^{\tau}-\beta\right)}{h(\tau)}\left(h'(\tau)\right)^{2}+\frac{\beta(1-h(t))}{e^{t}-\beta}.
\end{aligned}
$$

We solve the Cauchy problem (3.114) by the successive approximations. Denote $h_0(t)=g(t)$ and define $h_{m+1}(t)$, $m\geq 0$ as a solution of the linearized Cauchy problem

$$
\begin{cases} h'_{m+1}(t)=\frac{2\lambda}{n\varepsilon\theta}\int_{\mathbf{R}^{n}}\mathcal{N}(v(t,x))\,dx+Q_{m}(t), \\ h_{m+1}(0)=1, \end{cases} \tag{3.115}
$$

where

$$
\begin{aligned}
Q_m(t) &= \frac{2\lambda}{n\varepsilon\theta\left(e^{t}-\beta\right)}\left(\beta\int_{\mathbf{R}^{n}}\mathcal{N}(v(t,x))\,dx-\int_{\mathbf{R}^{n}}\mathcal{N}(v(0,x))\,dx\right.\\
&\left.-\int_{0}^{t}d\tau e^{\tau}\int_{\mathbf{R}^{n}}\partial_{\tau}\mathcal{N}(v(\tau,x))\,dx\right)\\
&+\frac{n+2}{2\left(e^{t}-\beta\right)}\int_{0}^{t}d\tau\frac{\left(e^{\tau}-\beta\right)}{h_m(\tau)}\left(h'_m(\tau)\right)^{2}+\frac{\beta(1-h_m(t))}{e^{t}-\beta}.
\end{aligned}
$$

We prove that for all $m\geq 0$

$$
|h_m(t)-g(t)|\leq C\varepsilon^{\frac{2}{n}}\log g(t),\ \ |h'_m(t)|\leq C\varepsilon^{\frac{2}{n}}\langle t\rangle^{-1}. \tag{3.116}
$$

For $m=0$ estimates (3.116) are valid. By induction we suppose that (3.116) is true for some $m\geq 0$. Then in view of the inequality $\|v\|_{\mathbf{X}}\leq C\varepsilon$, we see that $Q_m(t)$ has a better time decay

$$
|Q_m(t)|\leq C\varepsilon^{\frac{2}{n}}\langle t\rangle^{-\frac{3}{2}}
$$

for all $t>0$. Hence in view of (3.115)

$$
\begin{aligned}
&\left|h_{m+1}(t)-1-\frac{2\lambda}{\varepsilon n\theta}\int_{0}^{t}\int_{\mathbf{R}^{n}}\mathcal{N}(v(\tau,x))\,dxd\tau\right|\leq C\varepsilon^{\frac{2}{n}},\\
&\left|h'_{m+1}(t)\right|\leq C\varepsilon^{\frac{2}{n}}\langle t\rangle^{-\frac{3}{2}}+\varepsilon^{-1}\left|\int_{\mathbf{R}^{n}}\mathcal{N}(v(\tau,x))\,dxd\tau\right|.
\end{aligned} \tag{3.117}
$$

We write

$$
\begin{aligned}
&\int_{\mathbf{R}^{n}}\mathcal{N}(v(t,x))\,dx-(\varepsilon\theta)^{1+\frac{2}{n}}\int_{\mathbf{R}^{n}}\mathcal{N}(G(t,x))\,dx\\
&=\int_{\mathbf{R}^{n}}\left(\mathcal{N}(v(t,x))-\mathcal{N}(v_0)\right)dx\\
&+\int_{\mathbf{R}^{n}}\left(\mathcal{N}(v_0)-\mathcal{N}\left(\varepsilon\theta G\left(xt^{-\frac{1}{2}}\right)\right)\right)dx,
\end{aligned} \tag{3.118}
$$

where $G(t,x) \equiv (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$. By the condition

$$\|v(t) - v_0(t)\|_{\mathbf{L}^p} \leq C\varepsilon^{1+\frac{2}{n}} g^{-1}(t) \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

we obtain

$$\left| \int_{\mathbf{R}^n} (\mathcal{N}(v(t,x)) - \mathcal{N}(v_0)) \, dx \right| \leq C\varepsilon^{1+\frac{4}{n}} \langle t \rangle^{-1} g^{-1}(t) \tag{3.119}$$

and via Lemmas 3.34 and 3.35 we have

$$\left| \int_{\mathbf{R}^n} (\mathcal{N}(v_0) - \mathcal{N}(\varepsilon\theta G(t,x))) \, dx \right| \leq C\varepsilon^{1+\frac{2}{n}} \langle t \rangle^{-1-\gamma}. \tag{3.120}$$

A direct calculation shows

$$\frac{2\lambda}{\varepsilon n\theta} \int_{\mathbf{R}^n} \mathcal{N}(\varepsilon\theta G(t,x)) \, dx = \frac{2\lambda (\varepsilon\theta)^{\frac{2}{n}}}{n (4\pi t)^{1+\frac{n}{2}}} \int_{\mathbf{R}^n} e^{-\left(1+\frac{2}{n}\right)\frac{|x|^2}{4t}} dx = \frac{\kappa}{t}, \tag{3.121}$$

where $\kappa = \frac{\lambda}{2n\pi} (\varepsilon\theta)^{\frac{2}{n}} \left(\frac{n}{n+2}\right)^{\frac{n}{2}}$. Therefore by virtue of (3.118) - (3.121) we get

$$\int_{\mathbf{R}^n} \mathcal{N}(v(t,x)) \, dx = g(t) + C\varepsilon^{1+\frac{2}{n}} \log g(t);$$

hence by (3.117) the estimates (3.116) follow with m replaced by $m+1$. In the same manner we estimate the differences

$$|h_{m+1}(t) - h_m(t)| \leq \frac{1}{2} |h_m(t) - h_{m-1}(t)| \text{ and}$$
$$\left|h'_{m+1}(t) - h'_m(t)\right| \leq \frac{1}{2} \left|h'_m(t) - h'_{m-1}(t)\right|.$$

As a result there exists a unique solution $h(t) \in \mathbf{C}^1((0,\infty))$ of the Cauchy problem (3.110) satisfying estimates (3.111) for all $t > 0$. Lemma 3.37 is proved.

Proof of Theorem 3.32

As in the proof of Theorem 3.2 by changing the dependent variable $u(t,x) = e^{-\varphi(t)} v(t,x)$ in the damped wave equation (3.107) we get

$$\mathcal{L}v = f, \tag{3.122}$$

where $\mathcal{L} = \partial_t^2 + \partial_t - \Delta$ and

$$f = -\lambda e^{-\frac{2}{n}\varphi} \mathcal{N}(v) + 2\varphi' v_t + \left(\varphi'' - (\varphi')^2 + \varphi'\right) v.$$

Now we assume that $\varphi(t)$ satisfies the condition $\int_{\mathbf{R}^n} f(t,x) \, dx = 0$, that is

$$- \lambda e^{-\frac{2}{n}\varphi(t)} \int_{\mathbf{R}^n} \mathcal{N}\left(v\left(t,x\right)\right) dx + 2\varphi'\left(t\right) \int_{\mathbf{R}^n} v_t\left(t,x\right) dx$$
$$+ \left(\varphi''\left(t\right) - \left(\varphi'\left(t\right)\right)^2 + \varphi'\left(t\right)\right) \int_{\mathbf{R}^n} v\left(t,x\right) dx = 0, \tag{3.123}$$

and we also suppose that $\varphi\left(0\right) = \varphi'\left(0\right) = 0$. Integrating (3.122) with respect to x and using (3.123) we obtain

$$\frac{d}{dt} \int_{\mathbf{R}^n} \left(v_t\left(t,x\right) + v\left(t,x\right)\right) dx = 0$$

which implies

$$\int_{\mathbf{R}^n} \left(v_t\left(t,x\right) + v\left(t,x\right)\right) dx = \int_{\mathbf{R}^n} \left(v_t\left(0,x\right) + v\left(0,x\right)\right) dx \tag{3.124}$$
$$= \varepsilon \int_{\mathbf{R}^n} \left(u_0\left(x\right) + u_1\left(x\right)\right) dx = \varepsilon\theta,$$

since $u\left(0,x\right) = e^{-\varphi(0)} v\left(0,x\right)$ and

$$u_t\left(0,x\right) = -\varphi'\left(0\right) e^{-\varphi(0)} v\left(0,x\right) + e^{-\varphi(0)} v_t\left(0,x\right).$$

By (3.124) we have

$$e^{-t} \frac{d}{dt} \left(e^t \int_{\mathbf{R}^n} v\left(t,x\right) dx\right) = \varepsilon\theta,$$

so it follows that

$$\int_{\mathbf{R}^n} v\left(t,x\right) dx = \varepsilon\theta\left(1 - \beta e^{-t}\right),$$

where $\beta = \frac{1}{\theta} \int_{\mathbf{R}^n} u_1\left(x\right) dx$. By virtue of (3.124) and (3.123) we get

$$\varphi''\left(t\right)\left(1 - \beta e^{-t}\right) + \left(1 + \beta e^{-t}\right) \varphi'\left(t\right)$$
$$= \frac{\lambda}{\varepsilon\theta} e^{-\frac{2}{n}\varphi(t)} \int_{\mathbf{R}^n} \mathcal{N}\left(v\left(t,x\right)\right) dx + \left(\varphi'\left(t\right)\right)^2 \left(1 - \beta e^{-t}\right). \tag{3.125}$$

We put $h\left(t\right) = e^{\frac{2}{n}\varphi(t)}$, then multiplying (3.125) by $e^{t + \frac{2}{n}\varphi(t)}$ we find

$$\frac{d}{dt}\left(h'\left(t\right)\left(e^t - \beta\right)\right) = \frac{2\lambda}{n\varepsilon\theta} e^t \int_{\mathbf{R}^n} \mathcal{N}\left(v\left(t,x\right)\right) dx$$
$$+ \frac{n+2}{2h\left(t\right)} \left(h'\left(t\right)\right)^2 \left(e^t - \beta\right) - \beta h'\left(t\right), \tag{3.126}$$

with initial conditions $h\left(0\right) = 1$, $h'\left(0\right) = 0$. Thus instead of system (3.7) in the proof of Theorem 3.2 we obtain the following system of equations for $\left(v\left(t,x\right), h\left(t\right)\right)$

$$\begin{cases} \mathcal{L}v = f, \\ \frac{d}{dt}\left(h'(t)\left(e^t - \beta\right)\right) = \frac{2\lambda}{n\varepsilon\theta} e^t \int_{\mathbf{R}^n} \mathcal{N}(v(t,x))\,dx \\ \quad + \frac{n+2}{2h(t)}\left(h'(t)\right)^2 \left(e^t - \beta\right) - \beta h'(t), \\ v(0,x) = \varepsilon u_0(x),\ v_t(0,x) = \varepsilon u_1(x), \\ h(0) = 1, h'(0) = 0. \end{cases} \tag{3.127}$$

where

$$\begin{aligned} f &= -\frac{\lambda}{h}\mathcal{N}(v) + \varphi'(t)\,v + 2\varphi'(t)\,v_t + \left(\varphi''(t) - \left(\varphi'(t)\right)^2\right) v \\ &= -\frac{\lambda}{h}\mathcal{N}(v) - \frac{nh'(t)}{h(t)}\left(\frac{e^{-t}\beta}{\theta - e^{-t}\beta} v + v_t\right) \\ &+ \frac{\lambda v}{\varepsilon h(t)\left(\theta - \beta e^{-t}\right)} \int_{\mathbf{R}^n} \mathcal{N}(v(t,x))\,dx. \end{aligned}$$

We find a solution $(v(t,x), h(t))$ of the Cauchy problem (3.127) using the successive approximations method in the function space

$$\mathbf{X}_1 = \left\{(v,h) \in \mathbf{X} \times \mathbf{C}^1[0,\infty); \|(v,h)\|_{\mathbf{Z}} < \infty\right\},$$

where the norm

$$\begin{aligned} \|(v,h)\|_{\mathbf{X}_1} &\equiv \|v\|_{\mathbf{X}} + \sup_{t>0} \sup_{1\le p\le\infty} g(t)\,\langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{p}\right)} \|v(t) - v_0(t)\|_{\mathbf{L}^p} \\ &+ \sup_{t>0} \left(\log g(t)\right)^{-1} |h(t) - g(t)| + \sup_{t>0} \langle t\rangle\, |h'(t)|, \end{aligned}$$

$$\begin{aligned} \|\phi\|_{\mathbf{X}} &= \sup_{t>0} \sup_{0\le\alpha\le\delta} \langle t\rangle^{\frac{n}{4}+\frac{\alpha}{2}} \left\|(-\Delta)^{\frac{\alpha}{2}} \phi(t)\right\|_{\mathbf{L}^2} \\ &+ \sup_{t>0} \sup_{0\le\alpha\le\delta} \langle t\rangle^{\frac{n}{4}+\frac{\alpha+1}{2}} \left\|(-\Delta)^{\frac{\alpha}{2}} \langle\Delta\rangle^{-\frac{1}{2}} \partial_t\phi(t)\right\|_{\mathbf{L}^2} \\ &+ \sup_{t>0} \langle t\rangle^{\frac{n}{4}-\frac{\delta}{2}} \left\||\cdot|^{\delta} \phi(t)\right\|_{\mathbf{L}^2}, \end{aligned}$$

$$v_0(t) = \varepsilon \sum_{j=0}^{1} \mathcal{G}_j(t)\,u_j$$

and

$$g(t) = 1 + \kappa \log\langle t\rangle,\ \kappa = \frac{\lambda}{2n\pi}(\varepsilon\theta)^{\frac{2}{n}} \left(\frac{n}{n+2}\right)^{\frac{n}{2}}.$$

We now define for $\mathcal{G}_0 = \widetilde{\mathcal{G}}$, $\mathcal{G}_1 = \mathcal{G}$

$$v_0(t) = \varepsilon \sum_{j=0}^{1} \mathcal{G}_j(t)\,u_j, h_0(t) = g(t),$$

and for $(v_{m+1}(t), h_{m+1}(t))$, $m \geq 0$, we consider the linearized system of equations corresponding to (3.127)

$$\begin{cases} \mathcal{L}v_{m+1} = f_m, \\ \frac{d}{dt}\left(h'_{m+1}(t)\left(e^t - \beta\right)\right) = \frac{2\lambda}{n\varepsilon\theta} e^t \int_{\mathbf{R}^n} \mathcal{N}\left(v_m(t,x)\right) dx \\ \quad + \frac{n+2}{2h_{m+1}(t)}\left(h'_{m+1}(t)\right)^2 \left(e^t - \beta\right) - \beta h'_{m+1}(t), \\ v_{m+1}(0,x) = \varepsilon u_0(x),\ \partial_t v_{m+1}(0,x) = \varepsilon u_1(x), \\ h_{m+1}(0) = 1, h'_{m+1}(0) = 0, \end{cases} \tag{3.128}$$

where for $m \geq 1$

$$f_m = -\frac{\lambda}{h_{m+1}} \mathcal{N}(v_m) - \frac{n h'_{m+1}(t)}{h_{m+1}(t)} \left(\frac{e^{-t}\beta}{\theta - e^{-t}\beta} v_m + \partial_t v_m \right) + \frac{\lambda v_m}{\varepsilon h_{m+1}(t)\left(\theta - \beta e^{-t}\right)} \int_{\mathbf{R}^n} \mathcal{N}\left(v_m(t,x)\right) dx.$$

We now prove that for all $m \geq 0$

$$\|v_m\|_{\mathbf{X}} \leq C\varepsilon,\ \ \|v_m(t) - v_0(t)\|_{\mathbf{L}^p} \leq C\varepsilon^{1+\frac{2}{n}} g^{-1}(t) \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}, \tag{3.129}$$

and

$$|h_m(t) - g(t)| \leq C\varepsilon^{\frac{2}{n}} \log g(t),\ \ |\partial_t h_m(t)| \leq C\varepsilon^{\frac{2}{n}} \langle t \rangle^{-1} \tag{3.130}$$

for all $t > 0$, $1 \leq p \leq \infty$. By Lemmas 3.34 and 3.35 we see that (3.129) and (3.130) are valid for $m = 0$. We assume by induction that (3.129) and (3.130) are true for some m. By the definition of $h_m(t) = e^{\frac{2}{n}\varphi_m(t)}$, it follows that

$$\int_{\mathbf{R}^n} f_m(t,x)\, dx = 0$$

and

$$\int_{\mathbf{R}^n} v_{m+1}(t,x)\, dx = \varepsilon\theta\left(1 - \beta e^{-t}\right)$$

for all $t > 0$. We write equation $\mathcal{L}v_{m+1} = f_m$ in the integral form

$$v_{m+1} = v_0 + \int_0^t \mathcal{G}(t-\tau) f_m(\tau)\, d\tau$$

and apply Lemma 3.36 to get

$$\|(v_{m+1} - v_0) g\|_{\mathbf{X}} \leq C \|f_m g\|_{\mathbf{Y}} \leq C\varepsilon^{1+\frac{2}{n}}, \tag{3.131}$$

where

$$\|f\|_{\mathbf{Y}} = \sup_{t>0} \sup_{0 \leq \alpha \leq \delta} \langle t \rangle^{1+\frac{n}{4}+\frac{\alpha}{2}} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle \Delta \rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} + \sup_{t>0} \langle t \rangle^{1+\frac{n}{4}-\frac{\delta}{2}} \left\| |\cdot|^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2}.$$

Hence in view of the Sobolev Imbedding Theorem
1.4 it follows

$$\|v_{m+1}(t) - v_0(t)\|_{\mathbf{L}^p} \le C\varepsilon^{1+\frac{2}{n}} g^{-1}(t) \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}, \tag{3.132}$$

for all $t > 0$, $1 \le p \le \infty$. We also find by Lemma 3.37 that

$$|h_{m+1}(t) - g(t)| \le C\varepsilon^{\frac{2}{n}} \log g(t), \quad \left|h'_{m+1}(t)\right| \le C\varepsilon^{\frac{2}{n}} \langle t \rangle^{-1}. \tag{3.133}$$

Therefore the estimates (3.129) and (3.130) are valid for any m.

For the difference $w_m = v_{m+1} - v_m$ we get from (3.128)

$$\begin{cases} \mathcal{L}w_m = f_{m+1} - f_m \\ w_m(0,x) = 0, \ \partial_t w_m(0,x) = 0. \end{cases}$$

Since

$$\int_{\mathbf{R}^n} (f_{m+1}(t,x) - f_m(t,x))\, dx = 0,$$

by applying Lemma 3.36 we obtain

$$\|w_m\|_{\mathbf{X}} \le \frac{1}{2} \|w_{m-1}\|_{\mathbf{X}}$$

and by Lemma 3.37

$$\sup_{t>0} (g(t))^{-1} |h_{m+1}(t) - h_m(t)| \le \frac{1}{2} \|w_m\|_{\mathbf{X}}.$$

These estimates imply that there exists a unique solution $(v,h) \in \mathbf{Z}$ of the Cauchy problem (3.127) satisfying estimates

$$\|v(t) - v_0(t)\|_{\mathbf{L}^p} \le C\varepsilon^{1+\frac{2}{n}} g^{-1}(t) \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

and

$$|h(t) - g(t)| \le C\varepsilon^{\frac{2}{n}} \log g(t)$$

for all $t > 0$, $1 \le p \le \infty$. Since $u = e^{-\varphi} v$ and $h = e^{\frac{2}{n}\varphi}$ we have

$$\begin{aligned} &\left\|u(t)\, e^{\varphi(t)} - \varepsilon\theta G(t,x)\right\|_{\mathbf{L}^p} \\ &\le C \left\|u(t)\, e^{\varphi(t)} - v_0(t)\right\|_{\mathbf{L}^p} + C \|v_0(t) - \varepsilon\theta G(t,x)\|_{\mathbf{L}^p} \\ &\le C\varepsilon^{1+\frac{2}{n}} g^{-1}(t) \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}; \end{aligned}$$

hence

$$\left\|u(t) - \varepsilon\theta G(t,x)\, e^{-\varphi(t)}\right\|_{\mathbf{L}^p} \le C\varepsilon^{1+\frac{2}{n}} g^{-1-\frac{2}{n}}(t) \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$, $1 \le p \le \infty$. This completes the proof of Theorem 3.32.

3.5.2 Large initial data

This subsection is devoted to the study of the nonlinear damped wave equation

$$\begin{cases} u_{tt} + u_t - \Delta u = -\left|u\right|^{\sigma} u,\ x \in \mathbf{R}^n,\ t > 0, \\ u\left(0, x\right) = u_0\left(x\right),\ u_t\left(0, x\right) = u_1\left(x\right),\ x \in \mathbf{R}^n, \end{cases} \tag{3.134}$$

with a critical power $\sigma = \frac{2}{n}$ in any dimension $n \geq 1$. We will prove the large time asymptotic formulas for the solutions of the Cauchy problem (3.134) without any restriction on the size of the initial data.

Define

$$G_0\left(t, x\right) = \left(4\pi\left(1 + t\right)\right)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4(1+t)}}.$$

Denote $\eta = \left(4\pi\right)^{-1}\left(1 + \frac{2}{n}\right)^{-\frac{n}{2}}$. We define the space

$$\mathbf{X} = \left\{\phi \in \mathbf{C}\left(\left[0, \infty\right); \mathbf{H}^{1,k}\left(\mathbf{R}^n\right)\right) : \ \left\|\phi\right\|_{\mathbf{X}} < \infty\right\},$$

where the norm

$$\left\|\phi\right\|_{\mathbf{X}} = \sup_{t \geq 0}\Big(\left\langle t\right\rangle^{\frac{n}{4}-\frac{k}{2}}\left\|\phi\left(t\right)\right\|_{\mathbf{H}^{0,k}} + \left\langle t\right\rangle^{\frac{n}{4}}\left\|\phi\left(t\right)\right\|_{\mathbf{L}^2}$$
$$+ \left\langle t\right\rangle^{\frac{n}{4}+\frac{1}{2}-\frac{k}{2}}\left\|\nabla\phi\left(t\right)\right\|_{\mathbf{H}^{0,k}} + \left\langle t\right\rangle^{\frac{n}{4}+\frac{1}{2}}\left\|\nabla\phi\left(t\right)\right\|_{\mathbf{L}^2}\Big)$$

with $k > 6 + 3n$.

We will prove the following result.

Theorem 3.38. *Let $\sigma = \frac{2}{n}$. We assume that the initial data $u_0 \in \mathbf{H}^{2,k}\left(\mathbf{R}^n\right) \cap \mathbf{C}\left(\mathbf{R}^n\right)$, $u_1 \in \mathbf{H}^{1,k}\left(\mathbf{R}^n\right)$, with $k > 6 + 3n$. Then the Cauchy problem (3.134) has a unique global solution*

$$u \in \mathbf{C}\left(\left[0, \infty\right); \mathbf{H}^{1,k}\left(\mathbf{R}^n\right)\right) \cap \mathbf{C}^1\left(\left[0, \infty\right); \mathbf{H}^{0,k}\left(\mathbf{R}^n\right)\right).$$

Moreover the solution u has only one of the following asymptotics for large time $t \to \infty$

$$\left\|\log^{\frac{n}{2}+\gamma}(t)\left(u(t) - \left(\frac{n}{2\eta}\right)^{\frac{n}{2}} G_0\left(t\right)\log^{-\frac{n}{2}}(t)\right)\right\|_{\mathbf{X}} \leq C$$

where $0 < \gamma < \frac{1}{n}$, or

$$\left\|\log^{\frac{n}{2}+1}(t) u(t)\right\|_{\mathbf{X}} \leq C.$$

Remark 3.39. In the one dimensional case the asymptotic formulas stated in Theorem 3.38 are uniform with respect to $x \in \mathbf{R}$ by virtue of the Sobolev imbedding theorem. In the case of large initial data we can not give conditions on the data under which only one of the asymptotics occurs. This is why the result of Theorem 3.38 has the form of the alternative.

Below we obtain the weighted energy type estimates. Then applying the integral equation associated with Cauchy problem (3.134) we estimate the second derivative u_{tt}. By the maximum principle we find the optimal time decay estimates for the solutions. We describe the large time asymptotics of solutions for the nonlinear heat equation with a source in the critical case $\sigma = \frac{2}{n}$. Finally we then prove Theorem 3.38.

Preliminary estimates

We define the norm

$$\|\phi\|_{\mathbf{Y}} = \sup_{t\geq 0} \left(\langle t \rangle^{1+\frac{n}{4}-\frac{k}{2}} \|\phi(t)\|_{\mathbf{H}^{0,k}} + \langle t \rangle^{1+\frac{n}{4}} \|\phi(t)\|_{\mathbf{L}^2} \right).$$

Note that

$$\sup_{t\geq 0} \langle t \rangle^{\frac{1}{2}} \|\phi(t)\|_{\mathbf{L}^{1,1}} \leq \|\phi\|_{\mathbf{Y}}.$$

Define the Green operator $\mathcal{G}_0(t)$ of the linear heat equation

$$\mathcal{G}_0(t)\phi = \int_{\mathbf{R}^n} G(t, x-y)\phi(y)\,dy,$$

where the heat kernel $G(t,x)$ is

$$G(t,x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$

Also we denote the asymptotic kernel

$$G_0(t,x) = (4\pi(t+1))^{-\frac{n}{2}} e^{-\frac{|x|^2}{4(t+1)}}.$$

Lemma 3.40. *The triad* $(\mathbf{X}, \mathbf{Y}, \mathcal{G}_0)$ *is concordant, that is for any* ϕ *such that the mean value* $\int_{\mathbf{R}^n} \phi(x)\,dx = 0$ *the inequality*

$$\left\| g(t) \int_0^t \mathcal{G}_0(t-\tau)\phi(\tau)\,d\tau \right\|_{\mathbf{X}} \leq C \|g\phi\|_{\mathbf{Y}}$$

is valid, provided that the right-hand side is finite, where $g(t) = \log^{\gamma}(2+t)$ *for* $\gamma > 0$.

Proof. By virtue of Lemma 1.28 we obtain

$$\begin{aligned} &\left\| |\nabla|^{\rho} \int_0^t \mathcal{G}_0(t-\tau)\phi(\tau)\,d\tau \right\|_{\mathbf{L}^2} \\ &\leq C \int_0^t (t-\tau)^{-\frac{\rho}{2}} \langle \tau \rangle^{-\frac{n}{4}-1} g^{-1}(\tau)\,d\tau \sup_{\tau\geq 0} \langle \tau \rangle^{1+\frac{n}{4}} g(\tau) \|\phi(\tau)\|_{\mathbf{L}^2} \\ &\leq C \|g\phi\|_{\mathbf{Y}} \int_0^t (t-\tau)^{-\frac{\rho}{2}}\,d\tau \leq C g^{-1}(t) \langle t \rangle^{-\frac{\rho}{2}-\frac{n}{4}} \|g\phi\|_{\mathbf{Y}} \end{aligned}$$

for all $0 \le t \le 1$ and

$$\left\| |\nabla|^{\rho} \int_0^t \mathcal{G}_0 (t-\tau) \phi(\tau) \, d\tau \right\|_{\mathbf{L}^2}$$
$$\le C \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{\rho}{2}-\frac{n}{4}-\frac{1}{2}} \langle \tau \rangle^{-\frac{1}{2}} g^{-1}(\tau) \, d\tau \sup_{\tau \ge 0} \langle \tau \rangle^{\frac{1}{2}} g(\tau) \|\phi(\tau)\|_{\mathbf{L}^{1,1}}$$
$$+ C \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{\rho}{2}} \langle \tau \rangle^{-\frac{n}{4}-1} g^{-1}(\tau) \, d\tau \sup_{\tau \ge 0} \langle \tau \rangle^{1+\frac{n}{4}} g(\tau) \|\phi(\tau)\|_{\mathbf{L}^2}$$
$$\le C \langle t \rangle^{-\frac{\rho}{2}-\frac{n}{4}} \|g\phi\|_{\mathbf{Y}} \left(\langle t \rangle^{-\frac{1}{2}} \int_0^{\frac{t}{2}} \langle \tau \rangle^{-\frac{1}{2}} g^{-1}(\tau) \, d\tau + \langle t \rangle^{-1} \int_{\frac{t}{2}}^t g^{-1}(\tau) \, d\tau \right)$$
$$\le C g^{-1}(t) \langle t \rangle^{-\frac{\rho}{2}-\frac{n}{4}} \|g\phi\|_{\mathbf{Y}}$$

for all $t \ge 1$, $\rho = 0, 1$. In the same manner we have via Lemma 1.28

$$\left\| |x|^k |\nabla|^{\rho} \int_0^t \mathcal{G}_0 (t-\tau) \phi(\tau) \, d\tau \right\|_{\mathbf{L}^2}$$
$$\le C \int_0^t (t-\tau)^{\frac{k}{2}-\frac{\rho}{2}-\frac{n}{4}-\frac{1}{2}} \langle \tau \rangle^{-\frac{1}{2}} g^{-1}(\tau) \, d\tau \sup_{\tau \ge 0} \langle \tau \rangle^{\frac{1}{2}} g(\tau) \|\phi(\tau)\|_{\mathbf{L}^{1,a}}$$
$$+ C \int_0^t (t-\tau)^{-\frac{\rho}{2}} \langle \tau \rangle^{-1-\frac{n}{4}+\frac{k}{2}} g^{-1}(\tau) \, d\tau \sup_{\tau \ge 0} \langle \tau \rangle^{1+\frac{n}{4}-\frac{k}{2}} g(\tau) \|\phi(\tau)\|_{\mathbf{H}^{0,k}}$$
$$\le C \langle t \rangle^{\frac{k}{2}-\frac{\rho}{2}-\frac{n}{4}} \|g\phi\|_{\mathbf{Y}} \left(\langle t \rangle^{-\frac{1}{2}} \int_0^t \langle \tau \rangle^{-\frac{1}{2}} g^{-1}(\tau) \, d\tau + \langle t \rangle^{-1} \int_0^t g^{-1}(\tau) \, d\tau \right)$$
$$\le C g^{-1}(t) \langle t \rangle^{\frac{k}{2}-\frac{\rho}{2}-\frac{n}{4}} \|g\phi\|_{\mathbf{Y}}$$

for all $t \ge 0$, $\rho = 0, 1$. Hence the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G}_0)$ is concordant. Lemma 3.40 is proved.

Weighted energy type estimates

By applying a standard contraction mapping principle we have the following result.

Proposition 3.41. *Let $0 < \sigma \le \frac{2}{n}$. Suppose that the initial data $u_0 \in \mathbf{H}^{m+1,k}(\mathbf{R}^n)$, $u_1 \in \mathbf{H}^{m,k}(\mathbf{R}^n)$ with $k, m \ge 0$. Then there exists a positive time T and a unique solution $u \in \mathbf{C}\left([0,T]; \mathbf{H}^{m+1,k}\right) \cap \mathbf{C}^1\left([0,T]; \mathbf{H}^{m,k}\right)$ to the Cauchy problem (3.134).*

Using the idea of papers Ikehata et al. [2005] and Nishihara [2005] we obtain the following weighted energy type estimates (see also papers Nishihara and Zhao [2006], Ikehata and Tanizawa [2005], Ikehata et al. [2004], Todorova and Yordanov [2001]).

Lemma 3.42. *Let $0 < \sigma \leq \frac{2}{n}$. Suppose that the initial data $u_0 \in \mathbf{H}^{2,k}(\mathbf{R}^n)$, $u_1 \in \mathbf{H}^{1,k}(\mathbf{R}^n)$ with $k > 6\left(1 + \frac{2}{\sigma} - \frac{n}{2}\right)$. Then there exists a unique global solution $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^{2,k}(\mathbf{R}^n)\right) \cap \mathbf{C}^1\left([0,\infty);\mathbf{H}^{1,k}(\mathbf{R}^n)\right)$ to the Cauchy problem (3.134) which satisfies the a priori estimates*

$$\|u(t)\|_{\mathbf{L}^2}^2 + \langle t\rangle^{-k} \|u(t)\|_{\mathbf{H}^{0,k}}^2 \leq C \langle t\rangle^{\frac{n}{2}-\frac{2}{\sigma}}$$

and

$$\|u_t(t)\|_{\mathbf{H}^1}^2 + \langle t\rangle^{-k} \|u_t(t)\|_{\mathbf{H}^{1,k}}^2 + \|\nabla u(t)\|_{\mathbf{H}^1}^2 + \langle t\rangle^{-k} \|\nabla u(t)\|_{\mathbf{H}^{1,k}}^2 + \|u(t)\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2} \leq C \langle t\rangle^{-1+\frac{n}{2}-\frac{2}{\sigma}}$$

for all $t > 0$.

Remark 3.43. The estimates of Lemma 3.42 are optimal in the subcritical case $0 < \sigma < \frac{2}{n}$. In the critical case $\sigma = \frac{2}{n}$ under study in the present section the optimal time decay estimates contain logarithmic correction (see below Lemma 3.4.6).

Proof. Let u be a solution constructed in Proposition 3.41. We multiply equation (3.134) by $2\psi u_t + \phi u$ with arbitrary weight functions $\psi(t,x)$ and $\phi(t,x)$. Then integrating over $\mathbf{R}^n$ we get

$$0 = \int_{\mathbf{R}^n} \left(2\psi u_t u_{tt} + 2\psi u_t^2 - 2\psi u_t \Delta u + 2\psi |u|^\sigma u u_t + \phi u u_{tt} + \phi u u_t - \phi u \Delta u + \phi |u|^{\sigma+2}\right) dx.$$

Then integrating by parts with respect to x we find

$$\frac{dE}{dt} + H = \int_{\mathbf{R}^n} \frac{1}{\psi_t} |\psi_t \nabla u - u_t \nabla \psi|^2 \, dx, \tag{3.135}$$

where

$$E = \int_{\mathbf{R}^n} \left(\psi u_t^2 + \phi u u_t + \frac{\phi}{2} u^2 + \psi |\nabla u|^2 + \frac{2\psi}{\sigma+2} |u|^{\sigma+2}\right) dx,$$

$$H = \int_{\mathbf{R}^n} \left(\Phi u_t^2 + \phi |\nabla u|^2 - \frac{\phi_t}{2} u^2 + \Lambda |u|^{\sigma+2} - \phi_t u u_t + (\nabla\phi \cdot \nabla u) u\right) dx,$$

$$\Phi = 2\psi - \psi_t + \frac{1}{\psi_t} |\nabla\psi|^2 - \phi$$

and

$$\Lambda = \phi - \frac{2\psi_t}{\sigma+2}.$$

If the weight ψ satisfies the inequality $\psi_t \leq 0$, then equation (3.135) implies

$$\frac{dE}{dt} + H \le 0. \tag{3.136}$$

Multiplying inequality (3.136) by $(t+t_0)^\beta$ with $\beta = \frac{2}{\sigma} - \frac{n}{2} + 1$, we get

$$\frac{d}{dt}\left((t+t_0)^\beta E\right) \le -(t+t_0)^\beta \left(H - \frac{\beta}{t+t_0}E\right). \tag{3.137}$$

We now choose $\phi = \psi$, and consider the right-hand side of inequality (3.137)

$$H - \frac{\beta}{t+t_0}E = \int_{\mathbf{R}^n} \left(\left(\Phi - \frac{\beta\psi}{t+t_0}\right)u_t^2 - \left(\frac{\beta\psi}{t+t_0} + \psi_t\right)uu_t\right.$$
$$+\left(\psi - \frac{\beta\psi}{t+t_0}\right)|\nabla u|^2 + (\nabla\psi\cdot\nabla u)\,u$$
$$\left.+\frac{1}{2}\left(-\frac{\beta\psi}{t+t_0} - \psi_t\right)u^2 + \left(\Lambda - \frac{2\beta\psi}{(t+t_0)(\sigma+2)}\right)|u|^{\sigma+2}\right).$$

Applying the estimates

$$|uu_t| \le u_t^2 + \frac{1}{4}u^2$$

and

$$|(\nabla\psi\cdot\nabla u)\,u| \le \frac{2}{|\psi_t|}|\nabla\psi|^2|\nabla u|^2 + \frac{1}{8}|\psi_t|\,u^2,$$

we get

$$H - \frac{\beta}{t+t_0}E$$
$$\ge \int_{\mathbf{R}^n}\left(\left(\Phi - \frac{2\beta\psi}{t+t_0} + b\psi_t\right)u_t^2 + \left(\psi - \frac{\beta\psi}{t+t_0} - \frac{2}{|\psi_t|}|\nabla\psi|^2\right)|\nabla u|^2\right.$$
$$\left.+\frac{1}{8}\left(|\psi_t| - \frac{6\beta\psi}{t+t_0}\right)u^2 + \left(\Lambda - \frac{2\beta\psi}{(t+t_0)(\sigma+2)}\right)|u|^{\sigma+2}\right).$$

We choose the weight

$$\psi(t,x) = 1 + \frac{a|x|^{2k}}{(t+t_0)^k}$$

with some $a>0$, then for sufficiently large t_0 we find

$$\Phi - \frac{2\beta\psi}{t+t_0} + \psi_t = \left(1 - \frac{2\beta}{t+t_0}\right)\psi + \frac{1}{\psi_t}|\nabla\psi|^2$$
$$= \left(1 - \frac{2\beta}{t+t_0}\right)\left(1 + \frac{a|x|^{2k}}{(t+t_0)^k}\right) - \frac{4ka|x|^{2k-2}}{(t+t_0)^{k-1}}$$
$$\ge 1 + \frac{a|x|^{2k}}{(t+t_0)^k} - \frac{4ka|x|^{2k-2}}{(t+t_0)^{k-1}}$$
$$\ge 1 - 4a\,(8(k-1))^{k-1} + \frac{a|x|^{2k}}{2(t+t_0)^k} \ge \frac{1}{2}\psi$$

since by the Young inequality

$$\frac{4ka\,|x|^{2k-2}}{(t+t_0)^{k-1}} \le 4a\,(8\,(k-1))^{k-1} + \frac{a\,|x|^{2k}}{2\,(t+t_0)^k}$$

if $a \le \frac{1}{4(8(k-1))^{k-1}}$. Also we have

$$\begin{aligned}&\Lambda - \frac{2\beta\psi}{(t+t_0)(\sigma+2)} \\ &= \left(1 - \frac{2\beta}{(t+t_0)(\sigma+2)}\right)\psi - \frac{2\psi_t}{\sigma+2} \ge \frac{1}{2}\psi.\end{aligned}$$

Then by estimates

$$\psi - \frac{\beta\psi}{t+t_0} - \frac{2}{|\psi_t|}\,|\nabla\psi|^2 \ge \frac{1}{2}\psi$$

and

$$|\psi_t| - \frac{6\beta\psi}{t+t_0} \ge \frac{(k-6\beta)\,a\,|x|^{2k}}{(t+t_0)^{k+1}} - \frac{6\beta}{t+t_0}$$

we see that

$$\begin{aligned}H - \frac{\beta}{t+t_0}E \ge \int_{\mathbf{R}^n}&\left(\frac{1}{2}\psi\left(u_t^2 + |\nabla u|^2 + |u|^{\sigma+2}\right)\right. \\ &\left.+\frac{1}{8}\left(\frac{(k-6\beta)\,a\,|x|^{2k}}{(t+t_0)^{k+1}} - \frac{6\beta}{t+t_0}\right)u^2\right)dx \qquad (3.138)\end{aligned}$$

if we choose $k > 6\beta$.

Now we estimate the second summand in the right-hand side of (3.138). By the Hölder and Young inequalities we obtain

$$\begin{aligned}&-\frac{1}{8}\int_{\mathbf{R}^n}\left(\frac{(k-6\beta)\,a\,|x|^{2k}}{(t+t_0)^{k+1}} - \frac{6\beta}{t+t_0}\right)u^2dx \\ &\le \frac{\beta}{t+t_0}\int_{|x|\le\varrho\sqrt{t+t_0}} u^2dx \le C\,(t+t_0)^{-1}\,\|u\|^2_{\mathbf{L}^{\sigma+2}}\,\|1\|^2_{\mathbf{L}^{(\sigma+2)\frac{2}{\sigma}}\left(|x|\le\varrho\sqrt{t+t_0}\right)} \\ &\le C\,(t+t_0)^{\frac{\sigma n}{2(\sigma+2)}-1}\,\|u\|^2_{\mathbf{L}^{\sigma+2}} \le C\,(t+t_0)^{\frac{n}{2}-1-\frac{2}{\sigma}} + \frac{1}{2}\,\|u\|^{\sigma+2}_{\mathbf{L}^{\sigma+2}} \\ &\le C\,(t+t_0)^{\frac{n}{2}-1-\frac{2}{\sigma}} + \frac{1}{2}\int_{\mathbf{R}^n}|u|^{\sigma+2}\,\psi dx,\end{aligned}$$

where $\varrho > 0$ is sufficiently large. Thus

$$\begin{aligned}(t+t_0)^\beta\left(H - \frac{\beta}{t+t_0}E\right) \ge (t+t_0)^\beta\int_{\mathbf{R}^n}&\left(\frac{1}{2}\psi\left(u_t^2 + |\nabla u|^2 + |u|^{\sigma+2}\right)\right. \\ +\frac{1}{8}\left(\frac{(k-6\beta)\,a\,|x|^{2k}}{(t+t_0)^{k+1}} - \frac{6\beta}{t+t_0}\right)u^2&\left.\vphantom{\frac{1}{2}}\right)dx \ge -C\end{aligned}$$

since $\beta = \frac{n}{2} + 2\left(\frac{1}{\sigma} - \frac{n}{2}\right) + 1$. Then we obtain

$$\frac{d}{dt}\left(\left(t+t_0\right)^{\beta} E\right) + \frac{1}{2}\left(t+t_0\right)^{\beta} \int_{\mathbf{R}^n} \psi \left(u_t^2 + \left|\nabla u\right|^2 + \left|u\right|^{\sigma+2}\right) dx \leq C.$$

Integration of this inequality with respect to time yields

$$\begin{aligned} &\left(t+t_0\right)^{\beta} E\left(t\right) + \frac{1}{2}\int_0^t dt \left(t+t_0\right)^{\beta} \int_{\mathbf{R}^n} \left(u_t^2 + \left|\nabla u\right|^2 + \left|u\right|^{\sigma+2}\right) \psi dx \\ &\leq C\left(t+t_0\right) \end{aligned} \tag{3.139}$$

which imply the first estimate of the lemma.

Another estimate we obtain if we choose $\phi = 0$ and multiply inequality (3.136) by $\left(t+t_0\right)^{\beta+1}$

$$\frac{d}{dt}\left(\left(t+t_0\right)^{\beta+1} E_0\right) \leq -\left(t+t_0\right)^{\beta+1} H_0 + \left(\beta+1\right)\left(t+t_0\right)^{\beta} E_0, \tag{3.140}$$

where

$$E_0 = \int_{\mathbf{R}^n} \left(u_t^2 + \left|\nabla u\right|^2 + \frac{2}{\sigma+2}\left|u\right|^{\sigma+2}\right) \psi dx,$$

$$H_0 = \int_{\mathbf{R}^n} \left(\Phi_0 u_t^2 + \Lambda_0 \left|u\right|^{\sigma+2}\right) dx,$$

$$\Phi_0 = 2\psi - \psi_t + \frac{1}{\psi_t}\left|\nabla\psi\right|^2$$

and

$$\Lambda_0 = -\frac{2\psi_t}{\sigma+2}.$$

As above we have the estimates $\Phi_0 \geq 0$ and $\Lambda_0 \geq 0$. Therefore by (3.139) the integration of (3.140) with respect to time yields

$$\left(t+t_0\right)^{\beta+1} E_0 \leq C \int_0^t \left(t+t_0\right)^{\beta} E_0\left(t\right) dt \leq C\left(t+t_0\right).$$

Hence $E_0\left(t\right) \leq C\left(t+t_0\right)^{-\beta}$.

To prove the last estimate we apply the operator ∇ to equation (3.134) and multiply the result by $2\psi\nabla u_t + \psi\nabla u$. Then integrating by parts with respect to x in $\mathbf{R}^n$ we get

$$\begin{aligned} 0 = &\int_{\mathbf{R}^n} \psi \frac{d}{dt}\left(\left|\nabla u_t\right|^2 + \left(\Delta u\right)^2 + \frac{1}{2}\left|\nabla u\right|^2 + \left(\nabla u \cdot \nabla u_t\right)\right) dx \\ &+ \int_{\mathbf{R}^n} \left(\left|\nabla u_t\right|^2 + \left(\Delta u\right)^2 + \left(\sigma+1\right)\left|u\right|^{\sigma}\left|\nabla u\right|^2\right. \\ &\left.+2\left(\sigma+1\right)\left|u\right|^{\sigma}\left(\nabla u_t \cdot \nabla u\right)\right) \psi dx + \int_{\mathbf{R}^n} \left(\left(2\nabla u_t + \nabla u\right) \cdot \nabla\psi\right) \Delta u dx. \end{aligned}$$

Then we find

$$\frac{dE_1}{dt} + H_1 = 0, \tag{3.141}$$

where

$$E_1 = \int_{\mathbf{R}^n} \left(|\nabla u_t|^2 + (\Delta u)^2 + \frac{1}{2} |\nabla u|^2 + (\nabla u \cdot \nabla u_t) \right) \psi dx$$

and

$$\begin{aligned} H_1 = &\int_{\mathbf{R}^n} \Big(|\nabla u_t|^2 + (\Delta u)^2 + (\sigma+1) |u|^\sigma |\nabla u|^2 \\ &+2(\sigma+1)|u|^\sigma (\nabla u_t \cdot \nabla u) \Big) \psi dx + \int_{\mathbf{R}^n} ((2\nabla u_t + \nabla u) \cdot \nabla \psi) \Delta u dx \\ &- \int_{\mathbf{R}^n} \left(|\nabla u_t|^2 + (\Delta u)^2 + \frac{1}{2} |\nabla u|^2 + (\nabla u \cdot \nabla u_t) \right) \psi_t dx. \end{aligned}$$

Multiplying equation (3.141) by $(t+t_0)^{\beta+1}$ we get

$$\frac{d}{dt} \left((t+t_0)^{\beta+1} E_1 \right) = -(t+t_0)^{\beta+1} \left(H_1 - \frac{\beta+1}{t+t_0} E_1 \right). \tag{3.142}$$

By the Cauchy inequality we now estimate the right-hand side of (3.142)

$$\begin{aligned} &H_1 - \frac{\beta+1}{t+t_0} E_1 \\ &= \int_{\mathbf{R}^n} \Big(\left(\psi - \psi_t - \frac{\beta+1}{t+t_0} \psi \right) \left(|\nabla u_t|^2 + (\Delta u)^2 \right) + (\sigma+1) \psi |u|^\sigma |\nabla u|^2 \\ &+2(\sigma+1) \psi |u|^\sigma (\nabla u_t \cdot \nabla u) \Big) dx \\ &+ \int_{\mathbf{R}^n} (2(\nabla u_t \cdot \nabla \psi) \Delta u + (\nabla u \cdot \nabla \psi) \Delta u - (\nabla u \cdot \nabla u_t) \psi_t) dx \\ &+ \int_{\mathbf{R}^n} \left(\left(-\frac{1}{2} \psi_t - \frac{\beta+1}{2(t+t_0)} \psi \right) |\nabla u|^2 - \frac{\beta+1}{t+t_0} \psi (\nabla u \cdot \nabla u_t) \right) dx \\ &\geq \frac{1}{2} \int_{\mathbf{R}^n} \left(|\nabla u_t|^2 + (\Delta u)^2 \right) \psi dx - C(t+t_0)^{-1} \int_{\mathbf{R}^n} |\nabla u|^2 \psi dx \\ &- C \int_{\mathbf{R}^n} |u|^{2\sigma} |\nabla u|^2 \psi dx. \end{aligned} \tag{3.143}$$

By the Hölder inequality we have

$$\int_{\mathbf{R}^n} |u|^{2\sigma} |\nabla u|^2 \psi dx \leq \|u\|_{\mathbf{L}^{2+2\sigma}}^{2\sigma} \left\| \sqrt{\psi} |\nabla u| \right\|_{\mathbf{L}^{2+2\sigma}}^2.$$

By the Sobolev inequality

$$\|\phi\|_{\mathbf{L}^{2+2\sigma}} \leq C \|\nabla \phi\|_{\mathbf{L}^2}^{\frac{\sigma n}{2(1+\sigma)}} \|\phi\|_{\mathbf{L}^2}^{1-\frac{\sigma n}{2(1+\sigma)}}.$$

Then

$$\|u\|_{\mathbf{L}^{2+2\sigma}}^{2\sigma} \le C \|\nabla u\|_{\mathbf{L}^2}^{\frac{\sigma^2 n}{1+\sigma}} \|u\|_{\mathbf{L}^2}^{2\sigma-\frac{\sigma^2 n}{1+\sigma}} \le C t^{\sigma-\sigma\beta-\frac{\sigma^2 n}{2(1+\sigma)}}$$

and

$$\begin{aligned} \left\|\sqrt{\psi}\,|\nabla u|\right\|_{\mathbf{L}^{2+2\sigma}}^2 &\le C\left\|\sqrt{\psi}\Delta u\right\|_{\mathbf{L}^2}^{\frac{\sigma n}{1+\sigma}}\left\|\sqrt{\psi}\,|\nabla u|\right\|_{\mathbf{L}^2}^{2-\frac{\sigma n}{1+\sigma}} \\ &\quad + Ct^{-\frac{\sigma n}{2(1+\sigma)}}\left\|\sqrt{\psi}\,|\nabla u|\right\|_{\mathbf{L}^2}^2 \\ &\le Ct^{-\beta+\frac{\sigma n\beta}{2(1+\sigma)}}\left\|\sqrt{\psi}\Delta u\right\|_{\mathbf{L}^2}^{\frac{\sigma n}{1+\sigma}} + Ct^{-\frac{\sigma n}{2(1+\sigma)}-\beta}. \end{aligned}$$

Thus by the Young inequality

$$\begin{aligned} &\int_{\mathbf{R}^n} |u|^{2\sigma}|\nabla u|^2\psi dx \\ &\le Ct^{\sigma-(\sigma+1)\beta-\frac{\sigma n(\sigma-\beta)}{2(1+\sigma)}}\left\|\sqrt{\psi}\Delta u\right\|_{\mathbf{L}^2}^{\frac{\sigma n}{1+\sigma}} + Ct^{\sigma-(\sigma+1)\beta-\frac{\sigma n}{2}} \\ &\le \frac{1}{2}\left\|\sqrt{\psi}\Delta u\right\|_{\mathbf{L}^2}^2 + Ct^{-\beta-1}, \end{aligned} \tag{3.144}$$

since

$$\begin{aligned} &\left(\sigma-(\sigma+1)\beta-\frac{\sigma n(\sigma-\beta)}{2(1+\sigma)}\right)\left(\frac{1+\sigma}{1+\sigma-\frac{\sigma n}{2}}\right) \\ &= -\beta-1-\frac{\sigma+1}{1+\sigma-\frac{\sigma n}{2}} \le -\beta-1. \end{aligned}$$

By (3.142), (3.143) and (3.144) we get

$$\frac{d}{dt}\left((t+t_0)^{\beta+1}E_1\right) \le C + (t+t_0)^{\beta}\left\|\sqrt{\psi}\nabla u\right\|_{\mathbf{L}^2}^2 \tag{3.145}$$

Therefore by (3.139) the integration of (3.145) with respect to time yields

$$(t+t_0)^{\beta+1}E_1 \le C(t+t_0) + C\int_0^t (t+t_0)^{\beta}\left\|\sqrt{\psi}\nabla u\right\|_{\mathbf{L}^2}^2 dt \le C(t+t_0).$$

Hence $E_1(t) \le C(t+t_0)^{-\beta}$, which gives us the second estimate of the lemma. Lemma 3.42 is proved.

Estimates for the second derivative u_{tt}

Now we obtain the estimates for the second derivative u_{tt} in the norm

$$\|\phi\|_{\mathbf{Y}} = \sup_{t\ge 0}\langle t\rangle\left(\langle t\rangle^{\frac{n}{4}}\|\phi(t)\|_{\mathbf{L}^2} + \langle t\rangle^{\frac{n}{4}-\frac{k}{2}}\|\phi(t)\|_{\mathbf{H}^{0,k}}\right).$$

Lemma 3.44. *Suppose that the initial data* $u_0 \in \mathbf{H}^{2,k}(\mathbf{R}^n)$, $u_1 \in \mathbf{H}^{1,k}(\mathbf{R}^n)$ *with* $k > 6 + 3n$. *Let* u *be a global solution* $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^{2,k}\right) \cap \mathbf{C}^1\left([0,\infty)\right.$ $\left.;\mathbf{H}^{1,k}\right)$ *to the Cauchy problem (3.134) with* $\sigma = \frac{2}{n}$ *and satisfy the estimate*

$$\left\| \langle t\rangle^{-1} u \right\|_{\mathbf{Y}} + \left\| \langle t\rangle^{-\frac{1}{2}} u_t \right\|_{\mathbf{Y}} + \left\| \langle t\rangle^{-\frac{1}{2}} \nabla u \right\|_{\mathbf{Y}} \le C.$$

Then the estimate is true

$$\left\| \langle t\rangle^{\gamma} u_{tt} \right\|_{\mathbf{Y}} \le C,$$

where $\gamma > 0$.

Proof. We have by the integral representation (3.109)

$$\begin{aligned} \partial_t^2 u(t) &= (\partial_t + 1)\,\partial_t^2 \mathcal{G}(t)\, u_0 + \partial_t^2 \mathcal{G}(t)\, u_1 \\ &- \partial_t \mathcal{G}\left(\frac{t}{2}\right) \left| u\left(\frac{t}{2}\right)\right|^{\sigma} u\left(\frac{t}{2}\right) - \int_0^{\frac{t}{2}} \partial_t^2 \mathcal{G}(t-\tau) \left| u(\tau)\right|^{\sigma} u(\tau)\, d\tau \\ &- \int_{\frac{t}{2}}^{t} \partial_t \mathcal{G}(t-\tau)\, \partial_\tau \left| u(\tau)\right|^{\sigma} u(\tau)\, d\tau. \end{aligned}$$

By virtue of estimates of Lemma 3.34 we obtain

$$\left\| \partial_t \mathcal{G}(t)\, \phi \right\|_{\mathbf{L}^2} \le C \langle t\rangle^{-1} \left\| \phi \right\|_{\mathbf{L}^2}$$

and

$$\left\| \partial_t^2 \mathcal{G}(t)\, \phi \right\|_{\mathbf{L}^2} \le C \langle t\rangle^{-2-\frac{n}{4}} \left(\left\| \langle i\nabla\rangle \phi \right\|_{\mathbf{L}^1} + \left\| \phi \right\|_{\mathbf{H}^1} \right)$$

for all $t > 0$. Therefore

$$\begin{aligned} \left\| u_{tt}(t) \right\|_{\mathbf{L}^2} &\le C \langle t\rangle^{-2-\frac{n}{4}} \left(\left\| u_0 \right\|_{\mathbf{H}^{2,k}} + \left\| u_1 \right\|_{\mathbf{H}^{1,k}} \right) \\ &+ C \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-2-\frac{n}{4}} \left\| u(\tau) \right\|_{\mathbf{L}^{2+2\sigma}}^{\sigma} \left\| \langle i\nabla\rangle u(\tau) \right\|_{\mathbf{L}^{2+2\sigma}} d\tau \\ &+ C \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-2-\frac{n}{4}} \left\| u(\tau) \right\|_{\mathbf{L}^{1+\sigma}}^{\sigma} \left\| \langle i\nabla\rangle u(\tau) \right\|_{\mathbf{L}^{1+\sigma}} d\tau \\ &+ C \int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-1} \left\| u(\tau) \right\|_{\mathbf{L}^{2+2\sigma}}^{\sigma} \left\| u_\tau(\tau) \right\|_{\mathbf{L}^{2+2\sigma}} d\tau. \end{aligned}$$

By the Sobolev inequality we have

$$\left\| \phi \right\|_{\mathbf{L}^{2+2\sigma}} \le C \left\| \nabla \phi \right\|_{\mathbf{L}^2}^{\frac{\sigma n}{2(1+\sigma)}} \left\| \phi \right\|_{\mathbf{L}^2}^{1-\frac{\sigma n}{2(1+\sigma)}}.$$

Then

$$\left\| u(t) \right\|_{\mathbf{L}^{2+2\sigma}}^{\sigma} \le C \left\| \nabla u(t) \right\|_{\mathbf{L}^2}^{\frac{\sigma^2 n}{2(1+\sigma)}} \left\| u(t) \right\|_{\mathbf{L}^2}^{\sigma - \frac{\sigma^2 n}{2(1+\sigma)}} \le C \langle t\rangle^{-\frac{1}{2}-\frac{\sigma^2 n}{4(1+\sigma)}},$$

$$\left\| u(t) \right\|_{\mathbf{L}^{2+2\sigma}}^{\sigma} \left\| \langle i\nabla\rangle u(t) \right\|_{\mathbf{L}^{2+2\sigma}} \le C \langle t\rangle^{-1-\frac{n}{4}},$$

$$\|u(t)\|_{\mathbf{L}^{1+\sigma}}^{\sigma}\|\langle i\nabla\rangle u(t)\|_{\mathbf{L}^{1+\sigma}} \le C\langle t\rangle^{-1}$$

and

$$\|u(t)\|_{\mathbf{L}^{2+2\sigma}}^{\sigma}\|u_t(t)\|_{\mathbf{L}^{2+2\sigma}} \le C\langle t\rangle^{-\frac{3}{2}-\frac{n}{4}}.$$

Hence

$$\|u_{tt}(t)\|_{\mathbf{L}^2} \le C\langle t\rangle^{-2-\frac{n}{4}} + C\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-2-\frac{n}{4}}\langle\tau\rangle^{-1}d\tau$$
$$+ C\int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{-1}\langle\tau\rangle^{-\frac{3}{2}-\frac{n}{4}}d\tau \le C\langle t\rangle^{-1-\frac{n}{4}-\gamma}. \tag{3.146}$$

To prove the weighted estimate we write by using Lemma 3.34

$$\left\||\cdot|^k\partial_t\mathcal{G}(t)\phi\right\|_{\mathbf{L}^2} \le C\langle t\rangle^{\frac{k}{2}-1-\frac{n}{4}}\|\phi\|_{\mathbf{L}^1} + C\langle t\rangle^{-1}\left\|\langle\cdot\rangle^k\phi\right\|_{\mathbf{L}^2}$$

and

$$\left\||\cdot|^k\partial_t^2\mathcal{G}(t)\phi\right\|_{\mathbf{L}^2} \le C\langle t\rangle^{\frac{k}{2}-2-\frac{n}{4}}\|\langle i\nabla\rangle\phi\|_{\mathbf{L}^1} + C\langle t\rangle^{-2}\left\|\langle\cdot\rangle^k\langle i\nabla\rangle\phi\right\|_{\mathbf{L}^2}.$$

Hence

$$\left\||\cdot|^k\partial_t^2u(t)\right\|_{\mathbf{L}^2} \le C\langle t\rangle^{\frac{k}{2}-\frac{n}{4}-2}\left(\|u_0\|_{\mathbf{H}^{2,k}} + \|u_1\|_{\mathbf{H}^{1,k}}\right)$$
$$+ C\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{\frac{k}{2}-2-\frac{n}{4}}\|u(\tau)\|_{\mathbf{L}^{1+\sigma}}^{\sigma}\|\langle i\nabla\rangle u(\tau)\|_{\mathbf{L}^{1+\sigma}}d\tau$$
$$+ C\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-2}\|u(\tau)\|_{\mathbf{L}^{2+2\sigma}}^{\sigma}\left\|\langle\cdot\rangle^k\langle i\nabla\rangle u(\tau)\right\|_{\mathbf{L}^{2+2\sigma}}d\tau$$
$$+ C\int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{\frac{k}{2}-1-\frac{n}{4}}\|u(\tau)\|_{\mathbf{L}^{1+\sigma}}^{\sigma}\|u_\tau(\tau)\|_{\mathbf{L}^{1+\sigma}}d\tau$$
$$+ C\int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{-1}\|u(\tau)\|_{\mathbf{L}^{2+2\sigma}}^{\sigma}\left\|\langle\cdot\rangle^k u_\tau(\tau)\right\|_{\mathbf{L}^{2+2\sigma}}d\tau.$$

By the Sobolev inequality we have

$$\|u(t)\|_{\mathbf{L}^{2+2\sigma}}^{\sigma}\left\|\langle\cdot\rangle^k\langle i\nabla\rangle u(t)\right\|_{\mathbf{L}^{2+2\sigma}} \le C\langle t\rangle^{\frac{k}{2}-1-\frac{n}{4}},$$

$$\|u(t)\|_{\mathbf{L}^{1+\sigma}}^{\sigma}\|u_t(t)\|_{\mathbf{L}^{1+\sigma}} \le C\langle t\rangle^{-\frac{3}{2}}$$

and

$$\|u(t)\|_{\mathbf{L}^{2+2\sigma}}^{\sigma}\left\|\langle\cdot\rangle^k u_t(t)\right\|_{\mathbf{L}^{2+2\sigma}} \le C\langle t\rangle^{\frac{k}{2}-\frac{3}{2}-\frac{n}{4}}.$$

Hence

$$\begin{aligned}
&\left\| |\cdot|^k \partial_t^2 u(t) \right\|_{\mathbf{L}^2} \le C \langle t \rangle^{\frac{k}{2} - \frac{n}{4} - 2} \\
&+ C \int_0^{\frac{t}{2}} \left(\langle t - \tau \rangle^{\frac{k}{2} - 2 - \frac{n}{4}} \langle \tau \rangle^{-1} + \langle t - \tau \rangle^{-2} \langle \tau \rangle^{\frac{k}{2} - 1 - \frac{n}{4}} \right) d\tau \\
&+ C \int_{\frac{t}{2}}^t \left(\langle t - \tau \rangle^{\frac{k}{2} - 1 - \frac{n}{4}} \langle \tau \rangle^{-\frac{3}{2}} + \langle t - \tau \rangle^{-1} \langle \tau \rangle^{\frac{k}{2} - \frac{3}{2} - \frac{n}{4}} \right) d\tau \\
&\le C \langle t \rangle^{\frac{k}{2} - \frac{n}{4} - 1 - \gamma}. \qquad (3.147)
\end{aligned}$$

Thus the estimate of the lemma follows from (3.146) and (3.147). Lemma 3.44 is proved.

Optimal time decay estimates

We first compare the solutions of the following two problems

$$\begin{cases} u_t - \Delta u + |u|^\sigma u = \Psi, \ x \in \mathbf{R}^n, \ t > 0, \\ u(0, x) = u_0(x), \ x \in \mathbf{R}^n, \end{cases} \qquad (3.148)$$

and

$$\begin{cases} v_t - \Delta v + \epsilon v^{1+\sigma} = |\Psi|, \ x \in \mathbf{R}^n, \ t > 0, \\ v(0, x) = \mu |u_0(x)|, \ x \in \mathbf{R}^n, \end{cases} \qquad (3.149)$$

where $\sigma = \frac{2}{n}$.

Lemma 3.45. *Suppose that $\Psi \in \mathbf{C}((0, \infty) \times \mathbf{R}^n)$. Let u and v be classical solutions of (3.148) and (3.149) such that*

$$u, v \in \mathbf{C}\left((0, \infty); \mathbf{C}^2\right) \cap \mathbf{C}^1((0, \infty); \mathbf{C}).$$

Suppose that $0 \le \epsilon \le 1$ and $\mu \ge 1$. Then

$$|u(t, x)| \le v(t, x)$$

for all $t \ge 0$, $x \in \mathbf{R}^n$.

The proof of Lemma 3.45 is similar to that of Lemma 3.14 so we omit it.

In the next lemma we obtain optimal time decay estimates for problem (3.148) with $\Psi \in \mathbf{C}\left([0, \infty); \mathbf{L}^2(\mathbf{R}^n)\right)$. We remind that

$$\|\phi\|_{\mathbf{Y}} = \sup_{t \ge 0} \langle t \rangle \left(\langle t \rangle^{\frac{n}{4}} \|\phi(t)\|_{\mathbf{L}^2} + \langle t \rangle^{\frac{n}{4} - \frac{k}{2}} \|\phi(t)\|_{\mathbf{H}^{0,k}} \right).$$

Lemma 3.46. *Suppose that the initial data $u_0 \in \mathbf{C}(\mathbf{R}^n)$. Let the force $\Psi \in \mathbf{C}\left([0, \infty); \mathbf{L}^2(\mathbf{R}^n)\right)$ satisfy the estimate*

$$\|\langle t \rangle^\gamma \Psi\|_{\mathbf{Y}} \le C$$

for some $\gamma > 0$. *Suppose that there exists a solution* $u \in \mathbf{C}\left([0,\infty);\mathbf{L}^2(\mathbf{R}^n)\right)$ *of problem (3.148) such that*

$$\left\| \langle t \rangle^{-1} u \right\|_{\mathbf{Y}} \leq C.$$

Then the optimal time decay estimate is valid

$$\left\| (\log(2+t))^{\frac{n}{2}} u \right\|_{\mathbf{X}} \leq C. \tag{3.150}$$

Proof. Since the function $\Psi \in \mathbf{C}\left([0,\infty);\mathbf{L}^2(\mathbf{R}^n)\right)$ we can not apply Lemma 3.45 directly. Denote $\Psi_1 = \Psi - \theta_1(t) G_0(t)$, $\theta_1(t) = \int_{\mathbf{R}^n} \Psi(t,x)\,dx$, $v_1(t) = u(t)$ and

$$w_1(t) = \int_0^t \mathcal{G}_0(t-\tau)\Psi_1(\tau)\,d\tau.$$

Also we define v_m and w_m for $m \geq 2$ by the recurrent relations

$$v_m(t) = u(t) - \sum_{j=1}^{m-1} w_j(t)$$

and

$$w_m(t) = \int_0^t \mathcal{G}_0(t-\tau)\Psi_m(\tau)\,d\tau,$$

where

$$\Psi_m = |v_{m-1}|^\sigma v_{m-1} - |v_m|^\sigma v_m - \theta_m(t) G_0(t),$$
$$\theta_m(t) = \int_{\mathbf{R}^n} \left(|v_{m-1}|^\sigma v_{m-1} - |v_m|^\sigma v_m\right) dx.$$

Then we get

$$\begin{cases} \partial_t v_m - \Delta v_m = -|v_{m-1}|^\sigma v_{m-1} + \sum_{j=1}^{m-1} \theta_j(t) G_0(t), \; x \in \mathbf{R}^n, \; t > 0, \\ v_m(0,x) = u_0(x), \, x \in \mathbf{R}^n. \end{cases}$$

First let us prove by induction that

$$\left\| \langle t \rangle^\gamma w_j \right\|_{\mathbf{X}} \leq C. \tag{3.151}$$

By virtue of Lemma 3.40 we obtain

$$\left\| \langle t \rangle^\gamma w_j \right\|_{\mathbf{X}} \leq C \left\| \langle t \rangle^\gamma \Psi_j \right\|_{\mathbf{Y}}.$$

Since $v_j = v_{j-1} - w_{j-1}$ we have

$$\left| |v_{j-1}|^\sigma v_{j-1} - |v_j|^\sigma v_j \right| \leq C\left(|v_{j-1}| + |w_{j-1}|\right)^\sigma |w_{j-1}|.$$

Therefore

$$\|\langle t\rangle^{\gamma} \Psi_j\|_{\mathbf{Y}} \le C.$$

We now use the smoothing property of the heat kernel. We apply Lemma 3.35 to get the estimate

$$\begin{aligned}
&\|v_j(t)\|_{\mathbf{L}^{p_j}} \le \|\mathcal{G}_0(t) u_0\|_{\mathbf{L}^{p_j}} \\
&+ \int_0^t \left\| \mathcal{G}_0(t-\tau) \left(|v_{j-1}|^{\sigma} v_{j-1}(\tau) + \sum_{l=1}^{j-1} \theta_l(t) G_0(t) \right) \right\|_{\mathbf{L}^{p_j}} d\tau \\
&\le C \|u_0\|_{\mathbf{L}^{p_j}} + C \int_0^t (t-\tau)^{-\frac{n}{2}\left(\frac{1}{r_j} - \frac{1}{p_j}\right)} \|v_{j-1}(\tau)\|^{1+\sigma}_{\mathbf{L}^{r_j(1+\sigma)}} d\tau.
\end{aligned}$$

We choose $\frac{n}{2}\left(\frac{1}{r_j} - \frac{1}{p_j}\right) < 1$, and $v_{j-1} \in \mathbf{C}\left([0,\infty); \mathbf{L}^{p_{j-1}}(\mathbf{R}^n)\right)$, then we get $v_j \in \mathbf{C}\left([0,\infty); \mathbf{L}^{p_j}(\mathbf{R}^n)\right)$ with $\frac{n}{2}\left(\frac{1+\sigma}{p_{j-1}} - \frac{1}{p_j}\right) < 1$. Thus we choose $p_1 = 2$ and $\frac{1}{p_j} \le \frac{1+\sigma}{p_{j-1}} - \frac{2-\varepsilon}{n} \le \frac{1}{p_{j-1}} - \frac{1-\varepsilon}{n}$. Therefore we arrive at $p_j = \infty$ for some j. By the same considerations we obtain

$$v_m \in \mathbf{C}\left([0,\infty); \mathbf{C}(\mathbf{R}^n)\right) \cap \mathbf{C}\left((0,\infty); \mathbf{C}^2(\mathbf{R}^n)\right) \cap \mathbf{C}^1\left((0,\infty); \mathbf{C}(\mathbf{R}^n)\right)$$

for some m. Thus v_m satisfy (3.148) with a force

$$\widetilde{\Psi} = |v_m|^{\sigma} v_m - |v_{m-1}|^{\sigma} v_{m-1} + \sum_{j=1}^{m-1} \theta_j(t) G_0(t).$$

By virtue of (3.151) we have the following time decay estimate

$$\left\| \langle t\rangle^{\gamma} \widetilde{\Psi} \right\|_{\mathbf{Y}} \le C.$$

We now take a sufficiently small $\varepsilon > 0$ and consider the following two auxiliary Cauchy problems

$$\begin{cases} U_t - \Delta U + U^{\frac{2}{n}+1} = \varepsilon^2 \left|\widetilde{\Psi}\right|, \ x \in \mathbf{R}^n, \ t > 0, \\ U(0,x) = \varepsilon |u_0(x)|, \ x \in \mathbf{R}^n, \end{cases} \tag{3.152}$$

and

$$\begin{cases} V_t - \Delta V + \varepsilon^{\frac{4}{n}} V^{\frac{2}{n}+1} = \left|\widetilde{\Psi}\right|, \ x \in \mathbf{R}^n, \ t > 0, \\ V(0,x) = \frac{1}{\varepsilon} |u_0(x)|, \ x \in \mathbf{R}^n. \end{cases} \tag{3.153}$$

Note that problem (3.153) can be reduced to problem (3.152) by virtue of the change $V = \varepsilon^{-2} U$. And problem (3.152) has a sufficiently small initial data $\varepsilon |u_0(x)|$ and a small force $\varepsilon^2 \left|\widetilde{\Psi}\right|$. Moreover, the mean value $\theta = \int_{\mathbf{R}^n} u_0(x) dx$ $= O(\varepsilon)$. So that the term $\frac{1}{\theta} \varepsilon^2 \left|\widetilde{\Psi}\right| = O(\varepsilon)$ is also small. Therefore we can apply the results of Section 3.2 to calculate the large time asymptotic behavior of the functions $U(t,x)$ and $V(t,x)$. Hence by Lemma 3.45, we get $|v_m(t,x)| \leq V(t,x) = \varepsilon^{-2}U(t,x)$. Then via Lemma 3.45 we arrive at the optimal time decay estimate for the solution v_m

$$\|v_m(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon^{-2}\langle t\rangle^{-\frac{n}{2}}(\log(2+t))^{-\frac{n}{2}}$$

for all $t > 0$. Using the integral equation for the heat equation we find

$$\left\|(\log(2+t))^{\frac{n}{2}} v_m\right\|_{\mathbf{X}} \leq C. \tag{3.154}$$

Now by estimates (3.151) and (3.154) we find estimate (3.150). Lemma 3.46 is proved.

Nonlinear heat equation with a source

We start this section with the following auxiliary result. Consider the following equation

$$\frac{dy}{dt} = (1-|y|^\sigma)\, y + \phi(t), \tag{3.155}$$

where $\sigma > 0$. Denote $\psi(t) = 4(1-2^{-\sigma})^{-1}\sup_{\tau\geq t}|\phi(\tau)|$, hence $\psi(t)$ is monotonous.

Lemma 3.47. *Let* $\phi \in \mathbf{C}([0,\infty))$, *and* $e^{\gamma t}\psi(t) \in \mathbf{L}^2(0,\infty)$, *with* $0 < \gamma < \min\left(1, \frac{\sigma}{2}\right)$. *Then the solutions of equation (3.155) has only one of the asymptotic behavior*

$$|y(t)| \leq \psi(t) \tag{3.156}$$

or

$$y(t) = 1 + O\left(e^{-\gamma t}\right) \tag{3.157}$$

for all $t > 0$.

Proof. If $|y(t)| \leq \psi(t)$ for all $t > 0$, then we get (3.156). Consider $t \geq T > 0$ such that $\psi(t) \leq \frac{1}{2}$. Now we suppose that

$$|y(t_0)| > \psi(t_0)$$

for some $t_0 \geq T$. First consider the simplest case $y(t_0) > \frac{1}{2}$. Then let us prove that

$$y(t) > \frac{1}{2} \tag{3.158}$$

for all $t \geq t_0$. By the contrary in view of the continuity of the solution $y(t)$ we can find a maximal time $t_1 > t_0$ such that inequality (3.158) is true for all $t \in [t_0, t_1)$ and $y(t_1) = \frac{1}{2}$. Since $|\phi(t)| \leq \frac{1}{8}(1-2^{-\sigma})$, then by equation (3.155) we have

$$\frac{dy}{dt} \geq \frac{1}{2}(1-|y|^\sigma) - \frac{1}{4}\left(1-2^{-\sigma}\right) > 0,$$

when $t \in [t_0, t_1)$ is sufficiently close to t_1. Hence we see that (3.158) is true for all $t \in [t_0, t_1]$. The contradiction obtained proves that (3.158) is fulfilled for all $t \geq t_0$. Then changing the dependent variable $y = 1 + w$ and multiplying equation (3.155) by w we get

$$\frac{d}{dt} w^2 = 2y \left(1 - (1 + w)^{\sigma}\right) w + 2w\phi \leq -2\gamma w^2 + C\psi^2.$$

Therefore the integration with respect to time yields

$$w^2(t) \leq e^{-2\gamma t} \left(w^2(t_0) + C \int_{t_0}^{t} e^{-2\gamma\tau} \psi^2(\tau)\, d\tau \right).$$

Consequently $w^2(t) = O\left(e^{-2\gamma t}\right)$ for $t \to \infty$. Therefore asymptotics (3.157) is true.

Consider now the rest case

$$\psi(t_0) < y(t_0) \leq \frac{1}{2}$$

for some $t_0 > 0$. Then let us show that the solution $y(t)$ grows in time and there exists a finite time $t_1 > t_0$ such that

$$y(t_1) > \frac{1}{2} \tag{3.159}$$

(that is we arrive to the previous case.) By the contrary suppose that $y(t) \leq \frac{1}{2}$ for all $t > 0$. In view of the continuity of $y(t)$ we find a maximal time interval $t_2 > t_0$, such that

$$\psi(t) < y(t) \leq \frac{1}{2} \tag{3.160}$$

for all $t \in [t_0, t_2)$, and if $t_2 < \infty$, then $y(t_2) = \psi(t_2)$. Then by equation (3.155) we have

$$\frac{dy}{dt} = \left(1 - |y|^{\sigma}\right) y + \phi(t) > \left(1 - 2^{-\sigma}\right)(y - \psi(t)) > 0 \tag{3.161}$$

for all $t \in [t_0, t_2)$. Thus the solution $y(t)$ grows on the interval $[t_0, t_2)$, hence $y(t) > \psi(t)$ for all $t \geq t_0$. Consequently inequality (3.160) is true for all $t \geq t_0$. However the integration of (3.161) shows that the solution $y(t)$ becomes greater than $\frac{1}{2}$ after a finite time $t_1 > t_0$. The contradiction obtained proves (3.159). The case of the negative solutions can be treated in the same way. Lemma 3.47 is proved.

Now we consider the Cauchy problem for the nonlinear heat equation (3.148) in the critical case $\sigma = \frac{2}{n}$ with a source $\Psi \in \mathbf{C}\left((0, \infty); \mathbf{H}^{0,k}(\mathbf{R}^n)\right)$. The solutions to the Cauchy problem (3.134) satisfy (3.148) if we take $\Psi = -u_{tt}$. Denote $\eta = (4\pi)^{-1}\left(1 + \frac{2}{n}\right)^{-\frac{n}{2}}$ and $0 < \gamma < \frac{1}{n}$.

Theorem 3.48. *Assume that the initial data* $u_0 \in \mathbf{H}^{1,k}(\mathbf{R}^n)$, *with* $k > 6+3n$. *Suppose that the source* $\Psi \in \mathbf{C}\left((0,\infty);\mathbf{H}^{0,k}(\mathbf{R}^n)\right)$ *and the estimate*

$$\|\langle t\rangle^{\gamma} \Psi(t)\|_{\mathbf{Y}} \le C \tag{3.162}$$

is true. Assume that the Cauchy problem (3.148) has a solution

$$u(t,x) \in \mathbf{C}\left([0,\infty);\mathbf{H}^{1,k}(\mathbf{R}^n)\right) \cap \mathbf{C}^1\left([0,\infty);\mathbf{H}^{0,k}(\mathbf{R}^n)\right)$$

satisfying the a priori time decay estimate

$$\left\|(\log t)^{\frac{n}{2}} u(t)\right\|_{\mathbf{X}} \le C. \tag{3.163}$$

Then the solution u *has only one of the following asymptotics*

$$u(t) = \left(\frac{n}{2\eta}\right)^{\frac{n}{2}} G_0(t) \log^{-\frac{n}{2}} t + O\left(t^{-\frac{n}{2}} \log^{-\frac{n}{2}-\gamma} t\right),$$

or

$$u(t) = O\left(t^{-\frac{n}{2}} \log^{-\frac{n}{2}-1} t\right)$$

for large time $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}^n$.

Proof. We make a change

$$u(t,x) = \theta(t) G_0(t,x) + w(t,x).$$

Then from (3.148) we get for the new function $w(t.x)$

$$w_t - \Delta w = \mathcal{N}(u) - \theta' G_0, \tag{3.164}$$

where $\mathcal{N}(u) = -|u|^{\sigma} u + \Psi$. We choose the function $\theta(t)$ by the following equation

$$\theta'(t) = \int_{\mathbf{R}^n} \mathcal{N}(u(t,x))\, dx \tag{3.165}$$

with the initial condition $\theta(0) = \int_{\mathbf{R}^n} u_0(x)\, dx$. This implies that

$$\int_{\mathbf{R}^n} w(t,x)\, dx = 0$$

for all $t \ge 0$. In view of (3.165) the integral equation associated with equation (3.164) is

$$w(t) = \mathcal{G}_0(t) w_0 + \int_0^t \mathcal{G}_0(t-\tau)\left(\mathcal{N}(u(\tau)) - G_0(\tau)\int_{\mathbf{R}^n} \mathcal{N}(u(\tau))\, dx\right) d\tau. \tag{3.166}$$

We obtain the estimate

$$\left\| g^{1+\frac{n}{2}} \mathcal{N}(u) \right\|_{\mathbf{Y}} \leq C \left\| g^{\frac{n}{2}} u \right\|_{\mathbf{X}}^{1+\frac{2}{n}} + \left\| g^{1+\frac{n}{2}} \Psi \right\|_{\mathbf{Y}} \leq C. \tag{3.167}$$

and

$$\left\| g^{1+\frac{n}{2}} \mathcal{G}_0 w_0 \right\|_{\mathbf{X}} \leq C.$$

Since by Lemma 3.40 the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G}_0)$ is concordant, then by (3.167) we get from (3.166) the estimates

$$\begin{aligned} \left\| g^{1+\frac{n}{2}} w \right\|_{\mathbf{X}} &\leq \left\| g^{1+\frac{n}{2}} \mathcal{G}_0 w_0 \right\|_{\mathbf{X}} + \left\| g^{1+\frac{n}{2}} \int_0^t \mathcal{G}_0(t-\tau) \mathcal{N}(u(\tau)) \, d\tau \right\|_{\mathbf{X}} \\ &\leq C + C \left\| g^{1+\frac{n}{2}} \mathcal{N}(u) \right\|_{\mathbf{Y}} \leq C, \end{aligned}$$

which shows that $w(t)$ is a remainder term. Then since $\left\| g^{\frac{n}{2}} u \right\|_{\mathbf{X}} \leq C$ we see that

$$|\theta(t)| \leq C (\log(2+t))^{-\frac{n}{2}}.$$

Now we turn to equation (3.165)

$$\begin{aligned} \theta'(t) &= -\int_{\mathbf{R}^n} \mathcal{N}(u(t,x)) \, dx = -\int_{\mathbf{R}^n} |u|^{\sigma} u dx + \int_{\mathbf{R}^n} \Psi(t,x) \, dx \\ &= -|\theta(t)|^{\frac{2}{n}} \theta(t) \int_{\mathbf{R}^n} G_0^{1+\frac{2}{n}}(t,x) \, dx + O\left((1+t)^{-1} (\log(2+t))^{-\frac{n}{2}-2} \right). \end{aligned}$$

By a direct calculation we have

$$\begin{aligned} \int_{\mathbf{R}^n} G_0^{1+\frac{2}{n}}(t,x) \, dx &= (4\pi)^{-\frac{n}{2}-1} (t+1)^{-\frac{n}{2}-1} \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4(t+1)}\left(1+\frac{2}{n}\right)} dx \\ &= \eta (1+t)^{-1}, \end{aligned}$$

where $\eta = (4\pi)^{-1} \left(1+\frac{2}{n}\right)^{-\frac{n}{2}}$. Therefore we get

$$\theta'(t) = -\eta |\theta(t)|^{\frac{2}{n}} \theta(t) (1+t)^{-1} + O\left((1+t)^{-1} (\log(2+t))^{-\frac{n}{2}-2} \right). \tag{3.168}$$

We now make a change $\theta(t) = h(t) (\log(2+t))^{-\frac{n}{2}}$, then

$$\begin{aligned} h' &= \frac{\eta}{(1+t)\log(2+t)} h \left(\frac{n}{2\eta} - |h|^{\frac{2}{n}} \right) \\ &\quad + O\left((1+t)^{-1} (\log(2+t))^{-2} \right). \end{aligned} \tag{3.169}$$

Changing the dependent and independent variables $h(t) = \left(\frac{n}{2\eta}\right)^{\frac{n}{2}} y(\tau)$, $\tau = \frac{n}{2} \log\log(2+t)$, we obtain the following equation

$$\frac{d}{d\tau} y = (1 - |y|^{\sigma}) y + \phi(\tau),$$

where $\phi(\tau) = O\left(e^{-\frac{2}{n}\tau}\right)$. So we can apply Lemma 3.47 to equation (3.169), and arrive to the following possibilities for the function $\theta(t)$

$$\theta(t) = \left(\frac{n}{2\eta}\right)^{\frac{n}{2}} (\log t)^{-\frac{n}{2}} + O\left((\log t)^{-\frac{n}{2}-\gamma}\right) \tag{3.170}$$

or

$$|\theta(t)| \leq (\log t)^{-\frac{n}{2}-1} \tag{3.171}$$

for all $t > 0$, where $0 < \gamma < \frac{1}{n}$. The first possibility (3.170) leads to the first asymptotics formula of the theorem. In the case (3.171) we obtain a faster time decay estimate

$$\left\|(\log t)^{1+\frac{n}{2}} u\right\|_{\mathbf{X}} \leq C$$

for the solution. Theorem 3.48 is proved.

Proof of Theorem 3.38

Theorem 3.38 is now a consequence of Lemma 3.42, Lemma 3.44, Lemma 3.46 and Theorem 3.48.

3.6 Sobolev type equations

This section is devoted to the study of the Cauchy problem for the Sobolev type equation in the critical case

$$\begin{cases} \partial_t (u - \Delta u) - \alpha \Delta u = \lambda |u|^\sigma u, \ x \in \mathbf{R}^n, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^n, \end{cases} \tag{3.172}$$

where $\alpha > 0$, $\sigma = \frac{2}{n}$, $\lambda \in \mathbf{R}$. As in the supercritical case (see Chapter 2 section 2.4) we rewrite the solution of problem (3.172) in the form

$$u(t) = \mathcal{G}(t) u_0 + \lambda \int_0^t \mathcal{G}(t-\tau) \mathcal{B} |u|^\sigma u d\tau, \tag{3.173}$$

where the Green operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t)\phi = e^{-\alpha t} \overline{\mathcal{F}}_{\xi \to x} e^{\frac{\alpha t}{1+|\xi|^2}} \hat{\phi}(\xi)$$

and

$$\mathcal{B}\phi = \int_{\mathbf{R}^n} B(x-y)\phi(y)\,dy$$

with the Bessel-Macdonald kernel

$$B(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{i\xi x} \left(1 + |\xi|^2\right)^{-1} d\xi = |x|^{1-\frac{n}{2}} K_{\frac{n}{2}-1}(|x|).$$

Here

$$K_{\nu}(|x|) = K_{-\nu}(|x|) = 2^{-\nu-1}|x|^{\nu}\int_0^{\infty}\xi^{-\nu-1}e^{-\xi-\frac{|x|^2}{4\xi}}d\xi$$

is the Macdonald function (or modified Bessel function) of order $\nu \in \mathbf{R}$. Recall that for any $k \geq 0$

$$\left\|\mathcal{B}^k\phi\right\|_{\mathbf{L}^p} \leq C\|\phi\|_{\mathbf{L}^p} \tag{3.174}$$

for all $1 \leq p \leq \infty$ and

$$\left\|\mathcal{B}^k\phi\right\|_{\mathbf{L}^{1,a}} \leq C\|\phi\|_{\mathbf{L}^{1,a}} \tag{3.175}$$

for any $a \geq 0$. Denote

$$G_0(t,x) = (4\pi\alpha(t+1))^{-\frac{n}{2}}e^{-\frac{|x|^2}{4\alpha(t+1)}}.$$

We have the following result (see Lemma 1.31, and Lemma 1.33).

Lemma 3.49. *Suppose that the function $\phi \in \mathbf{L}^{\infty}(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, where $a \in (0,1)$. Then the estimates*

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^p} \leq e^{-at}\|\phi\|_{\mathbf{L}^p} + C\langle t\rangle^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|\phi\|_{\mathbf{L}^r}$$

$$\|\mathcal{G}(t)\phi - \vartheta G_0(t,x)\|_{\mathbf{L}^{\infty}} \leq Ct^{-\frac{n}{2}-\frac{a}{2}}\left(\|\phi\|_{\mathbf{L}^{\infty}} + \|\phi\|_{\mathbf{L}^{1,a}}\right)$$

and

$$\left\||\cdot|^b(\mathcal{G}(t)\phi - \vartheta G_0(t,x))\right\|_{\mathbf{L}^1} \leq Ct^{\frac{b-a}{2}}\|\phi\|_{\mathbf{L}^{1,a}}$$

are valid for all $t > 0$, where $1 \leq p \leq \infty$, $0 \leq b \leq a$ and $\vartheta = \int_{\mathbf{R}^n}\phi(x)\,dx$.

3.6.1 Small initial data

Define

$$\eta = 2(t+1)\int_{\mathbf{R}^n}(G_0(t,x))^{\frac{2}{n}+1}dx = \frac{1}{2\pi\alpha}\left(\frac{2}{n}+1\right)^{-\frac{n}{2}}.$$

Denote $g(t) = 1 + |\theta|^{\frac{2}{n}}\eta\log(1+t)$, $\theta = \int_{\mathbf{R}^n}u_0(x)dx$.

Now we state the results of this subsection.

Theorem 3.50. *Assume that $\lambda\theta < 0$. Let the initial data $u_0 \in \mathbf{L}^{\infty}(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, $a \in (0,1]$ are small $\|u_0\|_{\mathbf{L}^{\infty}} + \|u_0\|_{\mathbf{L}^{1,a}} \leq \varepsilon$, $\lambda\theta \leq -C\varepsilon < 0$. Then the Cauchy problem (3.172) has a unique global solution*

$$u(t,x) \in \mathbf{C}\left([0,\infty);\mathbf{L}^{\infty}(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)\right)$$

satisfying the asymptotics

$$u(t) = \theta G_0(t)g^{-\frac{n}{2}}(t) + O\left(\langle t\rangle^{-\frac{n}{2}}g^{-\frac{n}{2}-1}(t)\log\log t\right) \tag{3.176}$$

for $t \to \infty$ uniformly in $x \in \mathbf{R}^n$.

Proof of Theorem 3.50. We use Theorem 3.2 considering the integral formula (3.173) the Green operator should be changed by $\mathcal{G}(t-\tau)\mathcal{B}$. Define the norms

$$\|\phi\|_{\mathbf{Z}} = \|\phi(t)\|_{\mathbf{L}^\infty} + \|\phi(t)\|_{\mathbf{L}^{1,a}},$$

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0}\left(\langle t\rangle^{\frac{n}{2}}\|\phi(t)\|_{\mathbf{L}^\infty} + \langle t\rangle^{-\frac{a}{2}}\|\phi(t)\|_{\mathbf{L}^{1,a}}\right)$$

and

$$\|\phi\|_{\mathbf{Y}} = \|\langle t\rangle\phi(t)\|_{\mathbf{X}},$$

where $a \in (0,1)$. Note that the $\mathbf{L}^1$ norm is estimated by the norm $\mathbf{X}$

$$\begin{aligned}\|\phi(t)\|_{\mathbf{L}^1} &= \int_{|x|\le\langle t\rangle^{\frac{1}{2}}}|\phi(t,x)|\,dx + \int_{|x|>\langle t\rangle^{\frac{1}{2}}}|x|^{-\alpha}|x|^{\alpha}|\phi(t,x)|\,dx \\ &\le C\langle t\rangle^{\frac{n}{2}}\|\phi(t)\|_{\mathbf{L}^\infty} + C\langle t\rangle^{-\frac{a}{2}}\|\phi(t)\|_{\mathbf{L}^{1,a}} \le C\|\phi\|_{\mathbf{X}}. \qquad (3.177)\end{aligned}$$

By Lemma 3.49 we see that $G_0(t,x) = (4\pi\alpha(t+1))^{-\frac{n}{2}}e^{-\frac{|x|^2}{4\alpha(t+1)}}$ is the asymptotic kernel with a functional $f(\phi) = \int_{\mathbf{R}^n}\phi(x)\,dx$.

Now we prove that the triad $(\mathbf{X},\mathbf{Y},\mathcal{G}\mathcal{B})$ is concordant. In view of the estimate $g^{-1}(\tau) \le C$ and Lemma 3.49 we get

$$\begin{aligned}&\left\|\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)\mathcal{B}\phi(\tau)\,d\tau\right\|_{\mathbf{L}^\infty} + \left\|\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)\mathcal{B}\phi(\tau)\,d\tau\right\|_{\mathbf{L}^{1,a}} \\ &\le C\|\langle t\rangle\phi\|_{\mathbf{X}} \le C\|\langle t\rangle\phi\|_{\mathbf{X}}\,g^{-1}(t)\end{aligned}$$

for all $0 \le t \le 4$. We now consider $t > 4$. Via the condition of the lemma for the function $g(t)$ we have the estimate $\langle t\rangle^{-\frac{a}{4}} \le Cg^{-1}(t)$ and $\sup_{\tau\in[\sqrt{t},t]}g^{-1}(\tau) \le Cg^{-1}(t)$; hence, by virtue of Lemma 3.49 and (3.174) and (3.175), we obtain

$$\begin{aligned}&\left\|\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)\mathcal{B}f(\tau)\,d\tau\right\|_{\mathbf{L}^\infty} \\ &\le C\int_0^{\sqrt{t}}(t-\tau)^{-\frac{n}{2}-\frac{a}{2}}\left(\|\mathcal{B}f(\tau)\|_{\mathbf{L}^\infty} + \|\mathcal{B}f(\tau)\|_{\mathbf{L}^{1,a}}\right)d\tau \\ &+Cg^{-1}(t)\int_{\sqrt{t}}^{\frac{t}{2}}(t-\tau)^{-\frac{n}{2}-\frac{a}{2}}\left(\|\mathcal{B}f(\tau)\|_{\mathbf{L}^\infty} + \|\mathcal{B}f(\tau)\|_{\mathbf{L}^{1,a}}\right)d\tau \\ &+Cg^{-1}(t)\int_{\frac{t}{2}}^{t}\|\mathcal{B}f(\tau)\|_{\mathbf{L}^\infty}\,d\tau.\end{aligned}$$

Therefore, by using the definition of the norm $\mathbf{X}$ we get

$$\left\| \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau)\, \mathcal{B}f(\tau)\, d\tau \right\|_{\mathbf{L}^\infty}$$
$$\leq C \left\| \langle t \rangle f \right\|_{\mathbf{X}} \int_0^{\sqrt{t}} (t-\tau)^{-\frac{n}{2}-\frac{a}{2}} \langle \tau \rangle^{\frac{a}{2}-1} d\tau$$
$$+ C g^{-1}(t) \left\| \langle t \rangle f \right\|_{\mathbf{X}} \int_{\sqrt{t}}^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}-\frac{a}{2}} \langle \tau \rangle^{\frac{a}{2}-1} d\tau$$
$$+ C g^{-1}(t) \left\| \langle t \rangle f \right\|_{\mathbf{X}} \int_{\frac{t}{2}}^{t} \langle \tau \rangle^{-\frac{n}{2}-1} d\tau$$
$$\leq C \left(t^{-\frac{n}{2}-\frac{a}{4}} + g^{-1}(t)\, t^{-\frac{n}{2}} \right) \left\| \langle t \rangle f \right\|_{\mathbf{X}} \leq C g^{-1}(t)\, t^{-\frac{n}{2}} \left\| \langle t \rangle f \right\|_{\mathbf{X}}$$

and, similarly,

$$\left\| \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{L}^{1,a}}$$
$$\leq C \int_0^t g^{-1}(\tau) \left\| f(\tau) \right\|_{\mathbf{L}^{1,a}} d\tau$$
$$\leq C \left\| \langle t \rangle f \right\|_{\mathbf{X}} \int_0^{\sqrt{t}} \tau^{\frac{a}{2}-1} d\tau + C g^{-1}(t) \left\| \langle t \rangle f \right\|_{\mathbf{X}} \int_{\sqrt{t}}^{t} \tau^{\frac{a}{2}-1} d\tau$$
$$\leq C\varepsilon \left(t^{\frac{a}{4}} + g^{-1}(t)\, t^{\frac{a}{2}} \right) \left\| \langle t \rangle f \right\|_{\mathbf{X}} \leq C g^{-1}(t)\, t^{\frac{a}{2}} \left\| \langle t \rangle f \right\|_{\mathbf{X}}.$$

Thus the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G}\mathcal{B})$ is concordant.

Since by interpolation inequality (3.177)

$$\log(2+t) \left\| \mathcal{N}(v(t)) - \mathcal{N}(w(t)) \right\|_{\mathbf{L}^1}$$
$$\leq C \log(2+t) \left(\left\| v(t) \right\|_{\mathbf{L}^\infty}^{\sigma} + \left\| w(t) \right\|_{\mathbf{L}^\infty}^{\sigma} \right) \left\| v(t) - w(t) \right\|_{\mathbf{L}^1}$$
$$\leq C \langle t \rangle^{-1} \left\| \log(2+t)(v-w) \right\|_{\mathbf{X}} \left(\left\| v \right\|_{\mathbf{X}}^{\sigma} + \left\| w \right\|_{\mathbf{X}}^{\sigma} \right),$$

condition (3.4) is true. Also we have

$$\left\| \mathcal{K}(v) - \mathcal{K}(w) \right\|_{\mathbf{Y}} \leq \left\| \mathcal{N}(v) - \mathcal{N}(w) \right\|_{\mathbf{Y}}$$
$$+ \frac{1}{\theta} \left(\left\| v \right\|_{\mathbf{X}} + \left\| w \right\|_{\mathbf{X}} \right) \sup_{t>0} \langle t \rangle \left\| \mathcal{N}(v(t)) - \mathcal{N}(w(t)) \right\|_{\mathbf{L}^1}$$
$$+ \frac{1}{\theta} \left\| v - w \right\|_{\mathbf{X}} \sup_{t>0} \langle t \rangle \left(\left\| \mathcal{N}(v(t)) \right\|_{\mathbf{L}^1} + \left\| \mathcal{N}(w(t)) \right\|_{\mathbf{L}^1} \right)$$
$$\leq C \left\| v - w \right\|_{\mathbf{X}} \left(\left\| v \right\|_{\mathbf{X}}^{\sigma} + \left\| w \right\|_{\mathbf{X}}^{\sigma} \right) \left(1 + \frac{1}{\theta} \left(\left\| v \right\|_{\mathbf{X}} + \left\| w \right\|_{\mathbf{X}} \right) \right).$$

Since the triad $(\mathbf{X}, \mathbf{Y}, \mathcal{G}\mathcal{B})$ is concordant we see that condition (3.5) is fulfilled. Now by applying Theorem 3.2 we easily get the results of Theorem 3.50 which completes its proof.

3.6.2 Large initial data

We now remove the smallness condition on the initial data $u_0(x)$. Consider the Cauchy problem for the Sobolev type equation

$$\begin{cases} \partial_t (u - \Delta u) - \Delta u = - |u|^{\sigma} u, \ x \in \mathbf{R}^n, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^n \end{cases} \tag{3.178}$$

in the critical case $\sigma = \frac{2}{n}$. (That is we choose $\alpha = 1$ and $\lambda = -1$ in equation (3.172).)

Define the heat kernel

$$G_0(t,x) = (4\pi(1+t))^{-\frac{n}{2}} e^{-\frac{|x|^2}{4(1+t)}}.$$

Denote $\eta = (4\pi)^{-1} \left(1 + \frac{2}{n}\right)^{-\frac{n}{2}}$.

We will prove the following result.

Theorem 3.51. *Let $\sigma = \frac{2}{n}$. We assume that the initial data $u_0 \in \mathbf{W}^2_{\infty}(\mathbf{R}^n) \cap \mathbf{W}^{2,1}_1(\mathbf{R}^n) \cap \mathbf{H}^{2,k}(\mathbf{R}^n) \cap \mathbf{C}(\mathbf{R}^n)$, with $k > 6 + 3n$. Then the Cauchy problem (3.178) has a unique global solution $u \in \mathbf{C}\left([0,\infty); \mathbf{W}^2_{\infty}(\mathbf{R}^n) \cap \mathbf{H}^{2,k}(\mathbf{R}^n)\right)$. Moreover the solution u has only one of the following asymptotics for large time $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$*

$$u(t) = \left(\frac{n}{2\eta}\right)^{\frac{n}{2}} G_0(t) \log^{-\frac{n}{2}} t + O\left(t^{-\frac{n}{2}} \log^{-\frac{n}{2}-\gamma} t\right)$$

where $0 < \gamma < \frac{1}{n}$, or

$$u(t) = O\left(t^{-\frac{n}{2}} \log^{-\frac{n}{2}-1} t\right).$$

Before proving Theorem 3.51 we obtain the weighted energy type estimates and find the time decay estimates for the term Δu_t.

Weighted energy type estimates

By applying a standard contraction mapping principle we have the following result.

Proposition 3.52. *Let $0 < \sigma \leq \frac{2}{n}$. Suppose that the initial data $u_0 \in \mathbf{H}^{m,k}(\mathbf{R}^n)$ with $k, m \geq 0$. Then there exists a positive time T and a unique solution $u \in \mathbf{C}\left([0,T]; \mathbf{H}^{m,k}(\mathbf{R}^n)\right)$ to the Cauchy problem (3.178).*

Using the idea of paper Nishihara and Zhao [2006] we obtain the weighted energy type estimates (see also papers Ikehata et al. [2004], Ikehata and Tanizawa [2005], Todorova and Yordanov [2001]).

Lemma 3.53. *Let* $0 < \sigma \leq \frac{2}{n}$. *Suppose that the initial data* $u_0 \in \mathbf{H}^{2,k}(\mathbf{R}^n)$ *with* $k > 6\left(1 + \frac{2}{\sigma} - \frac{n}{2}\right)$. *Then there exists a unique global solution* $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^{2,k}(\mathbf{R}^n)\right)$ *to the Cauchy problem (3.178) which satisfies the a priori estimates*

$$\|u(t)\|_{\mathbf{L}^2}^2 + \langle t\rangle^{-k}\|u(t)\|_{\mathbf{H}^{0,k}}^2 \leq C\langle t\rangle^{\frac{n}{2}-\frac{1}{\frac{\sigma}{2}}}$$

and

$$\int_0^t (t+t_0)^{1+\frac{2}{\sigma}-\frac{n}{2}}\left(\|\nabla u(t)\|_{\mathbf{L}^2}^2 + \|u(t)\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2}\right)dt \leq C(t+t_0)$$

for all $t > 0$.

Proof. Let u be a solution constructed in Proposition 3.52. We multiply equation (3.178) by $2\psi u_t + \phi u$ with arbitrary weight functions $\psi(t,x)$ and $\phi(t,x)$. Then integrating over $\mathbf{R}^n$ we get

$$0 = \int_{\mathbf{R}^n}\left(2\psi u_t^2 - 2\psi u_t \Delta u_t - 2\psi u_t \Delta u + 2\psi|u|^{\sigma} u u_t\right.$$
$$\left. + \phi u u_t - \phi u \Delta u_t - \phi u \Delta u + \phi|u|^{\sigma+2}\right)dx.$$

Then integrating by parts with respect to x we find

$$\frac{dE}{dt} + H = 0, \tag{3.179}$$

where

$$E = \int_{\mathbf{R}^n}\left(\frac{\phi}{2}u^2 + \left(\frac{\phi}{2}+\psi\right)|\nabla u|^2 + \frac{\psi}{\frac{\sigma}{2}+1}|u|^{\sigma+2}\right)dx,$$

$$H = \int_{\mathbf{R}^n}\left(2\psi\left(u_t^2 + |\nabla u_t|^2\right) + \varPhi|\nabla u|^2 + \varLambda|u|^{\sigma+2} - \frac{\phi_t}{2}u^2\right.$$
$$\left. + (u+2u_t)(\nabla\psi\cdot\nabla u_t) + u(\nabla u\cdot\nabla\phi) + 2u_t(\nabla u\cdot\nabla\psi)\right)dx,$$

$$\varPhi = \phi - \frac{\phi_t}{2} - \psi_t$$

and

$$\varLambda = \phi - \frac{2\psi_t}{\sigma+2}.$$

Multiplying equality (3.179) by $(t+t_0)^{\beta}$ with $\beta = \frac{2}{\sigma} - \frac{n}{2} + 1$, we get

$$\frac{d}{dt}\left((t+t_0)^{\beta}E\right) = -(t+t_0)^{\beta}\left(H - \frac{\beta}{t+t_0}E\right). \tag{3.180}$$

We now choose $\psi = \alpha\phi$, and consider the right-hand side of equality (3.180)

$$H-\frac{\beta}{t+t_0}E=\int_{\mathbf{R}^n}\left(2\alpha\phi\left(u_t^2+|\nabla u_t|^2\right)+\left(\varPhi-\frac{\beta\left(1+2\alpha\right)\phi}{2\left(t+t_0\right)}\right)|\nabla u|^2\right.$$
$$+\left(\varLambda-\frac{2\beta\alpha\phi}{(t+t_0)\left(\sigma+2\right)}\right)|u|^{\sigma+2}+\frac{1}{2}\left(-\phi_t-\frac{\beta\phi}{t+t_0}\right)u^2$$
$$\left.+\alpha\left(u+2u_t\right)\left(\nabla u_t\cdot\nabla\phi\right)+\left(u+2\alpha u_t\right)\left(\nabla u\cdot\nabla\phi\right)\right)dx.$$

Applying the estimates

$$|\alpha u\left(\nabla u_t\cdot\nabla\phi\right)|\leq\frac{2\alpha^2}{|\phi_t|}|\nabla\phi|^2|\nabla u_t|^2+\frac{1}{8}|\phi_t|\,u^2,$$

$$|u\left(\nabla u\cdot\nabla\phi\right)|\leq\frac{2}{|\phi_t|}|\nabla\phi|^2|\nabla u|^2+\frac{1}{8}|\phi_t|\,u^2,$$

$$2\alpha\,|u_t\left(\nabla u_t\cdot\nabla\phi\right)|\leq\alpha\,|\nabla\phi|\left(u_t^2+|\nabla u_t|^2\right)$$

and

$$2\alpha\,|u_t\left(\nabla u\cdot\nabla\phi\right)|\leq\alpha\,|\nabla\phi|\left(u_t^2+|\nabla u|^2\right)$$

we get

$$H-\frac{\beta}{t+t_0}E\geq\int_{\mathbf{R}^n}\left(\left(\alpha\phi-\frac{2\alpha^2}{|\phi_t|}|\nabla\phi|^2-3\alpha\,|\nabla\phi|\right)\left(u_t^2+|\nabla u_t|^2\right)\right.$$
$$+\left(\varPhi-\frac{\beta\left(1+2\alpha\right)\phi}{2\left(t+t_0\right)}-\frac{2}{|\phi_t|}|\nabla\phi|^2-\alpha\,|\nabla\phi|\right)|\nabla u|^2$$
$$\left.+\left(\varLambda-\frac{\beta\alpha\phi}{(t+t_0)\left(\frac{\sigma}{2}+1\right)}\right)|u|^{\sigma+2}+\frac{1}{8}\left(|\phi_t|-\frac{4\beta\phi}{t+t_0}\right)u^2\right)dx.$$

We choose the weight

$$\phi\left(t,x\right)=1+\frac{a\,|x|^{2k}}{\left(t+t_0\right)^k}$$

with some $a>0$, then for sufficiently large t_0 we find

$$\varPhi-\frac{\beta\left(1+2\alpha\right)\phi}{2\left(t+t_0\right)}-\alpha\,|\nabla\phi|-\frac{2}{|\phi_t|}|\nabla\phi|^2$$
$$=\phi-\frac{1+2\alpha}{2}\left(\frac{\beta}{t+t_0}\phi+\phi_t\right)-\alpha\,|\nabla\phi|-\frac{2}{|\phi_t|}|\nabla\phi|^2$$
$$\geq\left(1-\frac{\left(1+2\alpha\right)\left(\beta+k\right)}{2\left(t+t_0\right)}-\frac{2\alpha k}{\sqrt{t+t_0}}\right)\phi$$
$$-\frac{8ka\,|x|^{2k-2}}{\left(t+t_0\right)^{k-1}}\geq\frac{1}{2}\phi-\frac{8ka\,|x|^{2k-2}}{\left(t+t_0\right)^{k-1}}$$
$$\geq\frac{1}{2}-4a\left(16\left(k-1\right)\right)^{k-1}+\frac{a\,|x|^{2k}}{4\left(t+t_0\right)^k}\geq\frac{1}{8}\phi$$

since by the Young inequality

$$\frac{8ka\,|x|^{2k-2}}{(t+t_0)^{k-1}} \leq 4a\,(16\,(k-1))^{k-1} + \frac{a\,|x|^{2k}}{4\,(t+t_0)^k}$$

if $a \leq \frac{1}{16(16(k-1))^{k-1}}$. In the same manner we have choosing $\alpha = \frac{1}{2}$

$$\begin{aligned}
&\Lambda - \frac{\beta\alpha\phi}{(t+t_0)\left(\frac{\sigma}{2}+1\right)} \\
&= \phi\left(1 - \frac{\beta\alpha}{(t+t_0)\left(\frac{\sigma}{2}+1\right)}\right) - \frac{\alpha\phi_t}{\frac{\sigma}{2}+1} \\
&= \left(\frac{1}{2} - \frac{\alpha\,(\beta+k)}{(t+t_0)\left(\frac{\sigma}{2}+1\right)}\right)\phi \geq \frac{1}{4}\phi.
\end{aligned}$$

Then by the estimates

$$\alpha\phi - \frac{2\alpha^2}{|\phi_t|}\,|\nabla\phi|^2 - 3\alpha\,|\nabla\phi| \geq \frac{\alpha}{8}\phi$$

and

$$|\phi_t| - \frac{4\beta\phi}{t+t_0} \geq \frac{(k-6\beta)\,a\,|x|^{2k}}{(t+t_0)^{k+1}} - \frac{6\beta}{t+t_0}$$

we see that

$$\begin{aligned}
H - \frac{\beta}{t+t_0}E \geq \frac{1}{8}\int_{\mathbf{R}^n} &\left(\alpha\phi\left(u_t^2 + |\nabla u_t|^2 + |\nabla u|^2 + |u|^{\sigma+2}\right)\right. \\
&\left. + \left(\frac{(k-6\beta)\,a\,|x|^{2k}}{(t+t_0)^{k+1}} - \frac{6\beta}{t+t_0}\right)u^2\right)dx \qquad (3.181)
\end{aligned}$$

if we choose $k > 6\beta$.

Now we estimate the second summand in the right-hand side of (3.181). By the Hölder and Young inequalities we obtain taking sufficiently large $\varrho > 0$

$$\begin{aligned}
&-\frac{1}{8}\int_{\mathbf{R}^n}\left(\frac{(k-6\beta)\,a\,|x|^{2k}}{(t+t_0)^{k+1}} - \frac{6\beta}{t+t_0}\right)u^2dx \\
&\leq \frac{\beta}{t+t_0}\int_{|x|\leq\varrho\sqrt{t+t_0}} u^2dx \leq C\,(t+t_0)^{-1}\,\|u\|^2_{\mathbf{L}^{\sigma+2}}\,\|1\|^2_{\mathbf{L}^{\left(\frac{\sigma}{2}+1\right)\frac{2}{\frac{\sigma}{2}}}\left(|x|\leq\varrho\sqrt{t+t_0}\right)} \\
&\leq C\,(t+t_0)^{\frac{\sigma n}{2(\sigma+2)}-1}\,\|u\|^2_{\mathbf{L}^{\sigma+2}} \leq C\,(t+t_0)^{-\beta} + \frac{\alpha}{16}\,\|u\|^{\sigma+2}_{\mathbf{L}^{\sigma+2}} \\
&\leq C\,(t+t_0)^{-\beta} + \frac{\alpha}{16}\int_{\mathbf{R}^n}|u|^{\sigma+2}\,\phi dx
\end{aligned}$$

since $\beta = \frac{2}{\sigma} - \frac{n}{2} + 1$. Thus

$$(t+t_0)^\beta \left(H - \frac{\beta}{t+t_0} E\right)$$
$$\geq \frac{\alpha}{16} (t+t_0)^\beta \int_{\mathbf{R}^n} \phi \left(u_t^2 + |\nabla u_t|^2 + |\nabla u|^2 + |u|^{\sigma+2}\right) dx - C.$$

Then we obtain from (3.180)

$$\frac{d}{dt}\left((t+t_0)^\beta E\right) + \frac{\alpha}{16} (t+t_0)^\beta \int_{\mathbf{R}^n} \phi \left(u_t^2 + |\nabla u_t|^2 + |\nabla u|^2 + |u|^{\sigma+2}\right) dx \leq C.$$

Integration of this inequality with respect to time yields

$$(t+t_0)^\beta E(t)$$
$$+ \frac{\alpha}{16} \int_0^t dt\, (t+t_0)^\beta \int_{\mathbf{R}^n} \phi \left(u_t^2 + |\nabla u_t|^2 + |\nabla u|^2 + |u|^{\sigma+2}\right) dx$$
$$\leq C(t+t_0) \tag{3.182}$$

which imply the estimate of the lemma. Lemma 3.53 is proved.

Estimates for Δu_t

We define the space

$$\mathbf{X} = \left\{\phi \in \mathbf{C}\left([0,\infty); \mathbf{H}^{1,k}(\mathbf{R}^n)\right) : \ \|\phi\|_{\mathbf{X}} < \infty\right\},$$

where the norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t \geq 0} \left(\langle t\rangle^{\frac{n}{4}-\frac{k}{2}} \|\phi(t)\|_{\mathbf{H}^{0,k}} + \langle t\rangle^{\frac{n}{2}} \|\phi(t)\|_{\mathbf{L}^\infty}\right)$$

with $k > 6+3n$. Also we define the norm $\|\phi\|_{\mathbf{Y}} = \|\langle t\rangle \phi(t)\|_{\mathbf{X}}$. Note that

$$\sup_{t\geq 0} \langle t\rangle^{\frac{1}{2}} \|\phi(t)\|_{\mathbf{L}^{1,1}} \leq \|\phi\|_{\mathbf{Y}}.$$

By Lemma 3.49 it follows that

Lemma 3.54. *The triad* $(\mathbf{X}, \mathbf{Y}, \mathcal{G}_0)$ *is concordant, that is for any* ϕ *such that the mean value* $\int_{\mathbf{R}^n} \phi(x)\, dx = 0$ *the inequality*

$$\left\| g(t) \int_0^t \mathcal{G}_0(t-\tau)\phi(\tau)\, d\tau \right\|_{\mathbf{X}} \leq C \|g\phi\|_{\mathbf{Y}}$$

is valid, provided that the right-hand side is finite, where $g(t) = \log^\gamma(2+t)$ *with* $\gamma > 0$.

Now we obtain the estimates for the term Δu_t in the norm $\mathbf{Y}$.

Lemma 3.55. *Suppose that the initial data* $u_0 \in \mathbf{W}^2_\infty(\mathbf{R}^n) \cap \mathbf{W}^{2,1}_1(\mathbf{R}^n)$. *Let* u *be a global solution* $u \in \mathbf{C}\left([0,\infty);\mathbf{W}^2_\infty \cap \mathbf{W}^{2,1}_1\right) \cap \mathbf{C}^1\left((0,\infty);\mathbf{W}^2_\infty \cap \mathbf{W}^{2,1}_1\right)$ *to the Cauchy problem (3.178) and satisfy the estimate*

$$\langle t\rangle^{\frac{n}{4}} \|u(t)\|_{\mathbf{L}^2} + \langle t\rangle^{-\frac{1}{2}} \|u(t)\|_{\mathbf{L}^{1,1}} \leq C,$$
$$\int_0^t \langle \tau\rangle^{1+\frac{2}{\sigma}-\frac{n}{2}} \|\nabla u(\tau)\|^2_{\mathbf{L}^2}\, d\tau \leq C\langle t\rangle \tag{3.183}$$

for all $t > 0$. *Then the estimate*

$$\left\|\langle t\rangle^{\gamma} \Delta u_t\right\|_{\mathbf{Y}} \leq C$$

is true with some $\gamma > 0$.

Proof. By the integral equation associated with the Cauchy problem (3.178) in view of Lemma 3.49 we find for $n = 1$

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-\frac{1}{2}} + C\int_{\frac{t}{2}}^t \|u(\tau)\|^3_{\mathbf{L}^\infty}\, d\tau$$
$$+ C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{1}{2}} \left(\|u(\tau)\|_{\mathbf{L}^\infty} \|u(\tau)\|^2_{\mathbf{L}^2} + \|u(\tau)\|^3_{\mathbf{L}^\infty}\right) d\tau.$$

Then by applying the Young inequality, the estimate $\|u\|_{\mathbf{L}^\infty} \leq \|u\|^{\frac{1}{2}}_{\mathbf{L}^2} \|u_x\|^{\frac{1}{2}}_{\mathbf{L}^2}$ and (3.183) we get

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-\frac{1}{2}} + C\langle t\rangle^{-\frac{1}{2}} \int_{\frac{t}{2}}^t \left(\langle t-\tau\rangle^{-1} + \langle \tau\rangle^{\frac{1}{2}} \|u_x(\tau)\|^2_{\mathbf{L}^2}\right) d\tau$$
$$+ C\langle t\rangle^{-\frac{1}{2}} \int_0^{\frac{t}{2}} \left(\langle \tau\rangle^{-1} + \langle \tau\rangle^{\frac{1}{2}} \|u_x(\tau)\|^2_{\mathbf{L}^2}\right) d\tau \leq C\langle t\rangle^{-\frac{1}{2}} \log\langle t\rangle$$

for all $t > 0$ in the case $n = 1$.

Now let us consider the case $n \geq 2$. Suppose that

$$\|u(t)\|_{\mathbf{L}^r} \leq C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{r}\right)} \log^\beta \langle t\rangle \tag{3.184}$$

for some $r \geq 2$ and let us estimate the norm $\mathbf{L}^p$ with $\frac{1}{p} \geq \frac{1}{r} - \frac{1}{2n}$. By the Sobolev imbedding inequality we have

$$\left\|\mathcal{B}\,|u|^\sigma u\right\|_{\mathbf{L}^p} \leq C\|u\|^{1+\sigma}_{\mathbf{L}^r}$$

and

$$\left\|\mathcal{B}^{\frac{1}{2}}\,|u|^{\frac{\sigma}{2}} u\right\|_{\mathbf{L}^p} \leq C\|u\|^{1+\frac{\sigma}{2}}_{\mathbf{L}^r}.$$

Hence by Lemma 3.49 we obtain

$$\begin{aligned}&\left\|\int_0^{\frac{t}{2}}\mathcal{G}(t-\tau)\mathcal{B}\left(|u|^{\sigma}u(\tau)\right)d\tau\right\|_{\mathbf{L}^p}\\&\le C\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}\left(\|u(\tau)\|_{\mathbf{L}^{1+\sigma}}^{1+\sigma}+\|u\|_{\mathbf{L}^r}^{1+\sigma}\right)d\tau\\&\le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}\int_0^{\frac{t}{2}}\langle\tau\rangle^{-1}\log^{2\beta}\langle\tau\rangle\,d\tau\le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}\log^{2\beta+1}\langle t\rangle\end{aligned}$$

and

$$\begin{aligned}&\left\|\int_{\frac{t}{2}}^{t}\mathcal{B}\mathcal{G}(t-\tau)|u|^{\sigma}u(\tau)\,d\tau\right\|_{\mathbf{L}^p}\\&\le C\int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1+\sigma}{r}-\frac{1}{p}\right)}\|u(\tau)\|_{\mathbf{L}^r}^{1+\sigma}\,d\tau\\&\le C\langle t\rangle^{-1-\frac{n}{2}+\frac{n(1+\sigma)}{2r}}\int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1+\sigma}{r}-\frac{1}{p}\right)}\log^{2\beta}\langle\tau\rangle\,d\tau\\&\le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}\log^{2\beta}\langle t\rangle\,.\end{aligned}$$

Therefore from (3.184) it follows that

$$\|u(t)\|_{\mathbf{L}^p}\le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}\log^{2\beta+1}\langle t\rangle \tag{3.185}$$

for $\frac{1}{p}\ge\frac{1}{r}-\frac{1}{2n}$. So starting from estimate (3.184) with $r=2$ and $\beta=0$ by n iterations we can arrive to the estimate of the $\mathbf{L}^{\infty}$ norm

$$\|u(t)\|_{\mathbf{L}^{\infty}}\le C\langle t\rangle^{-\frac{n}{2}}\log^{2^n}\langle t\rangle \tag{3.186}$$

for all $t>0$ and $n\ge 2$.

Now we estimate the derivatives by the estimates (3.183), (3.186) and by Lemma 3.49

$$\begin{aligned}&\int_0^{\frac{t}{2}}\|\nabla\mathcal{G}(t-\tau)\mathcal{B}|u|^{\sigma}u(\tau)\|_{\mathbf{L}^p}\,d\tau\\&\le C\langle t\rangle^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)}\int_0^{\frac{t}{2}}\left(\|u(\tau)\|_{\mathbf{L}^{\sigma+1}}^{\sigma+1}+\|u(\tau)\|_{\mathbf{L}^{(\sigma+1)p}}^{\sigma+1}\right)d\tau\\&\le C\langle t\rangle^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)}\int_0^{\frac{t}{2}}\langle\tau\rangle^{-1}\log^{2^{n+1}}\langle\tau\rangle\,d\tau\le C\langle t\rangle^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)}\log^{2^{n+1}}\langle t\rangle\end{aligned}$$

and

$$\int_{\frac{t}{2}}^{t} \left\| \nabla \mathcal{G}\left(t-\tau\right) \mathcal{B}\left|u\right|^{\sigma} u\left(\tau\right) \right\|_{\mathbf{L}^{p}} d\tau$$
$$\leq C \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-\frac{1}{2}} \left\| u\left(\tau\right) \right\|_{\mathbf{L}^{p(\sigma+1)}}^{\sigma+1} d\tau$$
$$\leq C \langle t \rangle^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)} \int_{0}^{\frac{t}{2}} \langle t-\tau \rangle^{-1} \log^{2^{n+1}} \langle \tau \rangle \, d\tau$$
$$\leq C \langle t \rangle^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)} \log^{2^{n+1}} \langle t \rangle .$$

Hence we find

$$\left\| \nabla u\left(t\right) \right\|_{\mathbf{L}^{p}} \leq C \langle t \rangle^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)} \log^{2^{n+1}} \langle t \rangle \tag{3.187}$$

for all $t > 0$ and $1 \leq p \leq \infty$.

Finally we estimate $\Delta u_t\left(t\right)$ by using the integral equation, estimates (3.183), (3.186), (3.187) and Lemma 3.49

$$\int_{0}^{\frac{t}{2}} \left\| \Delta \mathcal{B} \mathcal{G}_{t}\left(t-\tau\right) \left|u\right|^{\sigma} u\left(\tau\right) \right\|_{\mathbf{L}^{\infty}} d\tau$$
$$\leq C \langle t \rangle^{-2-\frac{n}{2}} \int_{0}^{\frac{t}{2}} \left(\left\| u \right\|_{\mathbf{L}^{\sigma+1}}^{\sigma+1} + \left\| u \right\|_{\mathbf{L}^{\infty}}^{\sigma+1} \right) d\tau$$
$$\leq C \langle t \rangle^{-2-\frac{n}{2}} \int_{0}^{\frac{t}{2}} \langle \tau \rangle^{-1} \log^{2^{n+2}} \langle t \rangle \, d\tau \leq C \langle t \rangle^{-2-\frac{n}{2}} \log^{2^{n+3}} \langle t \rangle$$

and

$$\int_{\frac{t}{2}}^{t} \left\| \nabla \mathcal{B} \mathcal{G}_{t}\left(t-\tau\right) \left|u\right|^{\sigma} \nabla u\left(\tau\right) \right\|_{\mathbf{L}^{\infty}} d\tau$$
$$\leq C \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-\frac{3}{2}} \left\| u\left(\tau\right) \right\|_{\mathbf{L}^{\infty}}^{\sigma} \left\| \nabla u\left(\tau\right) \right\|_{\mathbf{L}^{\infty}} d\tau$$
$$\leq C \langle t \rangle^{-\frac{3}{2}-\frac{n}{2}} \log^{2^{n+2}} \langle t \rangle \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-\frac{3}{2}} d\tau \leq C \langle t \rangle^{-\frac{3}{2}-\frac{n}{2}} \log^{2^{n+2}} \langle t \rangle$$

for all $t > 0$. Thus

$$\left\| \Delta u_t\left(t\right) \right\|_{\mathbf{L}^{\infty}} \leq \langle t \rangle^{-\gamma-\frac{n}{2}}$$

for all $t > 0$. The weighted norms are estimated in the same manner. Lemma 3.55 is proved.

Proof of Theorem 3.51

As in the proof of Theorem 3.38 in Subsection 3.5.2 now Theorem 3.51 is a consequence of Lemma 3.42, Lemma 3.55, Lemma 3.47 and Theorem 3.48.

3.7 Whitham type equations

In this section we study large time asymptotics of solutions to the Cauchy problem for dissipative equations

$$\begin{cases} u_t + \mathcal{N}(u) + \mathcal{L}u = 0, \; x \in \mathbf{R}, \; t > 0, \\ u(0,x) = u_0(x), \; x \in \mathbf{R}. \end{cases} \tag{3.188}$$

The linear part of equation (3.188) is a pseudodifferential operator defined by the Fourier transformation

$$\mathcal{L}u = \overline{\mathcal{F}}_{\xi \to x} \left(L(\xi) \widehat{u}(\xi) \right),$$

and the nonlinearity $\mathcal{N}(u)$ is a cubic pseudodifferential operator of nonconvective type

$$\begin{aligned} \mathcal{N}(u) = \overline{\mathcal{F}}_{\xi \to x} \int_{\mathbf{R}} a(t,\xi,y) \widehat{u}(t,\xi - y) \widehat{u}(t,y) \, dy \\ + \overline{\mathcal{F}}_{\xi \to x} \int_{\mathbf{R}^2} b(t,\xi,y,z) \widehat{u}(t,\xi - y) \widehat{u}(t,y - z) \widehat{u}(t,z) \, dydz, \end{aligned}$$

defined by the symbols $a(t,\xi,y)$ and $b(t,\xi,y,z)$. We consider here the real valued solutions $u(t,x)$.

We suppose that the symbols $a(t,\xi,y)$ and $b(t,\xi,y,z)$ are continuous functions with respect to time $t > 0$ and the operators $\mathcal{N}$ and $\mathcal{L}$ have a finite order, that is the symbols $a(t,\xi,y)$, $b(t,\xi,y,z)$ and $L(\xi)$ grow with respect to ξ, y and z no faster than a power of some order κ

$$\begin{aligned} |L(\xi)| \le C \langle \xi \rangle^{\kappa}, \; |a(t,\xi,y)| \le C \left(\langle \xi \rangle^{\kappa} + \langle y \rangle^{\kappa} \right), \\ |b(t,\xi,y,z)| \le C \left(\langle \xi \rangle^{\kappa} + \langle y \rangle^{\kappa} + \langle z \rangle^{\kappa} \right), \end{aligned}$$

where $C > 0$.

The particular case of model equation (3.188) is, for example, the cubic nonlinear heat equation

$$u_t + u^3 - u_{xx} = 0, \; x \in \mathbf{R}, \, t > 0, \tag{3.189}$$

when $\mathcal{N}(u) = u^3$, $\mathcal{L}u = -u_{xx}$, that is $a(t,\xi,y) = 0$, $b(t,\xi,y,z) = 1$ and $L(\xi) = \xi^2$. Another example is the potential Ott-Sudan-Ostrovsky equation

$$u_t + (u_x)^2 + u_{xxx} + \mathcal{H}u_{xxx} = 0, \; x \in \mathbf{R}, \, t > 0, \tag{3.190}$$

which follows from (3.188) if we take $\mathcal{N}(u) = (u_x)^2$, $\mathcal{L}u = u_{xxx} + \mathcal{H}u_{xxx}$, that is $a(t,\xi,y) = -(\xi - y) y$, $b(t,\xi,y,z) = 0$, $L(\xi) = |\xi|^3 - i\xi^3$. Here

$$\mathcal{H}(\phi) = \mathrm{PV}\frac{1}{\pi} \int \frac{\phi(y)}{x - y} dy$$

is the Hilbert transformation. Equation (3.190) comes from the Whitham (see Whitham [1999]) equation

$$v_t + vv_x + v_{xxx} + \mathcal{H}v_{xxx} = 0,\ x \in \mathbf{R},\ t > 0, \tag{3.191}$$

if we introduce a potential $u = \int_{-\infty}^{x} u(t,x)\,dx$, which vanishes as $x \to \infty$ if we consider the initial data $v(0,x)$ with zero total mass $\int_{\mathbf{R}} v(0,x)\,dx = 0$. Therefore $\int_{\mathbf{R}} v(t,x)\,dx = 0$ for all $t > 0$ in view of equation (3.191).

Suppose that the linear operator $\mathcal{L}$ satisfies the dissipation condition which in terms of the symbol $L(\xi)$ has the form

$$\operatorname{Re} L(\xi) \geq \mu \{\xi\}^{\delta} \langle\xi\rangle^{\nu} \tag{3.192}$$

for all $\xi \in \mathbf{R}$, where $\mu > 0$, $\nu \geq 0$, $\delta > 0$. Also we suppose that the symbol is smooth $L(\xi) \in \mathbf{C}^1(\mathbf{R}^n)$ and has the estimate

$$\left|\partial_\xi^l L(\xi)\right| \leq C\{\xi\}^{\delta - l} \langle\xi\rangle^{\nu} \tag{3.193}$$

for all $\xi \in \mathbf{R} \setminus \{0\}$, $l = 0, 1$.

To find the asymptotic formulas for the solution we assume that the symbol $L(\xi)$ has the following asymptotic representation in the origin

$$L(\xi) = L_0(\xi) + O\left(|\xi|^{\delta+\gamma}\right) \tag{3.194}$$

for all $|\xi| \leq 1$, where $L_0(\xi) = \mu_1 |\xi|^{\delta} + i\mu_2 |\xi|^{\delta-1}\xi$, $\mu_1 > 0$, $\mu_2 \in \mathbf{R}$, $\gamma \in (0,1)$.

We suppose that the symbols of the nonlinear operator $\mathcal{N}$ are such that

$$\left|\partial_\xi^l a(t,\xi,y)\right| \leq C\{\xi - y\}^{-l} \left(\{\xi - y\}^{\alpha} \langle\xi - y\rangle^{\sigma} + \{y\}^{\alpha} \langle y\rangle^{\sigma}\right) \tag{3.195}$$

for all $\xi, y \in \mathbf{R}$, $t > 0$, $l = 0, 1$, and

$$\begin{aligned} \left|\partial_\xi^l b(t,\xi,y,z)\right| \leq C\{\xi - y\}^{-l} \Big(&\{\xi - y\}^{\beta} \langle\xi - y\rangle^{\sigma} \\ &+ \{y - z\}^{\beta} \langle y - z\rangle^{\sigma} + \{z\}^{\beta} \langle z\rangle^{\sigma}\Big) \end{aligned} \tag{3.196}$$

for all $\xi, y, z \in \mathbf{R}$, $t > 0$, $l = 0, 1$, where $\alpha \geq 0, \beta \geq 0$, $\sigma = 0$ if $\nu = 0$ and $\sigma \in [0, \nu)$ if $\nu > 0$. We consider the case of nonlinearity of the nonconvective type, that is we suppose that

$$a(t,0,y) \neq 0 \text{ or } b(t,0,y,z) \neq 0.$$

Our aim is to obtain the large time asymptotic behavior of solutions to the Cauchy problem for nonlinear evolution equation (3.188) in the critical case. The critical case with respect to the large time asymptotic behavior of solutions means that

$$\delta = 1 + \alpha = 2 + \beta.$$

We assume that the symbols of the nonlinearity have the asymptotics

$$a(t,0,y) = a_0(y) + O\left(\{y\}^{\alpha+\gamma}\langle y\rangle^{\sigma}\right) \tag{3.197}$$

and

$$b(t,0,y,z) = b_0(y,z) + O\left((\{y\}+\{z\})^{\beta+\gamma}(\langle y\rangle+\langle z\rangle)^{\sigma}\right) \tag{3.198}$$

for all $y, z \in \mathbf{R}$, $t > 0$, where $\gamma \in (0,1)$, $a_0(y)$ is homogeneous of order α and $b_0(y,z)$ is homogeneous of order β. For example, the equation

$$\begin{aligned} & u_t + \left(\mu_1\left(-\partial_x^2\right)^{\frac{\delta}{2}} + \mu_2\left(-\partial_x^2\right)^{\frac{\delta-1}{2}}\partial_x + \mu_3\left(-\partial_x^2\right)^{\frac{\nu}{2}}\right)u \\ & + a_1\prod\nolimits_{j=1}^{2}\partial_x^{\alpha_j}u + b_1\prod\nolimits_{j=1}^{3}\partial_x^{\beta_j}u = 0, \end{aligned} \tag{3.199}$$

where $\mu_1 > 0$, $\mu_2, a_1, b_1 \in \mathbf{R}$, $\mu_3 \geq 0$, $0 < \delta < \nu$, $\alpha = \alpha_1 + \alpha_2$, $\beta = \beta_1 + \beta_2 + \beta_3$ satisfies conditions (3.194)-(3.198). Also we suppose the total mass of the initial data is not zero

$$\theta = \widehat{u}_0(0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} u_0(x)\,dx \neq 0.$$

Denote

$$\begin{aligned} \kappa \equiv \theta^2 & \int_{\mathbf{R}} a_0(y)\,e^{-L_0(-y)-L_0(y)}dy \\ & + \omega\theta^3\int_{\mathbf{R}^2} b_0(y,z)\,e^{-L_0(-y)-L_0(y-z)-L_0(z)}dydz, \end{aligned}$$

where $\omega = 0$ if $a_0 \neq 0$ and $\omega = 1$ if $a_0 \equiv 0$. To obtain asymptotics of solutions in the critical case we assume below that $\kappa > 0$. The condition $\kappa > 0$ implies the restriction on the nonlinearity and yields the positivity of the value $\int_{\mathbf{R}} \mathcal{N}(u_1)\,dx > 0$, where u_1 is the first approximation of the solution. We easily see that, for example, the equation

$$\begin{aligned} & u_t + \left(\mu_1\left(-\partial_x^2\right)^{\frac{\delta}{2}} + \mu_3\left(-\partial_x^2\right)^{\frac{\nu}{2}}\right)u \\ & + a_1\prod\nolimits_{j=1}^{2}\left(-\partial_x^2\right)^{\frac{\alpha_j}{2}}u + b_1\prod\nolimits_{j=1}^{3}\left(-\partial_x^2\right)^{\frac{\beta_j}{2}}u = 0 \end{aligned}$$

satisfies the condition $\kappa > 0$ if

$$\mu_1, \mu_3, a_1, b_1, \theta > 0,\ \delta < \nu,\ \alpha_1 + \alpha_2 = \delta - 1,\ \beta_1 + \beta_2 + \beta_3 = \delta - 2.$$

Define the norms

$$\|\varphi\|_{\mathbf{A}^{0,\infty}} = \|\widehat{\varphi}(\cdot)\|_{\mathbf{L}^{\infty}_{\xi}(|\xi|\leq 1)} \text{ and } \|\varphi\|_{\mathbf{B}^{0,1}} = \|\widehat{\varphi}(\cdot)\|_{\mathbf{L}^{1}_{\xi}(|\xi|\geq 1)}$$

and

$$\|\varphi\|_{\mathbf{D}^{0,0}} = \left\| |\partial_\xi|^\gamma \widehat{\varphi}(\cdot) \right\|_{\mathbf{L}^\infty_\xi}.$$

Denote

$$G_0(x) = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-L_0(\xi)}\right).$$

We prove the following result.

Theorem 3.56. *Assume that* $u_0 \in \mathbf{A}^{0,\infty} \cap \mathbf{B}^{0,1} \cap \mathbf{D}^{0,0}$ *with sufficiently small norm*

$$\|u_0\|_{\mathbf{A}^{0,\infty}} + \|u_0\|_{\mathbf{B}^{0,1}} + \|u_0\|_{\mathbf{D}^{0,0}} = \varepsilon.$$

Suppose that $\kappa > 0$ *and*

$$\theta = \widehat{u_0}(0) > 0.$$

Then there exists a unique solution

$$u(t,x) \in \mathbf{L}^\infty((0,\infty)\times\mathbf{R}) \cap \mathbf{C}\left([0,\infty); \mathbf{A}^{0,\infty} \cap \mathbf{B}^{0,1} \cap \mathbf{D}^{0,0}\right)$$

of the Cauchy problem (3.188) satisfying the following time decay estimate

$$\|u(t)\|_\infty \le C\langle t\rangle^{-\frac{1}{\delta}} (1+\kappa\log\langle t\rangle)^{\frac{\omega}{2}-1}.$$

Furthermore the following asymptotic formula

$$u(t,x) = \frac{\theta t^{-\frac{1}{\delta}}}{1+\kappa\log t} G_0\left(xt^{-\frac{1}{\delta}}\right) + O\left(\frac{t^{-\frac{1}{\delta}}}{(\log t)\log\log t}\right)$$

is valid for $t \ge 1$ *uniformly with respect to* $x \in \mathbf{R}$ *if* $a_0(y) \ne 0$. *In the case of* $a_0(y) \equiv 0$ *the asymptotics*

$$u(t,x) = \frac{\theta t^{-\frac{1}{\delta}}}{\sqrt{1+\kappa\log t}} G_0\left(xt^{-\frac{1}{\delta}}\right) + O\left(\frac{t^{-\frac{1}{\delta}}}{\sqrt{\log t}\log\log t}\right)$$

is true for $t \ge 1$ *uniformly with respect to* $x \in \mathbf{R}$.

Remark 3.57. The conditions of the theorem on the initial data u_0 can also be expressed in terms of the usual weighted Sobolev spaces as follows

$$\|u_0\|_{\mathbf{H}^{\rho,0}} + \|u_0\|_{\mathbf{H}^{0,\rho}} \le \varepsilon,$$

where $\rho > \frac{1}{2}$. However, the conditions on the initial data u_0 are described more precisely in the norm $\mathbf{A}^{0,\infty} \cap \mathbf{B}^{0,1} \cap \mathbf{D}^{0,0}$.

Remark 3.58. We give two examples of the application of Theorem 3.56: 1) In the case of cubic nonlinear heat equation (3.189) we have $a(t,\xi,y) = 0$, $b(t,\xi,y,z) = b_0(y,z) = 1$ and $L(\xi) = L_0(\xi) = \xi^2$. The conditions (3.196) and (3.198) are fulfilled with $\sigma = \beta = 0$, $\delta = \nu = 2$. Then for small initial data u_0 such that $\theta > 0$ and the norm

$$\|u_0\|_{\mathbf{H}^{\rho,0}} + \|u_0\|_{\mathbf{H}^{0,\rho}} \le \varepsilon, \rho > \frac{1}{2},$$

the asymptotics

$$u(t,x) = \frac{\theta}{\sqrt{4\pi t}\sqrt{1+\kappa \log t}} e^{-\frac{x^2}{2t}} + O\left(\frac{t^{-\frac{1}{2}}}{\sqrt{\log t}\log\log t}\right)$$

is true for large t. 2) For the potential Whitham equation (3.190) we have $a(t,\xi,y) = -(\xi - y)\,y$, $b(t,\xi,y,z) = 0$, $L(\xi) = |\xi|^3 - i\xi^3$, $a_0(y) = y^2$ and $L_0(\xi) = |\xi|^3 - i\xi^3$. The conditions (3.195) and (3.197) are fulfilled with $\sigma = \alpha = 2$, $\delta = \nu = 3$. Then for small initial data u_0 such that $\theta > 0$ and the norm

$$\|u_0\|_{\mathbf{H}^{\rho,0}} + \|u_0\|_{\mathbf{H}^{0,\rho}} \le \varepsilon, \rho > \frac{1}{2},$$

the asymptotics

$$u(t,x) = \frac{\theta t^{-\frac{1}{3}}}{1+\kappa\log t} G_0\left(xt^{-\frac{1}{3}}\right) + O\left(\frac{t^{-\frac{1}{3}}}{(\log t)\log\log t}\right)$$

is valid. Note that there is no blow up for the Whitham equation (3.190), that is all solutions exist globally in time (see Naumkin and Shishmarev [1994b]) even if $\theta < 0$. It is interesting to know the character of the large time asymptotic behavior of solutions in the case of $\theta < 0$. As far as we know this is an open problem.

3.7.1 Preliminary Lemmas

The Green operator $\mathcal{G}$ of the problem (3.188) is given by

$$\mathcal{G}(t)\,\phi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-L(\xi)t}\hat{\phi}(\xi)\right).$$

First we collect some preliminary estimates for the Green operator $\mathcal{G}(t)$ in the norms

$$\begin{aligned}
\|\varphi(t)\|_{\mathbf{A}^{\rho,p}} &= \||\cdot|^{\rho}\,\widehat{\varphi}(t,\cdot)\|_{\mathbf{L}^p_\xi(|\xi|\le 1)},\\
\|\varphi(t)\|_{\mathbf{B}^{s,p}} &= \||\cdot|^{s}\,\widehat{\varphi}(t,\cdot)\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)},\\
\|\varphi(t)\|_{\mathbf{D}^{\rho,s}} &= \||\partial_\xi|^{\gamma}\{\cdot\}^{\rho}\langle\cdot\rangle^{s}\,\widehat{\varphi}(t,\cdot)\|_{\mathbf{L}^\infty_\xi}
\end{aligned}$$

where $\rho, s \in \mathbf{R}$, $\gamma \in (0,1)$. The norm $\mathbf{A}^{\rho,p}$ is responsible for the large time asymptotic properties of solutions, and the norm $\mathbf{B}^{s,p}$ describes the regularity of solutions. Using result of Lemma 1.38 and Lemma 1.39 we effortlessly have the following result.

Lemma 3.59. *Let the linear operator $\mathcal{L}$ satisfy dissipation conditions (3.192) and (3.193), and $\widehat{\varphi}(0) = 0$. Then the estimates are valid for all $t > 0$*

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{A}^{\rho,p}} \leq C\langle t\rangle^{-\frac{1}{\delta}\left(\rho+\frac{1}{p}-\frac{1}{q}\right)}\|\varphi\|_{\mathbf{A}^{0,q}},$$

where $\rho \geq 0$, if $p = q$ and $\rho + \frac{1}{p} - \frac{1}{q} > 0$ if $1 \leq p < q \leq \infty$,

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{A}^{\rho,p}} \leq C\langle t\rangle^{-\frac{1}{\delta}\left(\rho+\gamma+\frac{1}{p}\right)}\|\varphi\|_{\mathbf{D}^{0,0}},$$

where $\rho + \gamma \geq 0$, if $p = \infty$ and $\rho + \gamma + \frac{1}{p} > 0$ if $1 \leq p < \infty$, and

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{B}^{s,p}} \leq Ce^{-\frac{\mu}{2}t}\{t\}^{-\frac{s}{\nu}}\|\varphi\|_{\mathbf{B}^{0,p}}$$

where $1 \leq p \leq \infty$, $s \geq 0$, if $\nu > 0$ and $s = 0$ if $\nu = 0$. In addition

$$\begin{aligned}&\|\mathcal{G}(t)\varphi\|_{\mathbf{D}^{\rho,s}} \leq C\langle t\rangle^{-\frac{\rho}{\delta}}\{t\}^{-\frac{s}{\nu}}\left(\|\varphi\|_{\mathbf{D}^{0,0}} + \|\varphi\|_{\mathbf{A}^{0,\infty}} + \|\varphi\|_{\mathbf{B}^{0,\infty}}\right)\\ &+ C\langle t\rangle^{-\frac{\rho}{\delta}+\frac{\gamma}{2\delta}}\{t\}^{-\frac{s}{\nu}}\left(\|\varphi\|_{\mathbf{A}^{0,\infty}} + \|\varphi\|_{\mathbf{B}^{0,\infty}}\right)^{\frac{1}{2}}\|\varphi\|_{\mathbf{D}^{0,0}}^{\frac{1}{2}}\end{aligned}$$

where $1 \leq p \leq q \leq \infty$, $s \geq 0$, $\rho \geq 0$, $\gamma \in [0,1)$ is such that $\gamma < \delta$ if $\rho = 0$ and $\gamma < \min(\rho,\delta)$ if $\rho > 0$.

Define $\mathcal{G}_0(t)$ by

$$\mathcal{G}_0(t)\phi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-L_0(\xi)t}\hat{\phi}(\xi)\right),$$

with the homogeneous symbol $L_0(\xi) = \mu_1|\xi|^\delta + i\mu_2|\xi|^{\delta-1}\xi$, μ_1, $\mu_2 \in \mathbf{R}$. Also denote

$$G_0(x) = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-L_0(\xi)}\right).$$

By Lemma 1.39 we have the estimates for a difference $\mathcal{G}(t) - \mathcal{G}_0(t)$.

Lemma 3.60. *Suppose that the linear operator $\mathcal{L}$ satisfies conditions (3.192) and (3.194). Then the estimates*

$$\|(\mathcal{G}(t) - \mathcal{G}_0(t))\phi\|_{\mathbf{A}^{\rho,p}} \leq C\langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{1}{\delta p}}\|\phi\|_{\mathbf{A}^{0,\infty}}$$

and

$$\left\|\mathcal{G}_0(t)\phi - t^{-\frac{1}{\delta}}\widehat{\phi}(0)\,G_0\left(t^{-\frac{1}{\delta}}(\cdot)\right)\right\|_{\mathbf{A}^{\rho,p}} \leq C\langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{1}{\delta p}}\|\phi\|_{\mathbf{D}^{0,0}}$$

are valid for all $t > 0$, where $1 \leq p \leq \infty$, $\rho \geq 0, \gamma \in [0,1)$ is such that $\gamma < \delta$ if $\rho = 0$ and $\gamma < \min(\rho,\delta)$ if $\rho > 0$.

In the next lemma we estimate the Green operator $\mathcal{G}(t)$ in our basic norms $\|\cdot\|_{\mathbf{X}}$ and $\|\cdot\|_{\mathbf{Y}}$

$$\|\phi\|_{\mathbf{X}} = \sup_{\rho\in[-\gamma,\alpha+\gamma]} \sup_{t>0} \langle t\rangle^{\frac{\rho+1}{\delta}} \|\phi(t)\|_{\mathbf{A}^{\rho,1}} + \sup_{\rho\in[0,\alpha+\gamma]} \sup_{t>0} \langle t\rangle^{\frac{\rho}{\delta}} \|\phi(t)\|_{\mathbf{A}^{\rho,\infty}}$$
$$+ \sup_{s\in[0,\sigma]} \sup_{1\le p\le\infty} \sup_{t>0} \{t\}^{\frac{s}{\nu}} \langle t\rangle^{1+\frac{\gamma}{\delta}+\frac{1}{\delta p}} \|\phi(t)\|_{\mathbf{B}^{s,p}}$$
$$+ \sup_{\rho=0,\alpha,\beta} \sup_{s\in[0,\sigma]} \sup_{t>0} \{t\}^{\frac{s}{\nu}} \langle t\rangle^{\frac{\rho-\gamma}{\delta}} \|\phi(t)\|_{\mathbf{D}^{\rho,s}},$$

and

$$\|\phi\|_{\mathbf{Y}} = \sup_{1\le p\le\infty} \sup_{t>0} \langle t\rangle^{1+\frac{1}{\delta p}} \|\phi(t)\|_{\mathbf{A}^{0,p}}$$
$$+ \sup_{1\le p\le\infty} \sup_{t>0} \langle t\rangle^{1+\frac{\gamma}{\delta}+\frac{1}{\delta p}} \{t\}^{\frac{\sigma}{\nu}} \|\phi(t)\|_{\mathbf{B}^{0,p}}$$
$$+ \sup_{t>0} \langle t\rangle^{1-\frac{\gamma}{\delta}} \{t\}^{\frac{\sigma}{\nu}} \|\phi(t)\|_{\mathbf{D}^{0,0}},$$

where $\gamma \in (0, \min(1,\delta))$ is such that $\gamma < \alpha$ if $\alpha > 0$ and $\gamma < \beta$ if $\beta > 0$. The norms $\|\cdot\|_{\mathbf{X}}$ and $\|\cdot\|_{\mathbf{Y}}$ depend on the order of the symbol $L(\xi)$ (see conditions (3.192) to (3.194)) and on the symbols a and b (see (3.195) to (3.198)), that is depend on the values $\delta, \nu, \alpha, \beta, \sigma$, and γ. Define the function $g(t)$

$$g(t) = 1 + \kappa \log\langle t\rangle$$

with some $\kappa > 0$.

Lemma 3.61. *Let the function $f(t,x)$ have a zero mean value $\widehat{f}(t,0) = 0$. Then the following inequality*

$$\left\| g(t) \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{X}} \le C \|f\|_{\mathbf{Y}}$$

is valid, provided that the right-hand side is finite.

Proof. Via the condition of the lemma for the function $g(t)$ we have the estimate $\langle t\rangle^{-\frac{\gamma}{4\delta}} \le Cg^{-1}(t)$ and

$$\sup_{\tau\in[\sqrt{t},t]} g^{-1}(\tau) \le C\left(1+\kappa\log\left(1+\sqrt{t}\right)\right)^{-1}$$
$$\le C\left(1+\frac{\kappa}{2}\log(1+t)\right)^{-1} \le Cg^{-1}(t); \tag{3.200}$$

hence, by virtue of the first two estimates of Lemma 3.59 and (3.200) we obtain for $\rho \in [-\gamma, \alpha+\gamma]$ if $p = 1$ and $\rho \in [0, \alpha+\gamma]$ if $p = \infty$

$$\left\| \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{A}^{\rho,p}}$$

$$\leq C \int_0^{\sqrt{t}} \langle t-\tau \rangle^{-\frac{\rho+\gamma}{\delta}-\frac{1}{\delta p}} \langle \tau \rangle^{\frac{\gamma}{\delta}-1} \{\tau\}^{-\frac{s}{\nu}} d\tau \sup_{\tau>0} \langle \tau \rangle^{1-\frac{\gamma}{\delta}} \{\tau\}^{\frac{s}{\nu}} \|f(\tau)\|_{\mathbf{D}^{0,0}}$$

$$+Cg^{-1}(t) \int_{\sqrt{t}}^{\frac{t}{2}} \langle t-\tau \rangle^{-\frac{\rho+\gamma}{\delta}-\frac{1}{\delta p}} \langle \tau \rangle^{\frac{\gamma}{\delta}-1} \{\tau\}^{-\frac{s}{\nu}} d\tau \sup_{\tau>0} \langle \tau \rangle^{1-\frac{\gamma}{\delta}} \{\tau\}^{\frac{s}{\nu}} \|f(\tau)\|_{\mathbf{D}^{0,0}}$$

$$+Cg^{-1}(t) \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-\frac{\rho}{\delta}} \langle \tau \rangle^{-\frac{1}{\delta p}-1} d\tau \sup_{\tau>0} \langle \tau \rangle^{1+\frac{1}{\delta p}} \|f(\tau)\|_{\mathbf{A}^{0,p}}$$

$$\leq C \langle t \rangle^{-\frac{\rho}{\delta}-\frac{1}{\delta p}} \left(\langle t \rangle^{-\frac{\gamma}{2\delta}} + g^{-1}(t) \right) \|f\|_{\mathbf{Y}} \leq Cg^{-1}(t) \langle t \rangle^{-\frac{\rho}{\delta}-\frac{1}{\delta p}} \|f\|_{\mathbf{Y}}.$$

Similarly by the third estimate of Lemma 3.59 we get for $s \in [0,\sigma]$, $1 \leq p \leq \infty$, $t > 0$

$$\left\| \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{B}^{s,p}}$$

$$\leq C \int_0^t e^{-\frac{\mu}{2}(t-\tau)} \langle \tau \rangle^{-1-\frac{\gamma}{\delta}-\frac{1}{\delta p}} \{t-\tau\}^{-\frac{s}{\nu}} \{\tau\}^{-\frac{\sigma}{\nu}} d\tau$$

$$\times \sup_{\tau>0} \langle \tau \rangle^{1+\frac{\gamma}{\delta}+\frac{1}{\delta p}} \{\tau\}^{\frac{\sigma}{\nu}} \|f(\tau)\|_{\mathbf{B}^{0,p}} \leq C \langle t \rangle^{-1-\frac{\gamma}{\delta}-\frac{1}{\delta p}} \{t\}^{-\frac{s}{\nu}} \|f\|_{\mathbf{Y}}.$$

Finally by the fourth estimate of Lemma 3.59 we find for $\rho = 0, \alpha, \beta$, $s \in [0,\sigma]$, $t > 0$

$$\left\| \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{D}^{\rho,s}}$$

$$\leq C \int_0^t d\tau \langle t-\tau \rangle^{-\frac{\rho}{\delta}} \{t-\tau\}^{-\frac{s}{\nu}} g^{-1}(\tau)$$

$$\times \left(\|f(\tau)\|_{\mathbf{D}^{0,0}} + \|f(\tau)\|_{\mathbf{A}^{0,\infty}} + \|f(\tau)\|_{\mathbf{B}^{0,\infty}} \right)$$

$$+ C \int_0^t d\tau \langle t-\tau \rangle^{-\frac{\rho}{\delta}+\frac{\gamma}{2\delta}} \{t-\tau\}^{-\frac{s}{\nu}} g^{-1}(\tau)$$

$$\times \left(\|f(\tau)\|_{\mathbf{A}^{0,\infty}} + \|f(\tau)\|_{\mathbf{B}^{0,\infty}} \right)^{\frac{1}{2}} \|f(\tau)\|_{\mathbf{D}^{0,0}}^{\frac{1}{2}}.$$

Using (3.200) and the norm $\mathbf{Y}$ we have

$$\left\| \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{D}^{\rho,s}}$$
$$\leq C \|f\|_{\mathbf{Y}} \int_0^{\sqrt{t}} \langle t-\tau\rangle^{-\frac{\rho}{\delta}} \langle \tau\rangle^{\frac{\gamma}{\delta}-1} \{t-\tau\}^{-\frac{s}{\nu}} \{\tau\}^{-\frac{\sigma}{\nu}} d\tau$$
$$+ C \|f\|_{\mathbf{Y}} g^{-1}(t) \int_{\sqrt{t}}^t \langle t-\tau\rangle^{-\frac{\rho}{\delta}} \langle \tau\rangle^{\frac{\gamma}{\delta}-1} \{t-\tau\}^{-\frac{s}{\nu}} \{\tau\}^{-\frac{\sigma}{\nu}} d\tau$$
$$+ C \|f\|_{\mathbf{Y}} \int_0^{\sqrt{t}} \langle t-\tau\rangle^{-\frac{\rho}{\delta}+\frac{\gamma}{2\delta}} \langle \tau\rangle^{\frac{\gamma}{2\delta}-1} \{t-\tau\}^{-\frac{s}{\nu}} \{\tau\}^{-\frac{\sigma}{\nu}} d\tau$$
$$+ C \|f\|_{\mathbf{Y}} g^{-1}(t) \int_{\sqrt{t}}^t \langle t-\tau\rangle^{-\frac{\rho}{\delta}+\frac{\gamma}{2\delta}} \langle \tau\rangle^{\frac{\gamma}{2\delta}-1} \{t-\tau\}^{-\frac{s}{\nu}} \{\tau\}^{-\frac{\sigma}{\nu}} d\tau$$
$$\leq C \|f\|_{\mathbf{Y}} \{t\}^{-\frac{s}{\nu}} \langle t\rangle^{\frac{\gamma-\rho}{\delta}} \left(\langle t\rangle^{-\frac{\gamma}{4\delta}} + g^{-1}(t) \right)$$
$$\leq C \|f\|_{\mathbf{Y}} g^{-1}(t) \{t\}^{-\frac{s}{\nu}} \langle t\rangle^{\frac{\gamma-\rho}{\delta}}.$$

Hence, the results of the lemma follow, and Lemma 3.61 is proved.

Now we estimate the nonlinearity $\mathcal{N}(u)$ in the norms $\mathbf{A}^{0,p}$, $\mathbf{B}^{0,p}$ and $\mathbf{D}^{0,0}$.

Lemma 3.62. *Let the nonlinear operator $\mathcal{N}$ satisfy conditions (3.195) and (3.196). Then the inequalities*

$$\|\mathcal{N}(\varphi)\|_{\mathbf{A}^{0,p}}$$
$$\leq C \left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}} \right) \left(\|\varphi(t)\|_{\mathbf{A}^{0,p}} + \|\varphi(t)\|_{\mathbf{B}^{0,\infty}} \right)$$
$$+ C \left(\|\varphi(t)\|_{\mathbf{A}^{\beta,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}} \right) \left(\|\varphi(t)\|_{\mathbf{A}^{0,1}} + \|\varphi(t)\|_{\mathbf{B}^{0,1}} \right)$$
$$\times \left(\|\varphi(t)\|_{\mathbf{A}^{0,p}} + \|\varphi(t)\|_{\mathbf{B}^{0,\infty}} \right),$$

$$\|\mathcal{N}(\varphi)\|_{\mathbf{B}^{0,p}}$$
$$\leq C \left(\|\varphi(t)\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}} \right) \left(\|\varphi(t)\|_{\mathbf{A}^{0,p}} + \|\varphi(t)\|_{\mathbf{B}^{0,p}} \right)$$
$$+ C \left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}} \right) \left(\|\varphi(t)\|_{\mathbf{A}^{\gamma,p}} + \|\varphi(t)\|_{\mathbf{B}^{0,p}} \right)$$
$$+ C \left(\|\varphi(t)\|_{\mathbf{A}^{\beta+\gamma,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}} \right) \left(\|\varphi(t)\|_{\mathbf{A}^{0,p}} + \|\varphi(t)\|_{\mathbf{B}^{0,p}} \right)$$
$$\times \left(\|\varphi(t)\|_{\mathbf{A}^{0,1}} + \|\varphi(t)\|_{\mathbf{B}^{0,1}} \right)$$
$$+ C \left(\|\varphi(t)\|_{\mathbf{A}^{\beta,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}} \right) \left(\|\varphi(t)\|_{\mathbf{A}^{\gamma,p}} + \|\varphi(t)\|_{\mathbf{B}^{0,p}} \right)$$
$$\times \left(\|\varphi(t)\|_{\mathbf{A}^{0,1}} + \|\varphi(t)\|_{\mathbf{B}^{0,1}} \right),$$

and

$$
\begin{aligned}
&\|\mathcal{N}(\varphi)\|_{\mathbf{D}^{0,0}} \le C\|\varphi(t)\|_{\mathbf{D}^{\alpha,\sigma}}\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\\
&+C\|\varphi(t)\|_{\mathbf{D}^{0,0}}\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\\
&+C\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha-\gamma,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{0,\infty}}+\|\varphi(t)\|_{\mathbf{B}^{0,\infty}}\right)\\
&+C\left(\|\varphi(t)\|_{\mathbf{A}^{-\gamma,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,\infty}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,\infty}}\right)\\
&+C\|\varphi(t)\|_{\mathbf{D}^{\beta,\sigma}}\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)^2\\
&+C\|\varphi(t)\|_{\mathbf{D}^{0,0}}\left(\|\varphi(t)\|_{\mathbf{A}^{\beta,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\\
&\times\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\\
&+C\left(\|\varphi(t)\|_{\mathbf{A}^{\beta-\gamma,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{0,\infty}}+\|\varphi(t)\|_{\mathbf{B}^{0,\infty}}\right)\\
&\times\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\\
&+C\left(\|\varphi(t)\|_{\mathbf{A}^{-\gamma,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\beta,\infty}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,\infty}}\right)\\
&\times\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)
\end{aligned}
$$

are valid for $1 \le p \le \infty$*, provided that the right-hand sides are bounded.*

Proof. By virtue of conditions (3.195), (3.196) and by the Young inequality we obtain

$$
\begin{aligned}
&\|\mathcal{N}(\varphi)\|_{\mathbf{A}^{0,p}} \le \left\|\int_{\mathbf{R}}|a(t,\cdot,y)|\,|\widehat{\varphi}(t,\cdot-y)\,\widehat{\varphi}(t,y)|\,dy\right\|_{\mathbf{L}^p_\xi(|\xi|\le 1)}\\
&+\left\|\int_{\mathbf{R}^2}|b(t,\cdot,y,z)|\,|\widehat{\varphi}(t,\cdot-y)\,\widehat{\varphi}(t,y-z)\,\widehat{\varphi}(t,z)|\,dydz\right\|_{\mathbf{L}^p_\xi(|\xi|\le 1)}\\
&\le C\left\|\int_{\mathbf{R}}\left(\langle\cdot-y\rangle^\sigma\{\cdot-y\}^\alpha+\langle y\rangle^\sigma\{y\}^\alpha\right)\right.\\
&\times\left.|\widehat{\varphi}(t,\cdot-y)\,\widehat{\varphi}(t,y)|\,dy\right\|_{\mathbf{L}^p_\xi(|\xi|\le 1)}\\
&+C\left\|\int_{\mathbf{R}^2}\left(\langle\cdot-y\rangle^\sigma\{\cdot-y\}^\beta+\langle y-z\rangle^\sigma\{y-z\}^\beta+\langle z\rangle^\sigma\{z\}^\beta\right)\right.\\
&\times\left.|\widehat{\varphi}(t,\cdot-y)\,\widehat{\varphi}(t,y-z)\,\widehat{\varphi}(t,z)|\,dydz\right\|_{\mathbf{L}^p_\xi(|\xi|\le 1)};
\end{aligned}
$$

hence

$$
\begin{aligned}
&\|\mathcal{N}(\varphi)\|_{\mathbf{A}^{0,p}} \le C\left\|\langle\cdot\rangle^\sigma\{\cdot\}^\alpha\,\widehat{\varphi}(t)\right\|_{\mathbf{L}^1_\xi}\Big(\|\widehat{\varphi}(t)\|_{\mathbf{L}^p_\xi(|\xi|\le 1)}\\
&+\|\widehat{\varphi}(t)\|_{\mathbf{L}^\infty_\xi(|\xi|>1)}\Big)\\
&+C\left\|\langle\cdot\rangle^\sigma\{\cdot\}^\beta\,\widehat{\varphi}(t)\right\|_{\mathbf{L}^1_\xi}\Big(\|\widehat{\varphi}(t)\|_{\mathbf{L}^p_\xi(|\xi|\le 1)}\\
&+\|\widehat{\varphi}(t)\|_{\mathbf{L}^\infty_\xi(|\xi|>1)}\Big)\|\widehat{\varphi}(t)\|_{\mathbf{L}^1_\xi}\\
&\le C\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{0,p}}+\|\varphi(t)\|_{\mathbf{B}^{0,\infty}}\right)\\
&+C\left(\|\varphi(t)\|_{\mathbf{A}^{\beta,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\\
&\times\left(\|\varphi(t)\|_{\mathbf{A}^{0,p}}+\|\varphi(t)\|_{\mathbf{B}^{0,\infty}}\right).
\end{aligned}
$$

As before

$$\|\mathcal{N}(\varphi)\|_{\mathbf{B}^{0,p}} \le \left\| \int_{|y|\le\frac{1}{2}} |a(t,\xi,y)| \left|\widehat{\varphi}(t,\xi-y)\widehat{\varphi}(t,y)\right| dy \right\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)}$$

$$+ \left\| \int_{|y|\ge\frac{1}{2}} |a(t,\cdot,y)| \left|\widehat{\varphi}(t,\cdot-y)\widehat{\varphi}(t,y)\right| dy \right\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)}$$

$$+ \left\| \int_{|y|+|z|\le\frac{1}{2}} |b(t,\cdot,y,z)| \left|\widehat{\varphi}(t,\cdot-y)\widehat{\varphi}(t,y-z)\widehat{\varphi}(t,z)\right| dydz \right\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)}$$

$$+ \left\| \int_{|y|+|z|\ge\frac{1}{2}} |b(t,\cdot,y,z)| \left|\widehat{\varphi}(t,\cdot-y)\widehat{\varphi}(t,y-z)\widehat{\varphi}(t,z)\right| dydz \right\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)},$$

so

$$\|\mathcal{N}(\varphi)\|_{\mathbf{B}^{0,p}} \le C \left\| \int_{|y|\le\frac{1}{2}} \{\cdot-y\}^{\gamma} \left(\langle\cdot-y\rangle^{\sigma}\{\cdot-y\}^{\alpha} + \{y\}^{\alpha}\right) \right.$$

$$\left. \times \left|\widehat{\varphi}(t,\cdot-y)\widehat{\varphi}(t,y)\right| dy \right\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)}$$

$$+ C \left\| \int_{|y|\ge\frac{1}{2}} \{y\}^{\gamma} \left(\langle\cdot-y\rangle^{\sigma}\{\cdot-y\}^{\alpha} + \langle y\rangle^{\sigma}\{y\}^{\alpha}\right) \right.$$

$$\left. \times \left|\widehat{\varphi}(t,\cdot-y)\widehat{\varphi}(t,y)\right| dy \right\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)}$$

$$+ C \left\| \int_{|y|+|z|\le\frac{1}{2}} \{\cdot-y\}^{\gamma} \left(\langle\cdot-y\rangle^{\sigma}\{\cdot-y\}^{\beta} + \{y-z\}^{\beta} + \{z\}^{\beta}\right) \right.$$

$$\left. \times \left|\widehat{\varphi}(t,\cdot-y)\widehat{\varphi}(t,y-z)\widehat{\varphi}(t,z)\right| dydz \right\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)}$$

$$+ C \left\| \int_{|y|+|z|\ge\frac{1}{2}} \left(\{y-z\}^{\gamma} + \{z\}^{\gamma}\right) \right.$$

$$\times \left(\langle\cdot-y\rangle^{\sigma}\{\cdot-y\}^{\beta} + \langle y-z\rangle^{\sigma}\{y-z\}^{\beta} + \langle z\rangle^{\sigma}\{z\}^{\beta}\right)$$

$$\left. \times \left|\widehat{\varphi}(t,\cdot-y)\widehat{\varphi}(t,y-z)\widehat{\varphi}(t,z)\right| dydz \right\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)}.$$

Therefore,

$$\begin{aligned}
&\|\mathcal{N}(\varphi)\|_{\mathbf{B}^{0,p}} \le C\left\|\langle\xi\rangle^{\sigma}\{\xi\}^{\alpha+\gamma}\widehat{\varphi}(t,\xi)\right\|_{\mathbf{L}^1_\xi}\|\widehat{\varphi}(t,\xi)\|_{\mathbf{L}^p_\xi}\\
&+C\|\{\xi\}^{\gamma}\widehat{\varphi}(t,\xi)\|_{\mathbf{L}^p_\xi}\|\langle\xi\rangle^{\sigma}\{\xi\}^{\alpha}\widehat{\varphi}(t,\xi)\|_{\mathbf{L}^1_\xi}\\
&\le C\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha+\gamma,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{0,p}}+\|\varphi(t)\|_{\mathbf{B}^{0,p}}\right)\\
&+C\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\gamma,p}}+\|\varphi(t)\|_{\mathbf{B}^{0,p}}\right)\\
&+C\left(\|\varphi(t)\|_{\mathbf{A}^{\beta+\gamma,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{0,p}}+\|\varphi(t)\|_{\mathbf{B}^{0,p}}\right)\\
&\times\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\\
&+C\left(\|\varphi(t)\|_{\mathbf{A}^{\beta,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\gamma,p}}+\|\varphi(t)\|_{\mathbf{B}^{0,p}}\right)\\
&\times\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right).
\end{aligned}$$

Thus the first two estimates of the lemma follow.

Denote

$$\begin{aligned}
\widetilde{a}(t,\xi,y) &= a(t,\xi,y)\left(\{\xi-y\}^{\alpha}\langle\xi-y\rangle^{\sigma}+\{y\}^{\alpha}\langle y\rangle^{\sigma}\right)^{-1},\\
\Phi(t,\xi,y) &= \left(\{\xi-y\}^{\alpha}\langle\xi-y\rangle^{\sigma}+\{y\}^{\alpha}\langle y\rangle^{\sigma}\right)\widehat{\varphi}(t,\xi-y)\,\widehat{\varphi}(t,y)\\
\widetilde{b}(t,\xi,y,z) &= b(t,\xi,y,z)\Big(\{\xi-y\}^{\beta}\langle\xi-y\rangle^{\sigma}\\
&\quad+\{y-z\}^{\beta}\langle y-z\rangle^{\sigma}+\{z\}^{\beta}\langle z\rangle^{\sigma}\Big)^{-1},\\
\Psi(t,\xi,y) &= \Big(\{\xi-y\}^{\beta}\langle\xi-y\rangle^{\sigma}+\{y-z\}^{\beta}\langle y-z\rangle^{\sigma}+\{z\}^{\beta}\langle z\rangle^{\sigma}\Big)\\
&\quad\times\widehat{\varphi}(t,\xi-y)\,\widehat{\varphi}(t,y-z)\,\widehat{\varphi}(t,z).
\end{aligned}$$

We have

$$\begin{aligned}
&\|\mathcal{N}(\varphi)\|_{\mathbf{D}^{0,0}} = \left\||\partial_\xi|^{\gamma}\int_{\mathbf{R}}\widetilde{a}(t,\cdot,y)\,\Phi(t,\cdot,y)\,dy\right\|_{\mathbf{L}^\infty_\xi}\\
&+\left\||\partial_\xi|^{\gamma}\int_{\mathbf{R}^2}\widetilde{b}(t,\cdot,y,z)\,\Psi(t,\xi,y,z)\,dydz\right\|_{\mathbf{L}^\infty_\xi}\\
&\le C\left\|\int_{\mathbf{R}}\Phi(t,\cdot,y)\,|\partial_\xi|^{\gamma}\,\widetilde{a}(t,\cdot,y)\,dy\right\|_{\mathbf{L}^\infty_\xi}\\
&+C\left\|\int_{\mathbf{R}}\left[|\partial_\xi|^{\gamma},\Phi(t,\cdot,y)\right]\widetilde{a}(t,\cdot,y)\,dy\right\|_{\mathbf{L}^\infty_\xi}\\
&+C\left\|\int_{\mathbf{R}^2}\Psi(t,\cdot,y,z)\,|\partial_\xi|^{\gamma}\,\widetilde{b}(t,\cdot,y,z)\,dydz\right\|_{\mathbf{L}^\infty_\xi}\\
&+C\left\|\int_{\mathbf{R}^2}\left[|\partial_\xi|^{\gamma},\Psi(t,\cdot,y,z)\right]\widetilde{b}(t,\cdot,y,z)\,dydz\right\|_{\mathbf{L}^\infty_\xi} \qquad (3.201)
\end{aligned}$$

where the commutators

$$\begin{aligned}&\left[|\partial_\xi|^\gamma, \Phi(t,\xi,y)\right]\widetilde{a}(t,\xi,y)\\ &\equiv \int_{\mathbf{R}} |\Phi(t,\xi-\eta,y) - \Phi(t,\xi,y)|\, \widetilde{a}(t,\xi-\eta,y)\, |\eta|^{-1-\gamma}\, d\eta\end{aligned}$$

and similarly

$$\begin{aligned}&\left[|\partial_\xi|^\gamma, \Psi(t,\xi,y,z)\right]\widetilde{b}(t,\xi,y,z)\\ &\equiv \int_{\mathbf{R}} |\Psi(t,\xi-\eta,y,z) - \Psi(t,\xi,y,z)|\, \widetilde{b}(t,\xi-\eta,y,z)\, |\eta|^{-1-\gamma}\, d\eta.\end{aligned}$$

By virtue of condition (3.195) we estimate the commutator

$$\begin{aligned}&\left\| \int_{\mathbf{R}} \left[|\partial_\xi|^\gamma, \Phi(t,\cdot,y)\right]\widetilde{a}(t,\cdot,y)\, dy \right\|_{\mathbf{L}^\infty_\xi}\\ &\leq C \left\| \int_{\mathbf{R}}\int_{\mathbf{R}} |\Phi(t,\cdot-\eta,y) - \Phi(t,\cdot,y)|\, |\eta|^{-1-\gamma}\, d\eta dy \right\|_{\mathbf{L}^\infty_\xi}\\ &\leq C \left\| \int_{\mathbf{R}} \left(|\partial_\xi|^\gamma \{\cdot - y\}^\alpha \langle \cdot - y\rangle^\sigma \widehat{\varphi}(t,\cdot-y)\right) \widehat{\varphi}(t,y)\, dy \right\|_{\mathbf{L}^\infty_\xi}\\ &+ C \left\| \int_{\mathbf{R}} \left(|\partial_\xi|^\gamma \widehat{\varphi}(t,\cdot-y)\right) \{y\}^\alpha \langle y\rangle^\sigma \widehat{\varphi}(t,y)\, dy \right\|_{\mathbf{L}^\infty_\xi}\\ &\leq C \left\| |\partial_\xi|^\gamma \{\cdot\}^\alpha \langle\cdot\rangle^\sigma \widehat{\varphi}(t,\cdot)\right\|_{\mathbf{L}^\infty_\xi} \|\widehat{\varphi}(t)\|_{\mathbf{L}^1_\xi}\\ &+ C \left\| |\partial_\xi|^\gamma \widehat{\varphi}(t,\cdot)\right\|_{\mathbf{L}^\infty_\xi} \left\|\{\cdot\}^\alpha \langle\cdot\rangle^\sigma \widehat{\varphi}(t,\cdot)\right\|_{\mathbf{L}^1_\xi}\\ &\leq C \|\varphi(t)\|_{\mathbf{D}^{\alpha,\sigma}} \left(\|\varphi(t)\|_{\mathbf{A}^{0,1}} + \|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\\ &+ C \|\varphi(t)\|_{\mathbf{D}^{0,0}} \left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right). \end{aligned} \tag{3.202}$$

By using (3.196) we get

$$\begin{aligned}&\left\| \int_{\mathbf{R}^2} \left[|\partial_\xi|^\gamma, \Psi(t,\cdot,y,z)\right]\widetilde{b}(t,\cdot,y,z)\, dydz \right\|_{\mathbf{L}^\infty_\xi}\\ &\leq C \|\varphi(t)\|_{\mathbf{D}^{\beta,\sigma}} \left(\|\varphi(t)\|_{\mathbf{A}^{0,1}} + \|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)^2\\ &+ C \|\varphi(t)\|_{\mathbf{D}^{0,0}} \left(\|\varphi(t)\|_{\mathbf{A}^{\beta,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\\ &\times \left(\|\varphi(t)\|_{\mathbf{A}^{0,1}} + \|\varphi(t)\|_{\mathbf{B}^{0,1}}\right). \end{aligned} \tag{3.203}$$

Via (3.195) we have

$$\begin{aligned}&\int_{|\eta| \geq \frac{1}{2}\{\xi - y\}} |\widetilde{a}(t,\xi-\eta,y) - \widetilde{a}(t,\xi,y)|\, \frac{d\eta}{|\eta|^{1+\gamma}}\\ &\leq C \int_{|\eta| \geq \frac{1}{2}\{\xi - y\}} \frac{d\eta}{|\eta|^{1+\gamma}} \leq C\{\xi - y\}^{-\gamma}\end{aligned}$$

and

$$\begin{aligned}
&\int_{|\eta|\le\frac{1}{2}\{\xi-y\}} \left|\widetilde{a}(t,\xi-\eta,y)-\widetilde{a}(t,\xi,y)\right| \frac{d\eta}{|\eta|^{1+\gamma}} \\
&\le C\int_{|\eta|\le\frac{1}{2}\{\xi-y\}} \left|\int_{\xi-y}^{\xi-y-\eta}\{\zeta\}^{-1}d\zeta\right| \frac{d\eta}{|\eta|^{1+\gamma}} \\
&\le C\{\xi-y\}^{-1}\int_{|\eta|\le\frac{1}{2}\{\xi-y\}} \frac{d\eta}{|\eta|^{\gamma}} \le C\{\xi-y\}^{-\gamma}.
\end{aligned}$$

Thus we have the estimate

$$\begin{aligned}
|\partial_\xi|^\gamma \widetilde{a}(t,\xi,y) &= \int_{\mathbf{R}} \left|\widetilde{a}(t,\xi-\eta,y)-\widetilde{a}(t,\xi,y)\right| |\eta|^{-1-\gamma}d\eta \\
&\le C\{\xi-y\}^{-\gamma}
\end{aligned}$$

for all $\xi, y\in\mathbf{R}$. Therefore

$$\begin{aligned}
&\left\|\int_{\mathbf{R}} \Phi(t,\cdot,y)\,|\partial_\xi|^\gamma \widetilde{a}(t,\cdot,y)\,dy\right\|_{\mathbf{L}^\infty_\xi} \\
&\le C\left\|\int_{\mathbf{R}} \{\cdot-y\}^{\alpha-\gamma}\langle\cdot-y\rangle^\sigma \widehat{\varphi}(t,\cdot-y)\,\widehat{\varphi}(t,y)\,dy\right\|_{\mathbf{L}^\infty_\xi} \\
&+C\left\|\int_{\mathbf{R}} \{\cdot-y\}^{-\gamma} \widehat{\varphi}(t,\cdot-y)\{y\}^\alpha\langle y\rangle^\sigma\,\widehat{\varphi}(t,y)\,dy\right\|_{\mathbf{L}^\infty_\xi} \\
&\le C\left\|\{\cdot\}^{\alpha-\gamma}\langle\cdot\rangle^\sigma\widehat{\varphi}(t)\right\|_{\mathbf{L}^1_\xi}\left\|\widehat{\varphi}(t)\right\|_{\mathbf{L}^\infty_\xi} \\
&+C\left\|\{\cdot\}^{-\gamma}\widehat{\varphi}(t)\right\|_{\mathbf{L}^1_\xi}\left\|\{\cdot\}^{\alpha}\langle\cdot\rangle^\sigma\widehat{\varphi}(t)\right\|_{\mathbf{L}^\infty_\xi} \\
&\le C\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha-\gamma,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{0,\infty}}+\|\varphi(t)\|_{\mathbf{B}^{0,\infty}}\right) \\
&+C\left(\|\varphi(t)\|_{\mathbf{A}^{-\gamma,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,\infty}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,\infty}}\right). \qquad (3.204)
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
&\left\|\int_{\mathbf{R}^2} \Psi(t,\cdot,y,z)\,|\partial_\xi|^\gamma \widetilde{b}(t,\cdot,y,z)\,dydz\right\|_{\mathbf{L}^\infty_\xi} \\
&\le C\left(\|\varphi(t)\|_{\mathbf{A}^{\beta-\gamma,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{0,\infty}}+\|\varphi(t)\|_{\mathbf{B}^{0,\infty}}\right) \\
&\times\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right) \\
&+C\left(\|\varphi(t)\|_{\mathbf{A}^{-\gamma,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\beta,\infty}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,\infty}}\right) \\
&\times\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right). \qquad (3.205)
\end{aligned}$$

In view of (3.201) through (3.205) we get

$$\begin{aligned}
&\|\mathcal{N}(\varphi)\|_{\mathbf{D}^{0,0}} \le C\|\varphi(t)\|_{\mathbf{D}^{\alpha,\sigma}}\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\\
&+C\|\varphi(t)\|_{\mathbf{D}^{0,0}}\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\\
&+C\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha-\gamma,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{0,\infty}}+\|\varphi(t)\|_{\mathbf{B}^{0,\infty}}\right)\\
&+C\left(\|\varphi(t)\|_{\mathbf{A}^{-\gamma,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,\infty}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,\infty}}\right)\\
&+C\|\varphi(t)\|_{\mathbf{D}^{\beta,\sigma}}\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)^{2}\\
&+C\|\varphi(t)\|_{\mathbf{D}^{0,0}}\left(\|\varphi(t)\|_{\mathbf{A}^{\beta,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\\
&\times\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\\
&+C\left(\|\varphi(t)\|_{\mathbf{A}^{\beta-\gamma,1}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{0,\infty}}+\|\varphi(t)\|_{\mathbf{B}^{0,\infty}}\right)\\
&\times\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\\
&+C\left(\|\varphi(t)\|_{\mathbf{A}^{-\gamma,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\beta,\infty}}+\|\varphi(t)\|_{\mathbf{B}^{\sigma,\infty}}\right)\\
&\times\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}}+\|\varphi(t)\|_{\mathbf{B}^{0,1}}\right).
\end{aligned}$$

Thus the third estimate of the lemma is true. Lemma 3.62 is proved.

The next lemma will be used in the proof of the theorem to evaluate large time behavior of the mean value of the nonlinearity in equation (3.188) in the norms $\mathbf{A}^{0,p}$, $\mathbf{B}^{0,p}$ and $\mathbf{D}^{0,0}$. We use the notations

$$\begin{aligned}
\mathcal{N}_0(\varphi) &= \overline{\mathcal{F}}_{\xi\to x}\int_{\mathbf{R}} a_0(y)\,\widehat{\varphi}(t,\xi-y)\,\widehat{\varphi}(t,y)\,dy\\
&\quad+\omega\overline{\mathcal{F}}_{\xi\to x}\int_{\mathbf{R}^2} b_0(y,z)\,\widehat{\varphi}(t,\xi-y)\,\widehat{\varphi}(t,y-z)\,\widehat{\varphi}(t,z)\,dydz,
\end{aligned}$$

where $\omega=0$ if $a_0\neq 0$ and $\omega=1$ if $a_0\equiv 0$. We also define as above

$$\begin{aligned}
\kappa &\equiv \theta^2\int_{\mathbf{R}} a_0(y)\,e^{-L_0(-y)-L_0(y)}dy\\
&\quad+\omega\theta^3\int_{\mathbf{R}^2} b_0(y,z)\,e^{-L_0(-y)-L_0(y-z)-L_0(z)}dydz>0,
\end{aligned}$$

where $\theta=\widehat{u}_0(0)$ and $g(t)=1+\kappa\log\langle t\rangle$.

Lemma 3.63. *Let the linear operator $\mathcal{L}$ satisfy conditions (3.192) and (3.194) and the nonlinear operator $\mathcal{N}$ satisfy conditions (3.195) - (3.198). Assume that u_0 is such that the norm $\|u_0\|_{\mathbf{A}^{0,\infty}}+\|u_0\|_{\mathbf{D}^{0,0}}=\varepsilon$. Let function $v(t,x)$ satisfy the estimates*

$$\|v\|_{\mathbf{X}}\le C\varepsilon \tag{3.206}$$

and

$$\|v(t)-\mathcal{G}(t)\,u_0\|_{\mathbf{A}^{\rho,p}}\le C\varepsilon^2 g^{-1}(t)\,\langle t\rangle^{-\frac{\rho}{\delta}-\frac{1}{\delta p}}, \tag{3.207}$$

where $\rho\in[0,\alpha]$, $1\le p\le\infty$.

Then the inequalities

$$\left|1+\int_0^t\widehat{\mathcal{N}_1(v)}(\tau,0)\,d\tau-\kappa\log t\right|\le\frac{C\varepsilon^3}{\kappa}\log(g(t))+C\varepsilon^2 \tag{3.208}$$

if $a_0 \neq 0$ *and*

$$\left|1+\int_0^t \widehat{\mathcal{N}_2(v)}(\tau,0)\,d\tau-\kappa\log t\right| \le \frac{C\varepsilon^4}{\kappa}\log\left(g(t)\right)+C\varepsilon^3 \tag{3.209}$$

if $a_0 \equiv 0$ *are valid for all* $t > 0$.

Proof. By Lemma 3.62 and in view of the condition (3.206) we get

$$\begin{aligned}\left|\int_0^t \widehat{\mathcal{N}(v)}(\tau,0)\,d\tau\right| &\le C\int_0^t \|\mathcal{N}(v(\tau))\|_{\mathbf{A}^{0,\infty}}\,d\tau\\ &\le C\|v\|_{\mathbf{X}}^2\int_0^t \{\tau\}^{-\frac{\sigma}{\nu}}\,d\tau \le C\varepsilon^2 t^{1-\frac{\sigma}{\nu}};\end{aligned} \tag{3.210}$$

hence, estimates (3.208) and (3.209) follow for all $0 < t < 1$. We now consider $t \ge 1$. By Lemma 3.60 and via condition of the lemma we get

$$\begin{aligned}&\left\|v(\tau)-\theta\tau^{-\frac{1}{\delta}}G_0\left(\tau^{-\frac{1}{\delta}}(\cdot)\right)\right\|_{\mathbf{A}^{\rho,p}}\\ &\le C\|v(\tau)-\mathcal{G}(\tau)u_0\|_{\mathbf{A}^{\rho,p}}+\left\|\mathcal{G}(\tau)u_0-\theta\tau^{-\frac{1}{\delta}}G_0\left(\tau^{-\frac{1}{\delta}}(\cdot)\right)\right\|_{\mathbf{A}^{\rho,p}}\\ &\le C\varepsilon^2 g^{-1}(\tau)\langle\tau\rangle^{-\frac{\rho}{\delta}-\frac{1}{\delta p}}+C\langle\tau\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{1}{\delta p}}\|u_0\|_{\mathbf{A}^{0,\infty}}\\ &+C\langle\tau\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{1}{\delta p}}\|u_0\|_{\mathbf{D}^{0,0}}\\ &\le C\varepsilon\langle\tau\rangle^{-\frac{\rho}{\delta}-\frac{1}{\delta p}}\left(\varepsilon g^{-1}(\tau)+\langle\tau\rangle^{-\frac{\gamma}{\delta}}+\langle\tau\rangle^{-\frac{\gamma}{\delta}}\right)\end{aligned} \tag{3.211}$$

for all $\tau > 0$, where $1 \le p \le \infty$, $\rho \ge 0$. By condition (3.195) we get

$$\begin{aligned}&\left|\int_{|y|\le 1}(a(\tau,0,y)-a_0(y))\,\widehat{v}(\tau,-y)\,\widehat{v}(\tau,y)\,dy\right|\\ &\le C\int_{|y|\le 1}|y|^{\alpha+\gamma}|\widehat{v}(\tau,-y)|\,|\widehat{v}(\tau,y)|\,dy\\ &\le C\|v(\tau)\|_{\mathbf{A}^{\alpha,1}}\|v(\tau)\|_{\mathbf{A}^{\gamma,\infty}} \le C\varepsilon^2\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{-1-\frac{\gamma}{\delta}}\end{aligned} \tag{3.212}$$

for all $\tau > 0$. Likewise via (3.196) we obtain

$$\begin{aligned}&\left|\int_{|y|+|z|\le 1}(b(\tau,0,y,z)-b_0(y,z))\,\widehat{v}(\tau,-y)\,\widehat{v}(\tau,y-z)\,\widehat{v}(\tau,z)\,dydz\right|\\ &\le C\varepsilon^3\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{-1-\frac{\gamma}{\delta}}.\end{aligned} \tag{3.213}$$

Further we find

$$
\begin{aligned}
&\left|\widehat{\mathcal{N}_1(v)}(\tau,0)-\mathcal{F}_{x\to\xi}\left(\mathcal{N}_0\left(\theta\tau^{-\frac{1}{\delta}}G_0\left(x\tau^{-\frac{1}{\delta}}\right)\right)\right)(\tau,0)\right| \\
&\le\left|\int_{|y|\le 1}(a(\tau,0,y)-a_0(y))\,\widehat{v}(\tau,-y)\,\widehat{v}(\tau,y)\,dy\right| \\
&+\left|\int_{|y|\le 1}a_0(y)\left(\widehat{v}(\tau,-y)\,\widehat{v}(\tau,y)-\theta^2e^{-\tau L_0(-y)-\tau L_0(y)}\right)dy\right| \\
&+\left|\int_{|y|\ge 1}a(\tau,0,y)\,\widehat{v}(\tau,-y)\,\widehat{v}(\tau,y)\,dy\right| \\
&+\theta^2\left|\int_{|y|\ge 1}a_0(y)\,e^{-\tau L_0(-y)-\tau L_0(y)}dy\right|
\end{aligned}
$$

if $a_0(y)\ne 0$. In the case of $a_0(y)\equiv 0$ we have

$$
\begin{aligned}
&\left|\widehat{\mathcal{N}_2(v)}(\tau,0)-\mathcal{F}_{x\to\xi}\left(\mathcal{N}_0\left(\theta\tau^{-\frac{1}{\delta}}G_0\left(x\tau^{-\frac{1}{\delta}}\right)\right)\right)(\tau,0)\right| \\
&\le\left|\int_{|y|+|z|\le 1}(b(\tau,0,y,z)-b_0(y,z))\,\widehat{v}(\tau,-y)\,\widehat{v}(\tau,y-z)\,\widehat{v}(\tau,z)\,dy\right| \\
&+\left|\int_{|y|+|z|\le 1}b_0(y,z)\left(\widehat{v}(\tau,-y)\,\widehat{v}(\tau,y-z)\,\widehat{v}(\tau,z)\right.\right. \\
&\left.\left.-\theta^3e^{-\tau L_0(-y)-\tau L_0(y-z)-\tau L_0(z)}\right)dydz\right| \\
&+\left|\int_{|y|+|z|\ge 1}b(\tau,0,y,z)\,\widehat{v}(\tau,-y)\,\widehat{v}(\tau,y-z)\,\widehat{v}(\tau,z)\,dydz\right| \\
&+\theta^3\left|\int_{|y|+|z|\ge 1}b_0(y,z)\,e^{-\tau L_0(-y)-\tau L_0(y-z)-\tau L_0(z)}dydz\right|.
\end{aligned}
$$

Applying (3.211) through (3.213) we obtain

$$
\begin{aligned}
&\left|\widehat{\mathcal{N}_1(v)}(\tau,0)-\mathcal{F}_{x\to\xi}\left(\mathcal{N}_0\left(\theta\tau^{-\frac{1}{\delta}}G_0\left(x\tau^{-\frac{1}{\delta}}\right)\right)\right)(\tau,0)\right| \\
&\le C\varepsilon^2\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{-1-\frac{\gamma}{\delta}}+C\left\|v(\tau)-\theta\tau^{-\frac{1}{\delta}}G_0\left(\tau^{-\frac{1}{\delta}}(\cdot)\right)\right\|_{\mathbf{A}^{0,\infty}} \\
&\times\left(\|v(\tau)\|_{\mathbf{A}^{\alpha,1}}+\left\|\theta\tau^{-\frac{1}{\delta}}G_0\left(\tau^{-\frac{1}{\delta}}(\cdot)\right)\right\|_{\mathbf{A}^{\alpha,1}}\right) \\
&+C\|v(\tau)\|_{\mathbf{B}^{\sigma,1}}\|v(\tau)\|_{\mathbf{B}^{0,\infty}} \\
&+C\theta^2\left\|\tau^{-\frac{1}{\delta}}G_0\left(\tau^{-\frac{1}{\delta}}(\cdot)\right)\right\|_{\mathbf{B}^{\alpha,1}}\left\|\tau^{-\frac{1}{\delta}}G_0\left(\tau^{-\frac{1}{\delta}}(\cdot)\right)\right\|_{\mathbf{B}^{0,\infty}},
\end{aligned}
$$

which yields

$$\left|\widehat{\mathcal{N}_1(v)}(\tau,0) - \mathcal{F}_{x\to\xi}\left(\mathcal{N}_0\left(\theta\tau^{-\frac{1}{\delta}}G_0\left(x\tau^{-\frac{1}{\delta}}\right)\right)\right)(\tau,0)\right|$$
$$\le C\varepsilon^2\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{-1-\frac{\gamma}{\delta}}$$
$$+C\varepsilon^2\langle\tau\rangle^{-1}\left(\varepsilon g^{-1}(\tau)+\langle\tau\rangle^{-\frac{\gamma}{\delta}}+\langle\tau\rangle^{-\frac{\gamma}{\delta}}\right)$$
$$+C\varepsilon^2\{t\}^{-\frac{\sigma}{\nu}}\langle t\rangle^{-1-\frac{\gamma}{\delta}}+C\varepsilon^2\{t\}^{-\frac{\alpha}{\delta}}\langle t\rangle^{-1-\frac{\gamma}{\delta}}.$$

Similarly, we get

$$\left|\widehat{\mathcal{N}_2(v)}(\tau,0) - \mathcal{F}_{x\to\xi}\left(\mathcal{N}_0\left(\theta\tau^{-\frac{1}{\delta}}G_0\left(x\tau^{-\frac{1}{\delta}}\right)\right)\right)(\tau,0)\right|$$
$$\le C\varepsilon^3\{\tau\}^{-\frac{\sigma}{\nu}}\langle\tau\rangle^{-1-\frac{\gamma}{\delta}}$$
$$+C\varepsilon^3\langle\tau\rangle^{-1}\left(\varepsilon g^{-1}(\tau)+\langle\tau\rangle^{-\frac{\gamma}{\delta}}+\langle\tau\rangle^{-\frac{\gamma}{\delta}}\right)$$
$$+C\varepsilon^3\{t\}^{-\frac{\sigma}{\nu}}\langle t\rangle^{-1-\frac{\gamma}{\delta}}+C\varepsilon^3\{t\}^{-\frac{\beta}{\delta}}\langle t\rangle^{-1-\frac{\gamma}{\delta}}$$

for all $\tau > 0$. By an explicit computation we have

$$\mathcal{F}_{x\to\xi}\left(\mathcal{N}_0\left(\theta\tau^{-\frac{1}{\delta}}G_0\left(x\tau^{-\frac{1}{\delta}}\right)\right)\right)(\tau,0)$$
$$=\theta^2\int_{\mathbf{R}}a_0(y)\,e^{-\tau L_0(-y)-\tau L_0(y)}dy$$
$$+\omega\theta^3\int_{\mathbf{R}^2}b_0(y,z)\,e^{-\tau L_0(-y)-\tau L_0(y-z)-\tau L_0(z)}dydz$$
$$=\theta^2\tau^{-1}\int_{\mathbf{R}}a_0(y)\,e^{-L_0(-y)-L_0(y)}dy$$
$$+\omega\theta^3\tau^{-1}\int_{\mathbf{R}^2}b_0(y,z)\,e^{-L_0(-y)-L_0(y-z)-L_0(z)}dydz$$
$$=\kappa\tau^{-1},$$

where $\omega = 0$ if $a_0 \neq 0$ and $\omega = 1$ if $a_0 \equiv 0$. Therefore we obtain

$$\left|\int_1^t\widehat{\mathcal{N}_1(v)}(\tau,0)\,d\tau-\kappa\log t\right|$$
$$\le C\varepsilon^3\int_1^t\frac{d\tau}{\tau(1+\kappa\log\langle t\rangle)}+C\varepsilon^2\int_1^t\tau^{-1-\frac{\gamma}{\delta}}d\tau+C\varepsilon^2\int_1^t\tau^{-1-\frac{\gamma}{\delta}}d\tau$$
$$\le\frac{C\varepsilon^3}{\kappa}\log(1+\kappa\log\langle t\rangle)+C\varepsilon^2$$

for all $t \ge 1$ if $a_0 \neq 0$ and

$$\left|\int_1^t\widehat{\mathcal{N}(v)}(\tau,0)\,d\tau-\kappa\log t\right|$$
$$\le C\varepsilon^4\int_1^t\frac{d\tau}{\tau(1+\kappa\log\langle t\rangle)}+C\varepsilon^3\int_1^t\tau^{-1-\frac{\gamma}{\delta}}d\tau+C\varepsilon^3\int_1^t\tau^{-1-\frac{\gamma}{\delta}}d\tau$$
$$\le\frac{C\varepsilon^4}{\kappa}\log(1+\kappa\log\langle t\rangle)+C\varepsilon^3$$

for all $t \geq 1$ if $a_0 \equiv 0$. Hence in view of (3.210) the result of the lemma follows. Lemma 3.63 is proved.

3.7.2 Proof of Theorem 3.56

For the local existence of classical solutions for the Cauchy problem (3.188) we refer to Chapter 2. We cannot apply directly the results of Theorem 3.2 , therefore we modify the method for case of problem (3.188) . We change the dependent variable $u(t,x) = e^{-\varphi(t)} v(t,x)$, then we get from (3.188)

$$v_t + \mathcal{L}v + e^{-\varphi(t)}\mathcal{N}_1(v) + e^{-2\varphi(t)}\mathcal{N}_2(v) - \varphi' v = 0, \tag{3.214}$$

where

$$\mathcal{N}_1(u) = \overline{\mathcal{F}}_{\xi\to x}\int_{\mathbf{R}} a(t,\xi,y)\,\widehat{u}(t,\xi-y)\,\widehat{u}(t,y)\,dy$$

and

$$\mathcal{N}_2(u) = \overline{\mathcal{F}}_{\xi\to x}\int_{\mathbf{R}^2} b(t,\xi,y,z)\,\widehat{u}(t,\xi-y)\,\widehat{u}(t,y-z)\,\widehat{u}(t,z)\,dydz.$$

Now we require that the real-valued function $\varphi(t)$ satisfies the following condition expressed in terms of the Fourier transform as follows

$$e^{-\varphi(t)}\widehat{\mathcal{N}_1}(v)(t,0) + e^{-2\varphi(t)}\widehat{\mathcal{N}_2}(v)(t,0) - \varphi'\widehat{v}(t,0) = 0;$$

hence, via equation (3.214), we get

$$\frac{d}{dt}\widehat{v}(t,0) = 0$$

for all $t > 0$. Therefore

$$e^{-\varphi(t)}\widehat{\mathcal{N}_1}(v)(t,0) + e^{-2\varphi(t)}\widehat{\mathcal{N}_2}(v)(t,0) = \varphi'\widehat{v}(0,0).$$

If we choose the initial conditions $\varphi(0) = 0$, we have

$$\widehat{v}(t,0) = \widehat{v}(0,0) = e^{\varphi(0)}\widehat{u}_0(0) = \theta,$$

and we obtain the following system

$$\begin{cases} v_t + \mathcal{L}v + e^{-\varphi(t)}\Big(\mathcal{N}(v) - \frac{v}{\theta}\widehat{\mathcal{N}_1}(v)(t,0) - \frac{v}{\theta}e^{-\varphi(t)}\widehat{\mathcal{N}_2}(v)(t,0)\Big) = 0, \\ \varphi'(t) = \frac{1}{\theta}e^{-\varphi(t)}\widehat{\mathcal{N}_1}(v)(t,0) + \frac{1}{\theta}e^{-2\varphi(t)}\widehat{\mathcal{N}_2}(v)(t,0), \\ v(0,x) = u_0(x),\ \varphi(0) = 0. \end{cases} \tag{3.215}$$

Multiplying the second equation of system (3.215) by the factor $e^{\varphi(t)}$, then integrating with respect to time $t > 0$ and making a change of the dependent variables $v = \mathcal{G}(t)u_0 + r$, and $e^{\varphi(t)} = h_1(t)$, we get the system of integral equations

$$\begin{cases} r = -\int_0^t h_1^{-1}(\tau)\,\mathcal{G}(t-\tau)\, f_1(\tau)\, d\tau, \\ h_1 = 1 + \frac{1}{\theta}\int_0^t \left(\widehat{\mathcal{N}_1}(v) + \frac{1}{h_1(\tau)}\widehat{\mathcal{N}_2}(v)\right)(\tau,0)\, d\tau, \end{cases} \tag{3.216}$$

where

$$f_1(t) = \mathcal{N}(v(t)) - \frac{v(t)}{\theta}\widehat{\mathcal{N}_1}(v)(t,0) - \frac{v(t)}{\theta h_1(t)}\widehat{\mathcal{N}_2}(v)(t,0).$$

In the case of $a_0 \equiv 0$ we denote $e^{2\varphi(t)} = h_2(t)$ and obtain the following system of integral equations:

$$\begin{cases} r = -\int_0^t h_2^{-1}(\tau)\,\mathcal{G}(t-\tau)\, f_2(\tau)\, d\tau, \\ h_2 = 1 + \frac{1}{2\theta}\int_0^t \left(\sqrt{h_2(\tau)}\widehat{\mathcal{N}_1}(v) + \widehat{\mathcal{N}_2}(v)\right)(\tau,0)\, d\tau \end{cases} \tag{3.217}$$

where

$$f_2(t) = \sqrt{h_2(t)}\mathcal{N}(v(t)) - \frac{v(t)}{\theta}\sqrt{h_2(t)}\widehat{\mathcal{N}_1}(v)(t,0) - \frac{v(t)}{\theta}\widehat{\mathcal{N}_2}(v)(t,0).$$

Denote

$$\mathcal{M}_j(r,h_j)(t) = -\int_0^t h_j^{-1}(\tau)\,\mathcal{G}(t-\tau)\, f_j(\tau)\, d\tau,$$
$$\mathcal{R}_1(r,h_1)(t) = 1 + \frac{1}{\theta}\int_0^t \left(\widehat{\mathcal{N}_1}(v) + \frac{1}{h_1(\tau)}\widehat{\mathcal{N}_2}(v)\right)(\tau,0)\, d\tau$$

in the case of $a_0 \neq 0$ and

$$\mathcal{R}_2(r,h_2)(t) = 1 + \frac{1}{2\theta}\int_0^t \left(\sqrt{h_2(\tau)}\widehat{\mathcal{N}_1}(v) + \widehat{\mathcal{N}_2}(v)\right)(\tau,0)\, d\tau$$

in the case of $a_0 \equiv 0$. Let us prove that $(\mathcal{M}_j, \mathcal{R}_j)$ is the contraction mapping in the set

$$\mathbf{X} = \{r \in \mathbf{C}((0,\infty);\mathbf{D}^{\alpha,\sigma}),\ h_j \in \mathbf{C}((0,\infty)) :$$
$$\|g(t)\, r(t)\|_{\mathbf{X}} \le C\varepsilon^2,\ \frac{1}{2}g(t) \le h_j(t) \le 2g(t) \text{ for all } t>0\},$$

where

$$\|\phi\|_{\mathbf{X}} = \sup_{\rho\in[-\gamma,\alpha+\gamma]}\sup_{t>0}\langle t\rangle^{\frac{\rho+1}{\delta}}\|\phi(t)\|_{\mathbf{A}^{\rho,1}} + \sup_{\rho\in[0,\alpha+\gamma]}\sup_{t>0}\langle t\rangle^{\frac{\rho}{\delta}}\|\phi(t)\|_{\mathbf{A}^{\rho,\infty}}$$
$$+ \sup_{s\in[0,\sigma]}\sup_{1\le p\le\infty}\sup_{t>0}\{t\}^{\frac{s}{\nu}}\langle t\rangle^{1+\frac{\gamma}{\delta}+\frac{1}{\delta p}}\|\phi(t)\|_{\mathbf{B}^{s,p}}$$
$$+ \sup_{\rho=0,\alpha,\beta}\sup_{s\in[0,\sigma]}\sup_{t>0}\{t\}^{\frac{s}{\nu}}\langle t\rangle^{\frac{\rho-\gamma}{\delta}}\|\phi(t)\|_{\mathbf{D}^{\rho,s}};$$

here $\gamma \in (0, \min(1, \delta))$ is such that $\gamma < \alpha$ if $\alpha > 0$ and $\gamma < \beta$ if $\beta > 0$. First we prove that the mapping $(\mathcal{M}_j, \mathcal{R}_j)$ transforms the set $\mathbf{X}$ into itself. When $(r, h_j) \in \mathbf{X}$ we get by Lemma 3.62

$$\|\mathcal{N}(v)\|_{\mathbf{Y}} \leq C\varepsilon^2$$

and

$$\left\| \frac{v(\tau)}{\theta} \widehat{\mathcal{N}}(v)(\tau, 0) \right\|_{\mathbf{Y}} \leq C\varepsilon^2,$$

where

$$\begin{aligned}\|\phi\|_{\mathbf{Y}} &= \sup_{1 \leq p \leq \infty} \sup_{t>0} \langle t \rangle^{1+\frac{1}{\delta p}} \|\phi(t)\|_{\mathbf{A}^{0,p}} \\ &+ \sup_{1 \leq p \leq \infty} \sup_{t>0} \langle t \rangle^{1+\frac{\gamma}{\delta}+\frac{1}{\delta p}} \{t\}^{\frac{\sigma}{\nu}} \|\phi(t)\|_{\mathbf{B}^{0,p}} \\ &+ \sup_{t>0} \langle t \rangle^{1-\frac{\gamma}{\delta}} \{t\}^{\frac{\sigma}{\nu}} \|\phi(t)\|_{\mathbf{D}^{0,0}}.\end{aligned}$$

Hence

$$\begin{aligned}\|f_1\|_{\mathbf{Y}} &= \left\| \mathcal{N}(v(t)) - \frac{v(t)}{\theta} \widehat{\mathcal{N}_1}(v)(t, 0) - \frac{v(t)}{\theta h_1(t)} \widehat{\mathcal{N}_2}(v)(t, 0) \right\|_{\mathbf{Y}} \\ &\leq C\varepsilon^2\end{aligned}$$

and

$$\begin{aligned}\|f_2\|_{\mathbf{Y}} &= \left\| \sqrt{h_2(t)} \mathcal{N}(v(t)) - \frac{v(t)}{\theta} \sqrt{h_2(t)} \widehat{\mathcal{N}_1}(v)(t, 0) - \frac{v(t)}{\theta} \widehat{\mathcal{N}_2}(v)(t, 0) \right\|_{\mathbf{Y}} \\ &\leq C\varepsilon^2.\end{aligned}$$

Therefore applying Lemma 3.61 we get the estimates

$$\begin{aligned}\|g(t) \mathcal{M}_j(r, h_j)(t)\|_{\mathbf{X}} &\leq C \left\| g(t) \int_0^t h_j^{-1}(\tau) \mathcal{G}(t - \tau) f_j(\tau) d\tau \right\|_{\mathbf{X}} \\ &\leq C \|f_j\|_{\mathbf{Y}} \leq C\varepsilon^2.\end{aligned}$$

Furthermore, when $r \in \mathbf{X}$ we have estimates $\|v\|_{\mathbf{X}} \leq C\varepsilon$ and

$$\|g(v - \mathcal{G}(t) u_0)\|_{\mathbf{X}} \leq C \|gr\|_{\mathbf{X}} \leq C\varepsilon^2.$$

Via Lemma 3.63 we obtain

$$\left| 1 + \int_0^t \widehat{\mathcal{N}_1(v)}(\tau, 0) d\tau - \kappa \log t \right| \leq \frac{C\varepsilon^3}{\kappa} \log(g(t)) + C\varepsilon^2$$

if $a_0 \neq 0$ and

$$\left| 1 + \int_0^t \widehat{\mathcal{N}_2(v)}(\tau, 0) d\tau - \kappa \log t \right| \leq \frac{C\varepsilon^4}{\kappa} \log(g(t)) + C\varepsilon^3$$

if $a_0 \equiv 0$. Hence

$$\frac{1}{2} g(t) \le h_j(t) \le 2g(t)$$

for all $t > 0$. Thus $(\mathcal{M}_j, \mathcal{R}_j)$ transforms the set $\mathbf{X}$ into itself.

In a similar manner we consider the differences $\mathcal{M}_j(r, h_j) - \mathcal{M}_j\left(\widetilde{r}, \widetilde{h}_j\right)$ and $\mathcal{R}_j(r, h_j) - \mathcal{R}_j\left(\widetilde{r}, \widetilde{h}_j\right)$ to see that the transformation $(\mathcal{M}_j, \mathcal{R}_j)$ is the contraction mapping. Therefore there exists a unique solution (r, h_j) of system of integral equations (3.216) in the space $\mathbf{X}$. From Lemma 3.63 we see that

$$h_j(t) = \kappa \log t + O(\log \log t)$$

for $t \to \infty$. Therefore via formulas $u(t,x) = e^{-\varphi(t)} v(t,x) = e^{-\varphi(t)} (\mathcal{G} v_0 + r)$ we obtain the asymptotic formula of the theorem. Theorem 3.56 is proved.

3.8 Comments

Section 3.1.

The asymptotic behavior in time of positive solutions to the critical nonlinear heat equation was studied in Escobedo and Kavian [1988], Escobedo et al. [1995], Galaktionov et al. [1985], Gmira and Véron [1984], Kamin and Peletier [1986], Kavian [1987]. In particular in papers Galaktionov et al. [1985], Gmira and Véron [1984] it was shown that if the initial data are not negative and $\sigma = \frac{2}{n}$, then the solution decays in time like $(t \log t)^{-\frac{n}{2}}$ for any $x \in \mathbf{R}^n$. This result was extended to the case of the porous media equation with critical exponents (see Gmira and Véron [1984]).

In Escobedo and Zuazua [1991] large time behavior of solutions to the convection-diffusion equation

$$u_t - \Delta u - (a, \nabla)(|u|^\sigma u) = 0 \tag{3.218}$$

was studied for the case of $\sigma \ge 0$ without smallness condition on the data and any restriction on space dimension. They showed that when $u_0 \in \mathbf{L}^1$, solutions of (3.218) behave like the heat kernel if $\sigma > \frac{1}{n}$ and the corresponding self-similar solutions if $\sigma = \frac{1}{n}$.

Section 3.2.

Local in time existence of solutions to the Cauchy problem (3.22) with $\rho = 2$ was studied by many authors (see, e.g. Ginibre and Velo [1996], Ginibre and Velo [1997] and references cited therein). In the case of $\rho \ne 2$, local in time existence can be clearly shown by the contraction mapping principle in L^2 framework. Nonlinear dissipative equations with a fractional power of the negative Laplacian in the principal part were studied extensively (see, e.g., Bardos et al. [1979], Biler et al. [2001a], Biler et al. [1998], Biler et al. [2001b], Shlesinger et al. [1995], Taylor [1992] and references cited therein). Blow-up in finite time of positive solutions to the Cauchy problem

$$\partial_t u + (-\Delta)^{\frac{\rho}{2}} u - u^{1+\sigma} = 0,\ u(0,x) = u_0(x) > 0$$

was proved in papers Fujita [1966], Weissler [1981] for the case of $0 < \sigma < \frac{2}{n}$, $\rho = 2$, in papers Hayakawa [1973], Kobayashi et al. [1977] for the case of $\sigma = \frac{2}{n}$, $\rho = 2$, and in paper Sugitani [1975] for the case of $0 < \rho \leq 2$, $0 < \sigma \leq \frac{\rho}{n}$. Their proofs of blow-up results are based on the positivity of the linear evolution operator $\overline{\mathcal{F}}_{\xi \to x} e^{-|\xi|^{\rho}}$, associated with equation (5.135), for $0 < \rho \leq 2$ (see book Yosida [1995]), and do not work for the case of $\rho > 2$, since $\overline{\mathcal{F}}_{\xi \to x} e^{-|\xi|^{\rho}}$ is not necessarily positive.

The result of Theorem 3.11 is applicable, in particular, to the Cauchy problem

$$\partial_t u + (-\Delta)^{\frac{\rho}{2}} u + \lambda u^{1+\sigma} = \mu u^{1+\kappa}, \ u(0,x) = u_0(x) > 0, \tag{3.219}$$

with $0 < \sigma < \kappa \leq \frac{\rho}{n}, \lambda, \mu > 0$. As we mentioned above the solutions of (3.219) blow up in finite time, when $\lambda = 0$, $\mu > 0$, $0 < \rho \leq 2$ and exist globally in time, when $\lambda > 0$, $\mu = 0$, $0 < \rho < \infty$. Thus the result of Theorem 3.11 shows that the dissipation term $u^{1+\sigma}$ in equation (3.219) is stronger than the blow-up term $u^{1+\kappa}$. Note that the problem of asymptotic behavior of solutions to (3.219) is still open for the subcritical case $0 < \kappa < \sigma \leq \frac{\rho}{n}$ even if $\rho = 2$. The proof of Theorem 3.11 in this section follows the paper Hayashi et al. [2004a]. Equation (3.25) with $\alpha = 2$ is a usual nonlinear heat equation, that was studied extensively (see papers Escobedo and Kavian [1988], Escobedo et al. [1995], Galaktionov et al. [1985], Gmira and Véron [1984], Kamin and Peletier [1986], Kavian [1987]). In the case of $\alpha \neq 2$, local in time existence can be easily shown by the contraction mapping principle in the L^2 - framework. Nonlinear dissipative equations with a fractional power of the negative Laplacian in the principal part were studied in papers Bardos et al. [1979], Biler et al. [1998], Komatsu [1984], Taylor [1992], Zhang [2001] (see also references cited therein). The results of this section were published in paper Hayashi et al. [2006b].

Section 3.3.

Equation (3.40) with $\alpha = 2$ is the nonlinear heat equation

$$u_t - \Delta u + |u|^{\sigma} u = 0.$$

It was studied in papers Galaktionov et al. [1985], Hayashi et al. [2003b] for the critical case of $\sigma = 2$. Nonlinear dissipative equations with a derivative of a fractional order in the principal part were studied extensively (see, Biler et al. [1998], Biler et al. [2000], Hayashi et al. [2000], Hayashi et al. [2004b], Hayashi et al. [2004a], Komatsu [1984], Shlesinger et al. [1995] and references cited therein). Large time behavior of solutions to problem (3.218) with $\mathcal{L} = -\partial_x^2 + |\partial_x|^{\alpha}$, $1 < \alpha < 2$ was studied in Biler et al. [2000] and Biler et al. [2001a]. Similar results to those of paper Escobedo et al. [1993b] were obtained in Biler et al. [2001b] for the supercritical case of $\sigma > \alpha$ and in Biler et al. [2000] for the critical case of $\sigma = \alpha$. Their method is based on the $\mathbf{L}^1$ - contraction property of the semigroup $\exp(-t|\partial_x|^{\alpha})$ if $1 < \alpha < 2$ (see Bardos et al. [1979]).

Section 3.4.

The blow-up phenomena of positive solutions to the semilinear parabolic equation

$$u_t - \Delta u = u^p$$

was obtained in Fujita [1966] for $1 < p < 1 + \frac{2}{n}$, in Hayakawa [1973] for $p = 1 + \frac{2}{n}$, $n = 1, 2$, and in Kobayashi et al. [1977] for $p = 1 + \frac{2}{n}$ and any space dimension n.

On the other hand, the asymptotic behavior in time of positive solutions to the equation

$$u_t - \Delta u = -u^{1+\frac{2}{n}}$$

was studied in papers Galaktionov et al. [1985], Gmira and Véron [1984], Mizoguchi and Yanagida [1998], where it was shown that if the initial data are not negative, then the solution decays in time as $(t \log t)^{-\frac{n}{2}}$ for any x. The first part of Theorem 3.22 includes this result. For the problem (3.68), the existence and uniqueness of solutions were considered in Ginibre and Velo [1996], Ginibre and Velo [1997], Okazawa and Yokota [2002] and the large time asymptotics of solutions was obtained in Hayashi et al. [2001] the one dimensional case $n = 1$. In Galaktionov et al. [1985] and Mizoguchi and Yanagida [1998] the asymptotic behavior in time of positive solutions to the Cauchy problem

$$\partial_t u - \partial_x^2 u + u^3 = 0, \quad u(0,x) = u_0(x) > 0 \tag{3.220}$$

was considered, and it was shown that the $\mathbf{L}^\infty$ norm of solutions decay in time faster than $t^{-1/2}$. However these methods do not work for the complex Landau - Ginzburg equation (3.68). The material of this section was taken from papers Hayashi et al. [2001], Hayashi et al. [2003a] and Hayashi et al. [2002].

Section 3.5.

Recently much attention has been drawn to nonlinear wave equations with dissipative terms. We mention here some recent works concerning the global existence and nonexistence of solutions to the Cauchy problem for damped nonlinear wave equations. The blow-up results were proved in Todorova and Yordanov [2001] for the case of $\mathcal{N}(u) = -|u|^p$, $p < 1 + \frac{2}{n}$, when the initial data are such that $\int_{\mathbf{R}^n} u_0(x)\,dx > 0$, $\int_{\mathbf{R}^n} u_1(x)\,dx > 0$. In the critical case the blow-up results were obtained in Li and Zhou [1995] for $n \leq 2$ and Zhang [2001], Ikehata and Ohta [2002] for general n . In Marcati and Nishihara [2003], a blow-up result was obtained for problem (3.107) with $\mathcal{N}(u) =$
$-|u|^{p-1} u$ in the space dimension $n = 3$ under the conditions $p \leq 1 + \frac{2}{n}$ and $u_0(x) = 0$, $u_1(x) \geq 0$, $\int_{\mathbf{R}^n} u_1(x)\,dx > 0$. Note that similar behavior first was discovered in Fujita [1966] for the nonlinear heat equation in the critical and subcritical cases $p \leq 1 + \frac{2}{n}$.

For the initial data from the usual Sobolev space $u_0 \in \mathbf{W}_1^1(\mathbf{R}^n) \cap \mathbf{W}_\infty^1(\mathbf{R}^n)$, $u_1 \in \mathbf{L}^1(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n)$, problem (3.107) was considered in Nishihara [2003]. By employing the fundamental solution of the linear problem the global existence of small solutions and large time decay estimates $\|u\|_{\mathbf{L}^q} \leq Ct^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}$, $1 \leq q \leq \infty$
for space dimension $n = 3$ were proved. Later these requirements on the initial data were relaxed in Ono [2003] as follows $u_0 \in \mathbf{L}^1(\mathbf{R}^n) \cap \mathbf{H}^1(\mathbf{R}^n)$, $u_1 \in \mathbf{L}^1(\mathbf{R}^n) \cap \mathbf{L}^2(\mathbf{R}^n)$, under the additional assumptions on p and q such that $p \leq 5$, $q \leq 6$ for the space dimension $n = 3$ and $q < \infty$ for the two dimensional case $n = 2$.

Recently in paper Nishihara and Zhao [2006] it was obtained an optimal time decay estimate of the $\mathbf{L}^p$ - norm of solutions to the Cauchy problem (3.134) in the sub critical case $\sigma \in \left(0, \frac{2}{n}\right)$ under the condition, that the initial data decay exponentially at infinity without any restriction on the size. Here we apply the idea of paper Nishihara and Zhao [2006] to obtain a priori weighted energy type estimates of the solutions. Then we follow the method of paper Hayashi and Naumkin [2006b] to find the large time asymptotics of solutions. In this section we follow the method of papers Hayashi et al. [2006a], Hayashi et al. [2004f].

Section 3.6.

The part of this section concerning small initial data was taken from paper Kaĭkina et al. [2005]. For obtaining the a priori weighted energy type estimates of the solutions in the case of large initial data we follow the idea of paper Nishihara and Zhao [2006].

Section 3.7.

The large time asymptotic behavior of solutions to the Cauchy problem for the nonlinear Schrödinger equation with dissipation

$$u_t + \mathcal{L}u + i|u|^2 u = 0,\ x \in \mathbf{R},\ t > 0$$

in the critical case was obtained in paper Hayashi et al. [2000]. Here the symbol $L(\xi)$ of the linear pseudodifferential operator $\mathcal{L}$ has the following asymptotic representation $L(\xi) \sim \mu\,|\xi|^2$ in the origin $\xi \to 0$, where $\operatorname{Re}\mu > 0$, $\operatorname{Im}\mu \geq 0$.

Papers Hayashi et al. [2001] and Hayashi et al. [2003a] considered the large time asymptotics for solutions of the complex Landau-Ginzburg equation

$$u_t - \mu\Delta u + a|u|^q u = 0,\ x \in \mathbf{R}^n,\ t > 0$$

in the critical case $q = \frac{2}{n}$. The asymptotic expansion of small solutions to the Cauchy problem for the complex Landau - Ginzburg equation was considered in paper Hayashi et al. [2002].

In this section we generalize the approach developed in papers Hayashi et al. [2005a], Hayashi et al. [2003b] and Hayashi et al. [2004c] which mainly considered the case of the Laplacian $\mathcal{L} = -\Delta$, the power nonlinearity $\mathcal{N}(u) = |u|^{q_1} u$, $0 < q_1 \leq \frac{2}{n}$ and the nonlinearity of the form $\partial_{x_1} |u|^{q_2+1}$, $0 < q_2 \leq \frac{1}{n}$. The $\mathbf{L}^p$ estimates of the Green operator $e^{t\Delta}$ were applied to show the positivity of the value $\int N(u_1)\,dx$, where u_1 is the first approximation of the solution. In comparison with works Hayashi et al. [2000] through Hayashi et al. [2003a] in this section we work in the Lebesgue spaces for the Fourier transform of the solution in order to treat the case of nonlocal nonlinearities of nonconvective type involving derivatives of unknown function and to show that $\int (N(u) - N(u_1))\,dx$ is the remainder term. To obtain the estimates of the remainder terms of the large time asymptotic formulas we assume that the initial data satisfy some decay condition at infinity.

4

Critical Convective Equations, Self-similar Solutions

In this chapter we study the large time asymptotic behavior of solutions to various critical dissipative equations with convective type nonlinearities. The famous Burgers equation $u_t + u\ u_x - u_{xx} = 0$, under the condition that the total mass of the initial data is nonzero, gives us an example of the convective equations. The character of the large time asymptotic behavior for convective type equations in the critical case is determined by the self-similar solutions. We also are interested in removing the smallness requirement on the initial data taking into account some additional properties such as the maximum principle, positivity of solutions or applying the energy type estimates.

4.1 General approach

Now we give a general approach for obtaining the large time asymptotic representation of solutions to the Cauchy problem (1.7)

$$\begin{cases} u_t + \mathcal{N}(u) + \mathcal{L}u = 0,\ x \in \mathbf{R}^n,\ t > 0, \\ \quad u(0,x) = u_0(x),\ x \in \mathbf{R}^n, \end{cases} \tag{4.1}$$

in the case of critical nonlinearity $\mathcal{N}(u)$ of the convective type. We fix a metric space $\mathbf{Z}$ of functions defined on $\mathbf{R}^n$ and a complete metric space $\mathbf{X}$ of functions defined on $[0,\infty) \times \mathbf{R}^n$. We denote as above by $G_0 \in \mathbf{X}$ the asymptotic kernel for the Green operator $\mathcal{G}$ in spaces $\mathbf{X}$, $\mathbf{Z}$ with a linear continuous functional f (see Definition 2.1.)

Definition 4.1. *We call the operator*

$$\mathcal{G}_0(t)\phi = t^{-\frac{n}{\delta}} \int_{\mathbf{R}^n} \widetilde{G_0}\left((x-y)\,t^{-\frac{1}{\delta}}\right)\phi(y)\,dy$$

with some $\delta > 0$ a self-similar asymptotic operator for the Green operator $\mathcal{G}$ in spaces $\mathbf{X}$, $\mathbf{Z}$ if the estimates are true

$$\|\mathcal{G}_0(t)\phi\|_{\mathbf{X}} + \|\langle t\rangle^{\gamma}(\mathcal{G}(t) - \mathcal{G}_0(t))\phi\|_{\mathbf{X}} \le C\|\phi\|_{\mathbf{Z}} \tag{4.2}$$

for any $\phi \in \mathbf{Z}$, *where* $\gamma > 0$. *Also we assume that the asymptotic kernel (see Definition 2.1) has a self-similar form* $G_0(t,x) = t^{-\alpha}\widetilde{G_0}\left(xt^{-\frac{1}{\delta}}\right)$ *with some* $\alpha > 0$, $\delta > 0$ *and the linear continuous functional* f *is such that*

$$f\left(t^{-\alpha}\phi\left((\cdot)\,t^{-\frac{1}{\delta}}\right)\right) = f(\phi) \tag{4.3}$$

for all $t > 0$, *and* $\phi \in \mathbf{Z}$.

We now fix a metric space $\mathbf{Q}$ of functions defined on $\mathbf{R}^n$, such that the norm of $\mathbf{Q}$ is induced by the norm of $\mathbf{X}$ by the relation

$$\|\phi\|_{\mathbf{Q}} = \left\|t^{-\alpha}\phi\left((\cdot)\,t^{-\frac{1}{\delta}}\right)\right\|_{\mathbf{X}}.$$

Definition 4.2. *We call the nonlinearity* $\mathcal{N}$ *in equation (4.1) a critical convective if* $f(\mathcal{N}(u)) = 0$ *for any* $u \in \mathbf{X}$, *and*

$$\begin{aligned} &t^{-\alpha}\int_0^1 \mathcal{G}_0(1-z)\left(xt^{-\frac{1}{\delta}}\right)\mathcal{N}\left(z^{-\alpha}\phi\left((\cdot)\,z^{-\frac{1}{\delta}}\right)\right)dz \\ &= \int_0^t \mathcal{G}_0(t-\tau)(x)\,\mathcal{N}\left(\tau^{-\alpha}\phi\left((\cdot)\,\tau^{-\frac{1}{\delta}}\right)\right)d\tau \end{aligned} \tag{4.4}$$

for any $\phi \in \mathbf{Q}$.

First we prove the existence of particular solutions of equation (4.1) having a self-similar form.

Lemma 4.3. *Suppose that the operator* $\mathcal{G}_0(t)$ *is a self-similar asymptotic operator for the Green operator* $\mathcal{G}$ *in spaces* $\mathbf{X}$, $\mathbf{Z}$ *and condition (4.3) is true. Let the nonlinearity* $\mathcal{N}$ *in equation (4.1) be the critical convective (see Definition 4.2). Also we assume that the estimate is true*

$$\begin{aligned} &\left\|\int_0^t \mathcal{G}_0(t-\tau)\left(\mathcal{N}(v(\tau)) - \mathcal{N}(w(\tau))\right)d\tau\right\|_{\mathbf{X}} \\ &\le C\|v-w\|_{\mathbf{X}}\left(\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}\right)^{\sigma} \end{aligned} \tag{4.5}$$

for any $v, w \in \mathbf{X}$. *Then for sufficiently small* θ ($|\theta| > 0$) *there exists a unique solution* $V \in \mathbf{Q}$ *to the integral equation*

$$V = \theta\widetilde{G_0} - \int_0^1 \mathcal{G}_0(1-z)\,\mathcal{N}\left(z^{-\alpha}V\left((\cdot)\,z^{-\frac{1}{\delta}}\right)\right)dz \tag{4.6}$$

such that $\|V\|_{\mathbf{Q}} \le C|\theta|$.

Proof. We prove the existence of the solution V for integral equation (4.6) by the contraction mapping principle. We define the transformation $\mathcal{R}(V)$ by the formula

$$\mathcal{R}(V) = \theta\widetilde{G_0} - \int_0^1 \mathcal{G}_0(1-z)\,\mathcal{N}\left(z^{-\alpha}V\left((\cdot)\,z^{-\frac{1}{\delta}}\right)\right)dz,$$

for any $V \in \mathbf{Q}_\rho$, where

$$\mathbf{Q}_\rho = \left\{V \in \mathbf{Q}: \ \|V\|_{\mathbf{Q}} \le \rho\right\}$$

and $C|\theta| \le \rho$ and $\rho > 0$ sufficiently small. First we check that the mapping $\mathcal{R}$ transforms the set $\mathbf{Q}_\rho$ into itself. We denote

$$v(t,x) = t^{-\alpha}V\left(xt^{-\frac{1}{\delta}}\right).$$

By property (4.4) and relation of the norms $\mathbf{Q}$ and $\mathbf{X}$ we have

$$\begin{aligned}
\|\mathcal{R}(V)\|_{\mathbf{Q}} &\le \left\|\theta\widetilde{G_0}\right\|_{\mathbf{Q}} + \left\|\int_0^1 \mathcal{G}_0(1-z,\xi)\,\mathcal{N}\left(z^{-\alpha}V\left((\cdot)\,z^{-\frac{1}{\delta}}\right)\right)dz\right\|_{\mathbf{Q}} \\
&\le C|\theta| + \left\|t^{\alpha}\int_0^t \mathcal{G}_0\left(t-\tau,\xi t^{\frac{1}{\delta}}\right)\mathcal{N}(v(\tau))\,d\tau\right\|_{\mathbf{Q}} \\
&= C|\theta| + \left\|\int_0^t \mathcal{G}_0(t-\tau)\,\mathcal{N}(v(\tau))\,d\tau\right\|_{\mathbf{X}}.
\end{aligned}$$

Because of (4.5), we get

$$\left\|\int_0^t \mathcal{G}_0(t-\tau)\,\mathcal{N}(v(\tau))\,d\tau\right\|_{\mathbf{X}} \le C\|v\|_{\mathbf{X}}^{\sigma+1} = C\|V\|_{\mathbf{Q}}^{\sigma+1} \le C\rho^{\sigma+1}.$$

Hence

$$\|\mathcal{R}(V)\|_{\mathbf{Q}} \le C|\theta| + C\rho^{\sigma+1} \le C\rho$$

if $\rho > 0$ is small. In the same manner we estimate the difference, denoting $w(t,x) = t^{-\alpha}W\left(xt^{-\frac{1}{\delta}}\right)$ and using the relation of the norms $\mathbf{Q}$ and $\mathbf{X}$

$$\begin{aligned}
&\|\mathcal{R}(V) - \mathcal{R}(W)\|_{\mathbf{Q}} \\
&\le \left\|\int_0^1 \mathcal{G}_0(1-z,\xi)\left(\mathcal{N}\left(z^{-\alpha}V\left((\cdot)\,z^{-\frac{1}{\delta}}\right)\right) - \mathcal{N}\left(z^{-\alpha}W\left((\cdot)\,z^{-\frac{1}{\delta}}\right)\right)\right)dz\right\|_{\mathbf{Q}} \\
&= \left\|\int_0^t \mathcal{G}_0(t-\tau)\left(\mathcal{N}(v(\tau)) - \mathcal{N}(w(\tau))\right)d\tau\right\|_{\mathbf{X}} \\
&\le \frac{1}{2}\|V-W\|_{\mathbf{Q}}\left(\|V\|_{\mathbf{Q}}^{\sigma} + \|W\|_{\mathbf{Q}}^{\sigma}\right) \le C\rho^{\sigma}\|V-W\|_{\mathbf{Q}} \le \frac{1}{2}\|V-W\|_{\mathbf{Q}}.
\end{aligned}$$

Therefore, $\mathcal{R}$ is a contraction mapping in the closed set $\mathbf{Q}_\rho$ of a complete metric space $\mathbf{X}$. Hence there exists a unique solution $V \in \mathbf{Q}_\rho$ to the integral equation (4.6). Lemma 4.3 is proved.

We now prove that self-similar solutions $V(\xi)$ obtained in Lemma 4.3 give us the asymptotics of solutions to the Cauchy problem (4.1). Denote $\theta = f(u_0)$.

Theorem 4.4. *Suppose that the initial data $u_0 \in \mathbf{Z}$ and the norm $\|u_0\|_{\mathbf{Z}} \le \varepsilon$, with sufficiently small $\varepsilon > 0$. Let the nonlinearity $\mathcal{N}(u)$ in equation (4.1) be the critical convective. Assume that the following estimates are true*

$$\left\| \langle t \rangle^{\gamma} \int_0^t \mathcal{G}(t-\tau)\left(\mathcal{N}(v(\tau)) - \mathcal{N}(w(\tau))\right) d\tau \right\|_{\mathbf{X}} \\ \le C \left\| \langle t \rangle^{\gamma} (v(t) - w(t)) \right\|_{\mathbf{X}} \left(\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}} \right)^{\sigma} \tag{4.7}$$

and

$$\left\| \langle t \rangle^{\gamma} \int_0^t \left(\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau)\right) \mathcal{N}(v(\tau))\, d\tau \right\|_{\mathbf{X}} \le C \|v\|_{\mathbf{X}}^{1+\sigma} \tag{4.8}$$

for any $v, w \in \mathbf{X}$, where $\sigma > 0$, $\gamma > 0$ and $\mathcal{G}_0$ is the asymptotic self-similar operator for the Green operator $\mathcal{G}$ in spaces $\mathbf{X}$, $\mathbf{Z}$. Suppose that there exists a unique global solution $u \in \mathbf{X}$ of the problem (4.1) with small norm $\|u\|_{\mathbf{X}} \le C\varepsilon$. Then for the solution $u \in \mathbf{X}$ of the Cauchy problem (4.1) the asymptotics

$$\left\| \langle t \rangle^{\gamma} \left(u(t) - t^{-\alpha} V\left((\cdot)\, t^{-\frac{1}{\delta}} \right) \right) \right\|_{\mathbf{X}} \le C \tag{4.9}$$

is valid for $t \to \infty$, where V is the solution of the integral equation (4.6) with $\theta = f(u_0)$.

Proof. Denote $v(t) = t^{-\alpha} \theta V\left((\cdot)\, t^{-\frac{1}{\delta}} \right)$, where V satisfies integral equation (4.6); we then have

$$v(t) = \theta G_0(t) - \int_0^t \mathcal{G}_0(t-\tau)\, \mathcal{N}(v(\tau))\, d\tau,$$

with $G_0(t) = t^{-\alpha} \widetilde{G}_0\left((\cdot)\, t^{-\frac{1}{\delta}} \right)$. Since $u(t)$ satisfies the integral equation (1.8)

$$u(t) = \mathcal{G}(t)\, u_0 - \int_0^t \mathcal{G}(t-\tau)\, \mathcal{N}(u(\tau))\, d\tau,$$

we write for the difference

$$\left\| \langle t \rangle^{\gamma} (u(t) - v(t)) \right\|_{\mathbf{X}} \le \left\| \langle t \rangle^{\gamma} \left(\mathcal{G}(t)\, u_0 - \theta G_0(t) \right) \right\|_{\mathbf{X}} \\ + \left\| \langle t \rangle^{\gamma} \int_0^t \mathcal{G}(t-\tau)\left(\mathcal{N}(u(\tau)) - \mathcal{N}(v(\tau)) \right) d\tau \right\|_{\mathbf{X}} \\ + \left\| \langle t \rangle^{\gamma} \int_0^t \left(\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau) \right) \mathcal{N}(v(\tau))\, d\tau \right\|_{\mathbf{X}} \\ \equiv I_1 + I_2 + I_3. \tag{4.10}$$

Note that in view of the relation of the norms $\mathbf{X}$ and $\mathbf{Q}$ we have

$$\|v\|_{\mathbf{X}} = |\theta| \, \|V\|_{\mathbf{Q}} \leq C \, |\theta| \leq C\varepsilon.$$

Then by the definition of the asymptotic kernel (see Definition 2.1) we obtain

$$I_1 = \left\| \langle t \rangle^{\gamma} \left(\mathcal{G}(t) \, u_0 - \theta G_0(t) \right) \right\|_{\mathbf{X}} \leq C \, \|u_0\|_{\mathbf{Z}} \leq C\varepsilon.$$

By virtue of condition (4.7) we get

$$\begin{aligned} I_2 &= \left\| \langle t \rangle^{\gamma} \int_0^t \mathcal{G}(t-\tau) \left(\mathcal{N}(u(\tau)) - \mathcal{N}(v(\tau)) \right) d\tau \right\|_{\mathbf{X}} \\ &\leq C \, \| \langle t \rangle^{\gamma} (u(t) - v(t)) \|_{\mathbf{X}} \left(\|u\|_{\mathbf{X}}^{\sigma} + \|v\|_{\mathbf{X}}^{\sigma} \right) \\ &\leq C\varepsilon^{\sigma} \, \| \langle t \rangle^{\gamma} (u(t) - v(t)) \|_{\mathbf{X}} , \end{aligned}$$

and via condition (4.8) we find

$$\begin{aligned} I_3 &= \left\| \langle t \rangle^{\gamma} \int_0^t \left(\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau) \right) \mathcal{N}(v(\tau)) \, d\tau \right\|_{\mathbf{X}} \\ &\leq C \, \|v\|_{\mathbf{X}}^{\sigma+1} \leq C \, \|V\|_{\mathbf{Q}}^{\sigma+1} \leq C \, |\theta|^{\sigma+1} \leq C\varepsilon^{\sigma+1}. \end{aligned}$$

Now (4.10) implies

$$\| \langle t \rangle^{\gamma} (u(t) - v(t)) \|_{\mathbf{X}} \leq C\varepsilon + C\varepsilon^{\sigma} \, \| \langle t \rangle^{\gamma} (u(t) - v(t)) \|_{\mathbf{X}} ,$$

so estimate (4.9) follows since $\varepsilon > 0$ is small enough. Theorem 4.4 is proved.

Example 4.5. **Large time asymptotics of solutions to the critical Burgers type equations**

We consider the Cauchy problem for the Burgers type equation (1.18)

$$\begin{cases} u_t - \Delta u = (\lambda \cdot \nabla) \, |u|^{\sigma} \, u, \ x \in \mathbf{R}^n, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^n, \end{cases} \tag{4.11}$$

where $\lambda \in \mathbf{R}^n$ and the critical $\sigma = \frac{1}{n}$. Define the space $\mathbf{Z} = \mathbf{L}^{1,a}(\mathbf{R}^n)$, where $a \in (0,1)$ and the space

$$\mathbf{X} = \left\{ \phi \in \mathbf{C}\left([0,\infty); \mathbf{L}^1(\mathbf{R}^n)\right) \cap \mathbf{C}\left((0,\infty); \mathbf{W}_{\infty}^1(\mathbf{R}^n)\right) : \|\phi\|_{\mathbf{X}} < \infty \right\},$$

where the norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0} \left(\|\phi(t)\|_{\mathbf{L}^1} + t^{\frac{n}{2}} \, \|\phi(t)\|_{\mathbf{L}^{\infty}} + t^{\frac{n+1}{2}} \, \|\nabla \phi(t)\|_{\mathbf{L}^{\infty}} \right).$$

Also we define the norm $\|\phi\|_{\mathbf{Q}} = \|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^{\infty}} + \|\nabla\phi\|_{\mathbf{L}^{\infty}}$, so that by a direct computation we obtain

$$\left\| t^{-\frac{n}{2}} \phi \left((\cdot)\, t^{-\frac{1}{2}} \right) \right\|_{\mathbf{X}} = \|\phi\|_{\mathbf{Q}} .$$

Note that the Green operator

$$\mathcal{G}_0(t)\,\phi = \mathcal{G}(t)\,\phi = \int_{\mathbf{R}^n} G(t, x-y)\,\phi(y)\,dy,$$

with the heat kernel

$$G(t,x) = (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{|x|^2}{4t}}$$

is the asymptotic self-similar operator in spaces $\mathbf{X}$, $\mathbf{Z}$ with asymptotic kernel

$$G_0(t) = (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{|x|^2}{4t}}$$

if we take $\delta = 2$, $\alpha = \frac{n}{2}$ and

$$f(\phi) = \int_{\mathbf{R}^n} \phi(x)\,dx.$$

Indeed in order to prove estimate (4.2) we write

$$\|G(t)\|_{\mathbf{L}^q} = Ct^{\frac{n}{2q}-\frac{n}{2}} \left(\int_{\mathbf{R}^n} e^{-|y|^2} dy \right)^{\frac{1}{q}} = Ct^{-\frac{n}{2}\left(1-\frac{1}{q}\right)},$$

and

$$\|\nabla G(t)\|_{\mathbf{L}^q} = Ct^{\frac{n}{2q}-\frac{n+1}{2}} \left(\int_{\mathbf{R}^n} e^{-|y|^2} dy \right)^{\frac{1}{q}} = Ct^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{q}\right)}$$

for all $t > 0$, where $1 \le q \le \infty$. Hence we see that the kernel $G \in \mathbf{X}$ and the norm

$$\begin{aligned} \|\mathcal{G}\phi\|_{\mathbf{X}} &= \sup_{t>0} \left(\|\mathcal{G}(t)\phi\|_{\mathbf{L}^1} + t^{\frac{n}{2}} \|\mathcal{G}(t)\phi\|_{\mathbf{L}^\infty} + t^{\frac{n+1}{2}} \|\nabla\mathcal{G}(t)\phi\|_{\mathbf{L}^\infty} \right) \\ &\le C\|\phi\|_{\mathbf{L}^1} \le C\|\phi\|_{\mathbf{Z}} . \end{aligned}$$

Thus estimate (4.2) is fulfilled. Equality (4.3) can be checked by changing the variables of integration $x = x' t^{\frac{1}{2}}$.

The nonlinear term $\mathcal{N}(u) = (\lambda \cdot \nabla)\,|u|^{\sigma}\, u$ in problem (4.11) has the form of the full derivative, so that it is convective $f(\mathcal{N}(u)) = 0$. We now check that it is critical, when $\sigma = \frac{1}{n}$. By changing the variables $\xi = xt^{-\frac{1}{2}}$, $\tau = zt$, $y = y' t^{\frac{1}{2}}$ we have

$$\begin{aligned} &\int_0^t \mathcal{G}(t-\tau)(x)\,\mathcal{N}\left(\tau^{-\frac{n}{2}} V\left((\cdot)\,\tau^{-\frac{1}{2}}\right)\right) d\tau \\ &= \int_0^t (t-\tau)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \widetilde{G_0}\left((x-y)(t-\tau)^{-\frac{1}{2}}\right) \mathcal{N}\left(\tau^{-\frac{n}{2}} V\left((\cdot)\,\tau^{-\frac{1}{2}}\right)\right) dy d\tau \\ &= t^{-\frac{n}{2}} \int_0^1 dz\,(1-z)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \widetilde{G_0}\left((\xi-y')(1-z)^{-\frac{1}{2}}\right) \mathcal{N}\left(z^{-\frac{n}{2}} V\left(y' z^{-\frac{1}{2}}\right)\right) dy' \\ &= t^{-\frac{n}{2}} \int_0^1 \mathcal{G}(1-z)\left(xt^{-\frac{1}{2}}\right) \mathcal{N}\left(z^{-\frac{n}{2}} V\left((\cdot)\,z^{-\frac{1}{2}}\right)\right) dz. \end{aligned}$$

Thus condition (4.4) is true, and, due to Definition 4.2, $\mathcal{N}$ is a critical convective nonlinearity.

Theorem 4.6. *Assume that the initial data* $u_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n)$, *with* $a \in (0,1)$, *and the norm* $\|u_0\|_{\mathbf{L}^{1,a}}$ *is sufficiently small. Then there exists a unique global solution* $u \in \mathbf{X}$ *to the Cauchy problem for the Burgers type equation (4.11), and this solution has asymptotics*

$$u(t) = t^{-\frac{n}{2}} V\left((\cdot)\, t^{-\frac{1}{2}}\right) + O\left(t^{-\frac{n}{2}-\gamma}\right)$$

for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}^n$, *where* $V \in \mathbf{L}^\infty(\mathbf{R}^n)$ *is the solution of the integral equation*

$$V(\xi) = \theta \widetilde{G}_0(\xi) - \int_0^1 \frac{dz}{z(1-z)^{\frac{n}{2}}} \int_{\mathbf{R}^n} \widetilde{G}_0\left(\frac{\xi - y z^{\frac{1}{2}}}{\sqrt{1-z}}\right) \mathcal{N}(V(y))\, dy. \tag{4.12}$$

Here $\widetilde{G}_0(\xi) = (4\pi)^{-\frac{n}{2}} e^{-\xi^2}$ *and a constant*

$$\theta = \int_{\mathbf{R}^n} u_0(x)\, dx.$$

Remark 4.7. The integral with respect to z near the origin in equation 4.12 is convergent by the fact that the nonlinearity $\mathcal{N}$ has the form of the full derivative, so that by integrating by parts we can see that the singularity of the integrand is like $z^{-\frac{1}{2}}$.

Before proving Theorem 4.6 we prepare the following lemma.

Lemma 4.8. *The estimate*

$$\left\| \langle t \rangle^\gamma \int_0^t \mathcal{G}(t-\tau)(\lambda \cdot \nabla)\phi(\tau) d\tau \right\|_{\mathbf{X}} \le C \left\| \sqrt{t}\, \langle t \rangle^\gamma \phi(t) \right\|_{\mathbf{X}}$$

is true, provided that the right-hand side is finite where $0 \le \gamma < \frac{1}{2}$.

Proof. Since $f(\phi) = 0$ by Lemma 1.28 with $\delta = \nu = 2$ we obtain

$$\left\| \partial_{x_j}^\beta \mathcal{G}(t)\, \phi \right\|_{\mathbf{L}^q} \le C t^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{\beta}{2}} \|\phi\|_{\mathbf{L}^r} \tag{4.13}$$

for all $t > 0$, where $1 \le r \le q \le \infty$, $\beta \ge 0$. Therefore we obtain the estimates

$$\begin{aligned} &\left\| \int_0^t \mathcal{G}(t-\tau)(\lambda \cdot \nabla)\phi(\tau) d\tau \right\|_{\mathbf{L}^1} \\ &\le \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \langle \tau \rangle^{-\gamma} d\tau \sup_{\tau>0} \sqrt{\tau}\, \langle \tau \rangle^\gamma \|\phi(\tau)\|_{\mathbf{L}^1} \\ &+ \int_{\frac{t}{2}}^t \tau^{-1} \langle \tau \rangle^{-\gamma} d\tau \sup_{\tau>0} \tau \langle \tau \rangle^\gamma \|\nabla \phi(\tau)\|_{\mathbf{L}^1} \\ &\le C \langle t \rangle^{-\gamma} \left\| \sqrt{t}\, \langle t \rangle^\gamma \phi(t) \right\|_{\mathbf{X}}, \end{aligned}$$

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}(t-\tau)\left(\lambda\cdot\nabla\right)\phi(\tau)d\tau \right\|_{\mathbf{L}^\infty} \\
&\le \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n+1}{2}} \tau^{-\frac{1}{2}} \langle \tau\rangle^{-\gamma} d\tau \sup_{\tau>0} \sqrt{\tau}\langle\tau\rangle^{\gamma} \|\phi(\tau)\|_{\mathbf{L}^1} \\
&+ \int_{\frac{t}{2}}^t \tau^{-1-\frac{n}{2}} \langle\tau\rangle^{-\gamma} d\tau \sup_{\tau>0} \tau^{1+\frac{n}{2}} \langle\tau\rangle^{\gamma} \|\nabla\phi(\tau)\|_{\mathbf{L}^\infty} \\
&\le C t^{-\frac{n}{2}} \langle t\rangle^{-\gamma} \left\| \sqrt{t}\langle t\rangle^{\gamma}\phi(t)\right\|_{\mathbf{X}},
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \nabla \int_0^t \mathcal{G}(t-\tau)\left(\lambda\cdot\nabla\right)\phi(\tau)d\tau \right\|_{\mathbf{L}^\infty} \\
&\le \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n+2}{2}} \tau^{-\frac{1}{2}} \langle \tau\rangle^{-\gamma} d\tau \sup_{\tau>0} \sqrt{\tau}\langle\tau\rangle^{\gamma} \|\phi(\tau)\|_{\mathbf{L}^1} \\
&+ \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}} \tau^{-1-\frac{n}{2}} \langle\tau\rangle^{-\gamma} d\tau \sup_{\tau>0} \tau^{1+\frac{n}{2}} \langle\tau\rangle^{\gamma} \|\nabla\phi(\tau)\|_{\mathbf{L}^\infty} \\
&\le C t^{-\frac{n+1}{2}} \langle t\rangle^{-\gamma} \left\| \sqrt{t}\langle t\rangle^{\gamma}\phi(t)\right\|_{\mathbf{X}}
\end{aligned}$$

for all $t > 0$. Hence the result of the lemma follows, and Lemma 4.8 is proved.

In view of the definition of norm $\mathbf{X}$ we obtain for $\sigma = \frac{1}{n}$

$$\begin{aligned}
&\left\| \sqrt{t}\langle t\rangle^{\gamma}\left(|v|^{\sigma} v(t) - |w|^{\sigma} w(t)\right)\right\|_{\mathbf{X}} \\
&\le C \left\|\langle t\rangle^{\gamma}\left(v(t)-w(t)\right)\right\|_{\mathbf{X}} \left(\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}\right)^{\sigma};
\end{aligned}$$

hence by taking $\phi = |v|^{\sigma} v - |w|^{\sigma} w$ in Lemma 4.8 we obtain estimates (4.5) and (4.7). Now existence of global solutions $u \in \mathbf{X}$ follows by application of Theorem 1.17. Thus we see that all the conditions of Theorem 4.4 are fulfilled, therefore we get the result of Theorem 4.6 with $\gamma = \frac{a}{2}$.

4.2 Whitham type equations

Consider the Cauchy problem for the Whitham type equations

$$\begin{cases} u_t + \mathcal{N}(u) + \mathcal{L}u = 0, \ x \in \mathbf{R}, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}, \end{cases} \tag{4.14}$$

where the linear part $\mathcal{L}$ is a pseudodifferential operator defined by the inverse Fourier transformation

$$\mathcal{L}u = \overline{\mathcal{F}}_{\xi\to x}\left(L(\xi)\,\widehat{u}(\xi)\right),$$

and the nonlinearity $\mathcal{N}(u)$ is a quadratic pseudodifferential operator defined by the symbols $a(t,\xi,y)$ written as

$$\mathcal{N}(u) = \overline{\mathcal{F}}_{\xi\to x} \int_{\mathbf{R}} a(t,\xi,y)\,\widehat{u}(t,\xi-y)\,\widehat{u}(t,y)\,dy.$$

We suppose that the symbols $a(t,\xi,y)$ are continuous functions with respect to time $t>0$ and the operators $\mathcal{N}$ and $\mathcal{L}$ have a finite order, that is the symbols $a(t,\xi,y)$ and $L(\xi)$ grow with respect to y and ξ no faster than a power of some order κ

$$|a(t,\xi,y)| \le C\left(\langle\xi\rangle^{\kappa} + \langle y\rangle^{\kappa}\right), |L(\xi)| \le C\langle\xi\rangle^{\kappa},$$

where $C>0$.

Model equation (4.14) combines many well-known equations of modern mathematical physics and describes various wave processes in different media. A particular case of (4.14) is the Whitham equation (see Whitham [1999]) $\left(\text{if we choose } a(t,\xi,y) = \frac{i\xi}{2}\right)$

$$u_t + uu_x + \mathcal{L}u = 0, \tag{4.15}$$

which contains many famous equations, such as Korteweg-de Vries, Burgers, Benjamin-Ono. When $\mathcal{N}(u) = uu_x$, $\mathcal{L}u = -u_{xx}$, that is $a(t,\xi,y) = \frac{i\xi}{2}$ and $L(\xi) = |\xi|^2$, equation (4.14) transforms into the Burgers equation Burgers [1948]

$$u_t + uu_x - u_{xx} = 0,\ x\in\mathbf{R},\ t>0, \tag{4.16}$$

which can be solved via the Hopf-Cole Hopf [1950] transformation $u = -2\partial_x \log(\phi)$, where

$$\phi(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}} e^{-\frac{(x-y)^2}{4t}} \exp\left(-\frac{1}{2}\int_{-\infty}^{y} u_0(z)\,dz\right) dy$$

satisfies the heat equation $\phi_t = \phi_{xx}$. A famous Korteweg–de Vries-Burgers equation

$$u_t + uu_x - u_{xx} + u_{xxx} = 0,\ x\in\mathbf{R},\ t>0 \tag{4.17}$$

is a particular case of (4.14) with $a(t,\xi,y) = \frac{i\xi}{2}$, $L(\xi) = |\xi|^2 - i\xi^3$. If we take

$$a(t,\xi,y) = \frac{i\xi}{2}\left(1+|\xi|^2\right)^{-1}, L(\xi) = \left(1+|\xi|^2\right)^{-1}\left(|\xi|^2 - i\xi^3\right)$$

in equation (4.14) then the Bengamin - Bona - Mahony - Peregrine - Burgers (BBMPB) equation (see Benjamin et al. [1972], Burgers [1948], Peregrine [1966]) follows

$$u_t + uu_x - u_{xx} + u_{xxx} - u_{xxt} = 0,\ x\in\mathbf{R},\ t>0. \tag{4.18}$$

Another example is the Benjamin-Ono-Burgers equation

$$u_t + uu_x - u_{xx} - \mathcal{H}(u_{xx}) = 0,\ x \in \mathbf{R},\, t > 0, \tag{4.19}$$

where

$$\mathcal{H}(\phi) = \frac{1}{\pi}\mathrm{PV}\int \frac{\phi(y)}{x-y}dy$$

is the Hilbert transformation. This equation follows from (4.14) if we choose $a(t,\xi,y) = \frac{i\xi}{2}$, $L(\xi) = |\xi|^2 + i|\xi|^2\mathrm{sign}\xi$.

We suppose that the symbol of the nonlinear operator $\mathcal{N}$ is such that

$$|a(t,\xi,y)| \le C\langle\xi\rangle^{\tilde{\theta}}\{\xi\}^{\omega}\left(\langle\xi-y\rangle^{\sigma}\{\xi-y\}^{\alpha} + \langle y\rangle^{\sigma}\{y\}^{\alpha}\right) \tag{4.20}$$

for all $\xi \in \mathbf{R}$, $y \in \mathbf{R}$, $t > 0$, where $\tilde{\theta}$, σ, $\alpha \ge 0$. We consider the case of nonlinearity of the type of the full derivative, that is we suppose that $\omega > 0$. We will show that this type of nonlinearity behaves asymptotically as a convective one. We consider the case of nonzero total mass of the initial data $\int_{\mathbf{R}} u_0(x)\,dx \equiv \theta \ne 0$.

Let the linear operator $\mathcal{L}$ satisfy the dissipation condition which in terms of $L(\xi)$ has the form

$$\mathrm{Re}\,L(\xi) \ge \mu\{\xi\}^{\delta}\langle\xi\rangle^{\nu} \tag{4.21}$$

for all $\xi \in \mathbf{R}$, where $\mu > 0$, $\nu \ge 0$. We study the large time asymptotic behavior of solutions to the Cauchy problem for nonlinear evolution equation (4.14) in the critical case. The critical case with respect to the large time asymptotic behavior of solutions means that the decay rates of the linear and nonlinear parts of the equation are balanced:

$$\delta = 1 + \alpha + \omega.$$

Define $\mathbf{X} = \{\phi \in \mathcal{S}' : \|\phi\|_{\mathbf{X}} < \infty\}$, where the norm

$$\begin{aligned}\|\phi\|_{\mathbf{X}} = &\sup_{\rho\in[0,\alpha+\gamma]}\ \sup_{1\le p\le\infty}\ \sup_{t>0}\langle t\rangle^{\frac{\rho}{\delta}+\frac{1}{\delta p}}\|\phi(t)\|_{\mathbf{A}^{\rho,p}}\\ &+\sup_{s\in[0,\sigma]}\ \sup_{1\le p\le\infty}\ \sup_{t>0}\{t\}^{\frac{s}{\nu}+\frac{b}{\nu p}}\langle t\rangle^{\frac{1+\alpha+\gamma}{\delta}+\frac{1}{\delta p}}\|\phi(t)\|_{\mathbf{B}^{s,p}},\end{aligned}$$

and $\gamma \in (0,\min(1,\omega))$, $b \in [0,1]$

$$\begin{aligned}\|\varphi(t)\|_{\mathbf{A}^{\rho,p}} &= \left\||\cdot|^{\rho}\widehat{\varphi}(t,\cdot)\right\|_{\mathbf{L}^p_{\xi}(|\xi|\le1)},\\ \|\varphi(t)\|_{\mathbf{B}^{s,p}} &= \left\||\cdot|^{s}\widehat{\varphi}(t,\cdot)\right\|_{\mathbf{L}^p_{\xi}(|\xi|\ge1)};\end{aligned}$$

the norm $\mathbf{A}^{\rho,p}$ is responsible for the large time asymptotic properties of solutions, and the norm $\mathbf{B}^{s,p}$ describes the regularity of solutions.

Theorem 4.9. *Let the linear operator $\mathcal{L}$ satisfy conditions (4.21) with $\delta = 1 + \alpha + \omega$, $\alpha \ge 0$, $\omega > 0$. Suppose that the nonlinear operator $\mathcal{N}$ satisfies estimates (4.20) with $\tilde{\theta} + \sigma \in [0,\nu)$ if $\nu > 0$ or $\sigma = 0 = \theta$ if $\nu = 0$. Let the initial data u_0 be such that*

$$\|u_0\|_{\mathbf{A}^{0,\infty}} + \|u_0\|_{\mathbf{B}^{0,\frac{1}{1-b}}} \leq \varepsilon,$$

where $\varepsilon > 0$ is sufficiently small. The value $b \in [0,1]$ is such that $b = 1$ if $\nu \geq \sigma + \tilde{\theta} + 1$, $b < 1 - \sqrt{1 + \sigma + \theta - \nu}$ if $\nu \in (0, \sigma + \theta + 1)$ and $b = 0$ if $\nu = 0$. Then there exists a unique solution $u \in \mathbf{X}$ of the Cauchy problem (4.14). Moreover, the solutions $u(t,x)$ have the time decay estimates

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C \{t\}^{-\frac{b}{\nu}} \langle t \rangle^{-\frac{1}{\delta}}$$

for all $t > 0$.

Remark 4.10. Note that in the case of zero total mass of the initial data $\theta = \int_{\mathbf{R}} u_0(x)\,dx = 0$ the solutions of the Cauchy problem for equation (4.14) obtain more rapid time decay rate

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C t^{-\frac{1+\min(1,\omega)}{\delta}}.$$

Thus the critical value is shifted $\delta_c = 1 + \alpha + \omega + \min(1,\omega)$ in this case.

4.2.1 Preliminary Lemmas

The Green operator $\mathcal{G}$ is given by

$$\mathcal{G}(t)\phi = \overline{\mathcal{F}}_{\xi \to x}\left(e^{-L(\xi)t}\hat{\phi}(\xi)\right).$$

Define

$$\langle \partial_x \rangle^\alpha \phi = \overline{\mathcal{F}}_{\xi \to x}\left(\langle \xi \rangle^\alpha \hat{\phi}(\xi)\right)$$

and

$$\{\partial_x\}^\alpha \phi = \overline{\mathcal{F}}_{\xi \to x}\left(\{\xi\}^\alpha \hat{\phi}(\xi)\right)$$

for $\alpha \in \mathbf{R}$. In the next lemma we estimate the Green operator $\mathcal{G}(t)$ in the norms $\mathbf{A}^{\rho,p}$ and $\mathbf{B}^{s,p}$ for $s, \rho \in \mathbf{R}$, $1 \leq p \leq \infty$.

Lemma 4.11. *Let the linear operator $\mathcal{L}$ satisfy the dissipation condition (4.21). Then the following estimates are valid for all $t > 0$*

$$\left\| \int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau \right\|_{\mathbf{A}^{\rho,p}}$$
$$\leq C \langle t \rangle^{1-\lambda-\frac{\rho+\omega}{\delta}-\frac{1}{\delta p}} \sup_{p \leq q \leq \infty} \sup_{\tau > 0} \left(\{\tau\}^\kappa \langle \tau \rangle^{\lambda + \frac{1}{\delta q}} \|\psi(\tau)\|_{\mathbf{A}^{-\omega,q}} \right)$$

where $\kappa < 1$, $\lambda < 1$, $\rho + \omega < \delta$; and

$$\left\| \int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau \right\|_{\mathbf{B}^{s,p}} \leq C \{t\}^{1-\kappa-\frac{s+\theta}{\nu}-\frac{b}{\nu r}-\frac{1}{\nu}\left(\frac{1}{p}-\frac{1}{r}\right)} \langle t \rangle^{-\widetilde{\lambda}-\frac{1}{\delta p}}$$
$$\times \sup_{p \leq q \leq \infty} \sup_{\tau > 0} \left(\{\tau\}^{\kappa + \frac{b}{\nu q}} \langle \tau \rangle^{\widetilde{\lambda} + \frac{1}{\delta q}} \|\psi(\tau)\|_{\mathbf{B}^{-\theta,q}} \right)$$

where $b \in [0,1]$, $\tilde{\lambda} \geq 0$. *Here* $r \in [p, \infty]$ *is such that* $\kappa + \frac{b}{\nu r} < 1$, *and* $s \geq 0$, $\theta \geq 0$ *are such that* $s + \theta < \nu$ *if* $\nu > 0$. *In the case of* $\nu = 0$ *we take* $b = 0$, $r = p$, $s = 0$, $\theta = 0$.

Proof. By virtue of dissipation condition (4.21) and by the result of Lemma 1.38 we have the following estimate

$$\|\mathcal{G}(t)\psi(\tau)\|_{\mathbf{A}^{\rho,p}} \leq C\langle t\rangle^{-\frac{\rho}{\delta}-\frac{1}{\delta}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\psi(\tau)\|_{\mathbf{A}^{0,q}} \tag{4.22}$$

for $\rho \geq 0$, $1 \leq p \leq q \leq \infty$. In view of (4.22) with $q = \infty$ and $q = p$ we get

$$\begin{aligned}
&\left\|\int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau\right\|_{\mathbf{A}^{\rho,p}} \leq C\left\|\{\partial_x\}^{-\omega}\int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau\right\|_{\mathbf{A}^{\rho+\omega,p}} \\
&\leq C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{\rho+\omega}{\delta}-\frac{1}{\delta p}}\left\|\{\partial_x\}^{-\omega}\psi(\tau)\right\|_{\mathbf{A}^{0,\infty}} d\tau \\
&+C\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\frac{\rho+\omega}{\delta}}\left\|\{\partial_x\}^{-\omega}\psi(\tau)\right\|_{\mathbf{A}^{0,p}} d\tau \\
&\leq C\sup_{p\leq q\leq\infty}\sup_{\tau>0}\left(\{\tau\}^{\kappa}\langle\tau\rangle^{\lambda+\frac{1}{\delta q}}\|\psi(\tau)\|_{\mathbf{A}^{-\omega,q}}\right) \\
&\times\left(\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-\frac{\rho+\omega}{\delta}-\frac{1}{\delta p}}\{\tau\}^{-\kappa}\langle\tau\rangle^{-\lambda}\,d\tau\right. \\
&\left.+\int_{\frac{t}{2}}^t\langle t-\tau\rangle^{-\frac{\rho+\omega}{\delta}}\{\tau\}^{-\kappa}\langle\tau\rangle^{-\lambda-\frac{1}{\delta p}}\,d\tau\right) \\
&\leq C\langle t\rangle^{1-\lambda-\frac{\rho+\omega}{\delta}-\frac{1}{\delta p}}\sup_{p\leq q\leq\infty}\sup_{\tau>0}\left(\{\tau\}^{\kappa}\langle\tau\rangle^{\lambda+\frac{1}{\delta q}}\|\psi(\tau)\|_{\mathbf{A}^{-\omega,q}}\right)
\end{aligned}$$

for all $t > 0$, where $\kappa < 1$, $\lambda < 1$ and $\rho + \omega < \delta$. Thus the first estimate of the lemma is true.

Similarly, from condition (4.21) we find

$$\|\mathcal{G}(t)\psi(\tau)\|_{\mathbf{B}^{s,p}} \leq Ce^{-\frac{\mu}{2}t}\{t\}^{-\frac{s}{\nu}-\frac{1}{\nu}\left(\frac{1}{p}-\frac{1}{r}\right)}\|\psi(\tau)\|_{\mathbf{B}^{0,r}}, \tag{4.23}$$

where $s \geq 0$, $p \leq r \leq \infty$ if $\nu > 0$ and $s = 0$, $r = p$ if $\nu = 0$. Then by (4.23) we have

$$\begin{aligned}
&\left\|\int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau\right\|_{\mathbf{B}^{s,p}} \leq C\left\|\langle\partial_x\rangle^{-\theta}\int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau\right\|_{\mathbf{B}^{s+\theta,p}} \\
&\leq C\int_0^{\frac{t}{2}} e^{-\frac{\mu}{2}(t-\tau)}\{t-\tau\}^{-\frac{s+\theta}{\nu}-\frac{1}{\nu}\left(\frac{1}{p}-\frac{1}{r}\right)}\left\|\langle\partial_x\rangle^{-\theta}\psi(\tau)\right\|_{\mathbf{B}^{0,r}} d\tau \\
&+C\int_{\frac{t}{2}}^t e^{-\frac{\mu}{2}(t-\tau)}\{t-\tau\}^{-\frac{s+\theta}{\nu}}\left\|\langle\partial_x\rangle^{-\theta}\psi(\tau)\right\|_{\mathbf{B}^{0,p}} d\tau;
\end{aligned}$$

hence, we obtain

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau \right\|_{\mathbf{B}^{s,p}} \\
&\le C\int_0^{\frac{t}{2}} e^{-\frac{\mu}{2}(t-\tau)} \{t-\tau\}^{-\frac{s+\theta}{\nu}-\frac{1}{\nu}\left(\frac{1}{p}-\frac{1}{r}\right)} \{\tau\}^{-\kappa-\frac{b}{\nu r}}\,d\tau \\
&\times \sup_{\tau>0}\left(\{\tau\}^{\kappa+\frac{b}{\nu r}} \|\psi(\tau)\|_{\mathbf{B}^{-\theta,r}} \right) \\
&+ C\int_{\frac{t}{2}}^t e^{-\frac{\mu}{2}(t-\tau)} \{t-\tau\}^{-\frac{s+\theta}{\nu}} \{\tau\}^{-\kappa-\frac{b}{\nu p}} \langle\tau\rangle^{-\widetilde{\lambda}-\frac{1}{\delta p}}\,d\tau \\
&\times \sup_{\tau>0}\left(\{\tau\}^{\kappa+\frac{b}{\nu p}} \langle\tau\rangle^{\widetilde{\lambda}+\frac{1}{\delta p}} \|\psi(\tau)\|_{\mathbf{B}^{-\theta,p}} \right) \\
&\le C\{t\}^{1-\kappa-\frac{s+\theta}{\nu}-\frac{b}{\nu r}-\frac{1}{\nu}\left(\frac{1}{p}-\frac{1}{r}\right)} \langle t\rangle^{-\widetilde{\lambda}-\frac{1}{\delta p}} \\
&\times \sup_{p\le q\le\infty}\sup_{\tau>0}\left(\{\tau\}^{\kappa+\frac{b}{\nu q}} \langle\tau\rangle^{\widetilde{\lambda}+\frac{1}{\delta q}} \|\psi(\tau)\|_{\mathbf{B}^{-\theta,q}} \right).
\end{aligned}$$

Since $\kappa+\frac{b}{\nu r}<1$ we have

$$\begin{aligned}
&\int_0^{\frac{t}{2}} e^{-\frac{\mu}{2}(t-\tau)} \{t-\tau\}^{-\frac{s+\theta}{\nu}-\frac{1}{\nu}\left(\frac{1}{p}-\frac{1}{r}\right)} \{\tau\}^{-\kappa-\frac{b}{\nu r}}\,d\tau \\
&\le Ce^{-Ct}\{t\}^{1-\kappa-\frac{s+\theta}{\nu}-\frac{b}{\nu r}-\frac{1}{\nu}\left(\frac{1}{p}-\frac{1}{r}\right)}
\end{aligned}$$

and since $s+\theta<\nu$ and $-\frac{b}{\nu p}\ge -\frac{b}{\nu r}-\frac{1}{\nu}\left(\frac{1}{p}-\frac{1}{r}\right)$ we get

$$\begin{aligned}
&\int_{\frac{t}{2}}^t e^{-\frac{\mu}{2}(t-\tau)} \{t-\tau\}^{-\frac{s+\theta}{\nu}} \{\tau\}^{-\kappa-\frac{b}{\nu p}} \langle\tau\rangle^{-\widetilde{\lambda}-\frac{1}{\delta p}}\,d\tau \\
&\le C\{t\}^{1-\kappa-\frac{s+\theta}{\nu}-\frac{b}{\nu p}} \langle t\rangle^{-\widetilde{\lambda}-\frac{1}{\delta p}} \le C\{t\}^{1-\kappa-\frac{s+\theta}{\nu}-\frac{b}{\nu r}-\frac{1}{\nu}\left(\frac{1}{p}-\frac{1}{r}\right)} \langle t\rangle^{-\widetilde{\lambda}-\frac{1}{\delta p}}
\end{aligned}$$

for all $t>0$. Thus the second estimate of the lemma follows, and Lemma 4.11 is proved.

Now we estimate the nonlinearity

$$\mathcal{N}(\varphi,\phi) = \overline{\mathcal{F}}_{\xi\to x}\int_{\mathbf{R}} a(t,\xi,y)\,\widehat{\varphi}(t,\xi-y)\,\widehat{\phi}(t,y)\,dy,$$

in the norms $\|\cdot\|_{\mathbf{A}^{-\omega,p}}$ and $\|\cdot\|_{\mathbf{B}^{-\theta,p}}$.

Lemma 4.12. *Let the nonlinear operator $\mathcal{N}$ satisfy the condition (4.20). Then the inequalities*

$$\begin{aligned}
&\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{A}^{-\omega,p}} \le C\left(\|\varphi\|_{\mathbf{A}^{\alpha,1}}+\|\varphi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}}+\|\phi\|_{\mathbf{B}^{0,\infty}}\right) \\
&+ C\left(\|\phi\|_{\mathbf{A}^{\alpha,1}}+\|\phi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}}+\|\varphi\|_{\mathbf{B}^{0,\infty}}\right)
\end{aligned}$$

and

$$\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{B}^{-\theta,p}} \le C\left(\|\varphi\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|\varphi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}} + \|\phi\|_{\mathbf{B}^{0,p}}\right)$$
$$+ C\left(\|\phi\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|\phi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}} + \|\varphi\|_{\mathbf{B}^{0,p}}\right)$$

are valid for $1 \le p \le \infty$, provided that the right-hand sides are bounded.

Proof. By virtue of condition (4.20) and by the Young inequality, we obtain

$$\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{A}^{-\omega,p}} \le \left\| \int_{\mathbf{R}} |\cdot|^{-\omega} |a(t,\cdot,y)| \left|\widehat{\varphi}(t,\cdot-y)\widehat{\phi}(t,y)\right| dy \right\|_{\mathbf{L}^p_\xi(|\xi|\le 1)}$$
$$\le C \left\| \int_{\mathbf{R}} \left(\langle\cdot-y\rangle^{\sigma}\{\cdot-y\}^{\alpha} + \langle y\rangle^{\sigma}\{y\}^{\alpha}\right) \left|\widehat{\varphi}(t,\cdot-y)\widehat{\phi}(t,y)\right| dy \right\|_{\mathbf{L}^p_\xi(|\xi|\le 1)}$$
$$\le C \left\|\langle\cdot\rangle^{\sigma}\{\cdot\}^{\alpha}\widehat{\varphi}\right\|_{\mathbf{L}^1_\xi} \left(\left\|\widehat{\phi}\right\|_{\mathbf{L}^p_\xi(|\xi|\le 1)} + \left\|\widehat{\phi}\right\|_{\mathbf{L}^\infty_\xi(|\xi|>1)} \right)$$
$$+ C \left\|\langle\cdot\rangle^{\sigma}\{\cdot\}^{\alpha}\widehat{\phi}\right\|_{\mathbf{L}^1_\xi} \left(\|\widehat{\varphi}\|_{\mathbf{L}^p_\xi(|\xi|\le 1)} + \|\widehat{\varphi}\|_{\mathbf{L}^\infty_\xi(|\xi|>1)} \right)$$
$$\le C\left(\|\varphi\|_{\mathbf{A}^{\alpha,1}} + \|\varphi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}} + \|\phi\|_{\mathbf{B}^{0,\infty}}\right)$$
$$+ C\left(\|\phi\|_{\mathbf{A}^{\alpha,1}} + \|\phi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}} + \|\varphi\|_{\mathbf{B}^{0,\infty}}\right),$$

and

$$\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{B}^{-\theta,p}} \le \left\| \int_{\mathbf{R}} |\cdot|^{-\theta}\{\cdot\}^{\gamma} |a(t,\cdot,y)| \left|\widehat{\varphi}(t,\cdot-y)\widehat{\phi}(t,y)\right| dy \right\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)}$$
$$\le C \left\| \int_{\mathbf{R}} \left(\langle\cdot-y\rangle^{\sigma}\{\cdot-y\}^{\alpha+\gamma} + \langle y\rangle^{\sigma}\{y\}^{\alpha+\gamma}\right)\right.$$
$$\times \left.\left|\widehat{\varphi}(t,\cdot-y)\widehat{\phi}(t,y)\right| dy \right\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)}$$
$$\le C \left\|\langle\cdot\rangle^{\sigma}\{\cdot\}^{\alpha+\gamma}\widehat{\varphi}\right\|_{\mathbf{L}^1_\xi} \left\|\widehat{\phi}\right\|_{\mathbf{L}^p_\xi} + C\|\widehat{\varphi}\|_{\mathbf{L}^p_\xi} \left\|\langle\cdot\rangle^{\sigma}\{\cdot\}^{\alpha+\gamma}\widehat{\phi}\right\|_{\mathbf{L}^1_\xi}$$
$$\le C\left(\|\varphi\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|\varphi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}} + \|\phi\|_{\mathbf{B}^{0,p}}\right)$$
$$+ C\left(\|\phi\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|\phi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}} + \|\varphi\|_{\mathbf{B}^{0,p}}\right).$$

Therefore the estimates of the lemma follow, and Lemma 4.12 is proved.

4.2.2 Proof of Theorem 4.9

Denote $\mathbf{X} = \{\phi \in \mathcal{S}' : \|\phi\|_{\mathbf{X}} < \infty\}$, where the norm

$$\|\phi\|_{\mathbf{X}} = \sup_{\rho\in[0,\alpha+\gamma]} \sup_{1\le p\le\infty} \sup_{t>0} \langle t\rangle^{\frac{\rho}{\delta}+\frac{1}{\delta p}} \|\phi(t)\|_{\mathbf{A}^{\rho,p}}$$
$$+ \sup_{s\in[0,\sigma]} \sup_{1\le p\le\infty} \sup_{t>0} \{t\}^{\frac{s}{\nu}+\frac{b}{\nu p}} \langle t\rangle^{\frac{1+\alpha+\gamma}{\delta}+\frac{1}{\delta p}} \|\phi(t)\|_{\mathbf{B}^{s,p}},$$

where $b \in [0,1]$ is such that $b = 1$ if $\nu \geq 1 + \sigma + \theta$, $b < 1 - \sqrt{1 + \sigma + \theta - \nu}$ if $\nu \in (0, 1 + \sigma + \theta)$ and $b = 0$ if $\nu = 0$. Using estimates (4.22) and (4.23) we get

$$\|\mathcal{G}(t) u_0\|_{\mathbf{X}} \leq C \|u_0\|_{\mathbf{A}^{0,\infty}} + C \|u_0\|_{\mathbf{B}^{0,\frac{p}{1-b}}}.$$

Denote also

$$\|\psi\|_{\mathbf{Y}} = \sup_{1 \leq p \leq \infty} \sup_{t>0} \{t\}^{\kappa} \langle t \rangle^{\lambda + \frac{1}{\delta p}} \left(\|\psi(t)\|_{\mathbf{A}^{-\omega,p}} + \{t\}^{\frac{b}{\nu p}} \langle t \rangle^{\frac{\gamma}{\delta}} \|\psi(t)\|_{\mathbf{B}^{-\theta,p}} \right),$$

where $\kappa = 0$ if $\nu = 0$, and $\kappa = \frac{\sigma + b}{\nu} < 1$ if $\nu > 0$; $\lambda = 1 - \frac{\omega}{\delta} < 1$. Then by virtue of Lemma 4.11 we have for $\rho \in [0, \alpha + \gamma]$, $1 \leq p \leq \infty$

$$\begin{aligned} &\langle t \rangle^{\frac{\rho}{\delta} + \frac{1}{\delta p}} \left\| \int_0^t \mathcal{G}(t-\tau) \mathcal{N}(v,v)(\tau) \, d\tau \right\|_{\mathbf{A}^{\rho,p}} \\ &\leq C \sup_{p \leq q \leq \infty} \sup_{\tau > 0} \left(\{\tau\}^{\kappa} \langle \tau \rangle^{\lambda + \frac{1}{\delta q}} \|\mathcal{N}(v,v)(\tau)\|_{\mathbf{A}^{-\omega,q}} \right) \leq C \|\mathcal{N}(v,v)\|_{\mathbf{Y}} \end{aligned}$$

and for $s \in [0, \sigma]$, $1 \leq p \leq \infty$ taking $\widetilde{\lambda} = 1 - \frac{\omega - \gamma}{\delta} = \frac{1 + \alpha + \gamma}{\delta}$

$$\begin{aligned} &\{t\}^{\frac{s}{\nu} + \frac{b}{\nu p}} \langle t \rangle^{\frac{1+\alpha+\gamma}{\delta} + \frac{1}{\delta p}} \left\| \int_0^t \mathcal{G}(t-\tau) \mathcal{N}(v,v)(\tau) \, d\tau \right\|_{\mathbf{B}^{s,p}} \\ &\leq C \{t\}^{1 - \kappa - \frac{\theta}{\nu} - \frac{1-b}{\nu}\left(\frac{1}{p} - \frac{1}{r}\right)} \\ &\times \sup_{p \leq q \leq \infty} \sup_{\tau > 0} \left(\{\tau\}^{\kappa + \frac{b}{\nu q}} \langle \tau \rangle^{\widetilde{\lambda} + \frac{1}{\delta q}} \|\mathcal{N}(v,v)(\tau)\|_{\mathbf{B}^{-\theta,q}} \right) \\ &\leq C \|\mathcal{N}(v,v)\|_{\mathbf{Y}}, \end{aligned}$$

for all $t > 0$. Here we suppose that $1 - \kappa - \frac{\theta}{\nu} - \frac{1-b}{\nu}\left(1 - \frac{1}{r}\right) \geq 0$. By conditions of Lemma 4.11 it follows that $b \in [0,1]$, $r \in [p, \infty]$ satisfy $\kappa + \frac{b}{\nu r} < 1$, and $s \geq 0$, $\theta \geq 0$ satisfy $s + \theta < \nu$ if $\nu > 0$. In the case of $\nu = 0$ we take $b = 0$, $r = p$, $s = 0$ and $\theta = 0$. Solving the conditions for b, r

$$\nu - \sigma - b - \theta - (1-b)\left(1 - \frac{1}{r}\right) \geq 0, \; \sigma + b + \frac{b}{r} < \nu$$

we see that we can take $b = 1$ and $r = \infty$ if $\nu \geq 1 + \sigma - \theta$ and we can choose $b = \sigma = \theta = 0$, $r = 1$ if $\nu = 0$. Otherwise we can take $0 \leq b < 1 - \sqrt{1 + \sigma + \theta - \nu}$ and $r = (1 + \sigma + \theta - \nu)^{-\frac{1}{2}}$ if $0 < \nu < 1 + \sigma + \theta$.

Therefore

$$\left\| \int_0^t \mathcal{G}(t-\tau) \mathcal{N}(v)(\tau) \, d\tau \right\|_{\mathbf{X}} \leq C \|\mathcal{N}(v)\|_{\mathbf{Y}}.$$

Now by Lemma 4.12 we find

$$\sup_{1\le p\le\infty}\sup_{t>0}\{t\}^{\kappa}\langle t\rangle^{\lambda+\frac{1}{\delta p}}\|\mathcal{N}(v)(t)\|_{\mathbf{A}^{-\omega,p}}$$
$$\le C\sup_{1\le p\le\infty}\sup_{t>0}\{t\}^{\kappa}\langle t\rangle^{\lambda}\left(\|v(t)\|_{\mathbf{A}^{\alpha,1}}+\|v(t)\|_{\mathbf{B}^{\sigma,1}}\right)$$
$$\times\langle t\rangle^{\frac{1}{\delta p}}\left(\|v(t)\|_{\mathbf{A}^{0,p}}+\|v(t)\|_{\mathbf{B}^{0,\infty}}\right)$$

and

$$\sup_{1\le p\le\infty}\sup_{t>0}\{t\}^{\kappa+\frac{b}{\nu p}}\langle t\rangle^{\widetilde{\lambda}+\frac{1}{\delta p}}\|\mathcal{N}(v)(t)\|_{\mathbf{B}^{-\theta,p}}$$
$$\le C\sup_{1\le p\le\infty}\sup_{t>0}\{t\}^{\kappa}\langle t\rangle^{\widetilde{\lambda}}\left(\|v(t)\|_{\mathbf{A}^{\alpha+\gamma,1}}+\|v(t)\|_{\mathbf{B}^{\sigma,1}}\right)$$
$$\times\{t\}^{\frac{b}{\nu p}}\langle t\rangle^{\frac{1}{\delta p}}\left(\|v(t)\|_{\mathbf{A}^{0,p}}+\|v(t)\|_{\mathbf{B}^{0,p}}\right);$$

hence

$$\|\mathcal{N}(v)\|_{\mathbf{Y}}=\sup_{1\le p\le\infty}\sup_{t>0}\{t\}^{\kappa}\langle t\rangle^{\lambda+\frac{1}{\delta p}}\|\mathcal{N}(v)(t)\|_{\mathbf{A}^{-\omega,p}}$$
$$+\sup_{1\le p\le\infty}\sup_{t>0}\{t\}^{\kappa+\frac{b}{\nu p}}\langle t\rangle^{\widetilde{\lambda}+\frac{1}{\delta p}}\|\mathcal{N}(v)(t)\|_{\mathbf{B}^{-\theta,p}}$$
$$\le C\sup_{1\le p\le\infty}\sup_{t>0}\Bigg(\langle t\rangle^{\lambda}\|v(t)\|_{\mathbf{A}^{\alpha,1}}+\langle t\rangle^{\widetilde{\lambda}}\|v(t)\|_{\mathbf{A}^{\alpha+\gamma,1}}$$
$$+\{t\}^{\kappa}\langle t\rangle^{\widetilde{\lambda}}\|v(t)\|_{\mathbf{B}^{\sigma,1}}\Bigg)$$
$$\times\langle t\rangle^{\frac{1}{\delta p}}\left(\|v(t)\|_{\mathbf{A}^{0,p}}+\{t\}^{\frac{b}{\nu p}}\|v(t)\|_{\mathbf{B}^{0,p}}+\|v(t)\|_{\mathbf{B}^{0,\infty}}\right).$$

Therefore

$$\|\mathcal{N}(v)\|_{\mathbf{Y}}\le C\sup_{t>0}\Bigg(\langle t\rangle^{\frac{1+\alpha}{\delta}}\|v(t)\|_{\mathbf{A}^{\alpha,1}}+\langle t\rangle^{\frac{1+\alpha+\gamma}{\delta}}\|v(t)\|_{\mathbf{A}^{\alpha+\gamma,1}}$$
$$+\{t\}^{\kappa}\langle t\rangle^{\frac{2+\alpha+\gamma}{\delta}}\|v(t)\|_{\mathbf{B}^{\sigma,1}}\Bigg)$$
$$\times\sup_{1\le p\le\infty}\sup_{t>0}\left(\langle t\rangle^{\frac{1}{\delta p}}\|v(t)\|_{\mathbf{A}^{0,p}}+\{t\}^{\frac{b}{\nu p}}\langle t\rangle^{\widetilde{\lambda}+\frac{1}{\delta p}}\|v(t)\|_{\mathbf{B}^{0,p}}\right)\le C\|v\|_{\mathbf{X}}^{2},$$

since we have chosen $\kappa=\frac{\sigma+b}{\nu}$, $\lambda=1-\frac{\omega}{\delta}=\frac{\alpha+1}{\delta}$, $\omega>0$.

Thus we have

$$\left\|\int_0^t\mathcal{G}(t-\tau)\mathcal{N}(v)(\tau)\,d\tau\right\|_{\mathbf{X}}\le C\|v\|_{\mathbf{X}}^{2}.$$

In the same manner we prove estimate

$$\left\|\int_0^t\mathcal{G}(t-\tau)\left(\mathcal{N}(v_1)(\tau)-\mathcal{N}(v_2)(\tau)\right)d\tau\right\|_{\mathbf{X}}\le\frac{1}{2}\|v_1-v_2\|_{\mathbf{X}}. \qquad (4.24)$$

Therefore, as in the proof of Theorem 4.4, by applying the contraction mapping in $\mathbf{X}$ we prove the existence of a unique global solution $u(t,x) \in \mathbf{X}$ to the Cauchy problem (4.14). Since

$$\|u(t)\|_{\mathbf{L}^\infty} \le \|u(t)\|_{\mathbf{A}^{0,1}} + \|u(t)\|_{\mathbf{B}^{0,1}}$$

and $u(t,x) \in \mathbf{X}$ we have the estimate

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\{t\}^{-\frac{b}{\nu}} \langle t\rangle^{-\frac{1}{\delta}}$$

for all $t > 0$. Thus the result of the theorem is true, and Theorem 4.9 is proved.

4.2.3 Self-similar solutions

Along with the Cauchy problem for equation (4.14) we consider also the Cauchy problem

$$\begin{cases} w_t + \mathcal{N}_0(w) + \mathcal{L}_0 w = 0, \ x \in \mathbf{R}, \ t > 0, \\ w(0,x) = u_0(x), x \in \mathbf{R}, \end{cases} \tag{4.25}$$

where $\mathcal{L}_0$ has a symbol $L_0(\xi)$, which is a homogeneous function of order δ, that is $L_0(t\xi) = t^\delta L_0(\xi)$ for all $\xi \in \mathbf{R}$, $t > 0$. We assume that the asymptotic representation at the origin

$$L(\xi) = L_0(\xi) + O\left(|\xi|^{\delta+\gamma}\right) \tag{4.26}$$

is fulfilled for all $|\xi| \le 1$, where $\gamma > 0$. Let $\mathcal{N}_0$ have a symbol $a_0(\xi,y)$ which is homogeneous with respect to ξ and y of order $\omega + \alpha$, that is $a_0(t\xi,ty) = t^{\alpha+\omega}a_0(\xi,y)$ for all $\xi, y \in \mathbf{R}$, $t > 0$. We suppose that the asymptotic relation

$$\begin{aligned} &|a(t,\xi,y) - a_0(\xi,y)| \\ &\le C\langle\xi\rangle^\theta \{\xi\}^\omega \left(\langle\xi-y\rangle^\sigma \{\xi-y\}^{\alpha+\gamma} + \langle y\rangle^\sigma \{y\}^{\alpha+\gamma}\right) \\ &+ C\langle t\rangle^{-\frac{\gamma}{\delta}} \langle\xi\rangle^{\tilde{\theta}} \{\xi\}^\omega \left(\langle\xi-y\rangle^\sigma \{\xi-y\}^\alpha + \langle y\rangle^\sigma \{y\}^\alpha\right) \end{aligned} \tag{4.27}$$

is true for all ξ, $y \in \mathbf{R}$, $t > 0$, where $\tilde{\theta}$, σ, $\alpha \ge 0$, ω, $\gamma > 0$. Note that $a_0(\xi,y)$ also satisfies estimates (4.20).

Consider self-similar solutions $w(t,x) = t^{-\frac{1}{\delta}} f_\theta\left(xt^{-\frac{1}{\delta}}\right)$ for equation (4.25) which have a total mass

$$\theta = \int_{\mathbf{R}} t^{-\frac{1}{\delta}} f_\theta\left(xt^{-\frac{1}{\delta}}\right) dx = \int_{\mathbf{R}} f_\theta(x)\, dx \ne 0.$$

The existence of a unique self-similar solution $w(t,x) = t^{-\frac{1}{\delta}} f\left(xt^{-\frac{1}{\delta}}\right)$, where $f(x)$ is such that

$$\sup_{\rho\in[0,\alpha]}\left\|\left|\cdot\right|^{\rho}\widehat{f}\right\|_{\mathbf{L}^1}+\sup_{\rho\in[0,\alpha+\gamma]}\left\|\left|\cdot\right|^{\rho}\widehat{f}\right\|_{\mathbf{L}^\infty}<\infty$$

will be given below, provided that $|\theta|$ is sufficiently small.

We construct the self-similar solution for equation (4.25) taking the initial data $\theta\delta_0(x)$, where $\delta_0(x)$ is the Dirac delta function. We write the Cauchy problem for (4.25) as the integral equation

$$w(t)=\theta\mathcal{G}_0(t)\,\delta_0(x)-\int_0^t\mathcal{G}_0(t-\tau)\,\mathcal{N}_0(w)(\tau)\,d\tau, \tag{4.28}$$

where

$$\mathcal{G}_0(t)\,\phi=\overline{\mathcal{F}}_{\xi\to x}\left(e^{-tL_0(\xi)}\hat{\phi}(\xi)\right),$$

and by changing the variable of integration $\xi t^{\frac{1}{\delta}}=\eta$ we have

$$\mathcal{G}_0(t)\,\delta_0(x)=(2\pi)^{-\frac{1}{2}}\int_{\mathbf{R}}e^{i\xi x-tL_0(\xi)}d\xi=t^{-\frac{1}{\delta}}G\left(xt^{-\frac{1}{\delta}}\right)$$

and

$$G(x)=(2\pi)^{-\frac{1}{2}}\int_{\mathbf{R}}e^{i\eta x-L_0(\eta)}d\eta.$$

We look for the self-similar solution $w(t,x)=t^{-\frac{1}{\delta}}f\left(xt^{-\frac{1}{\delta}}\right)$. Applying the Fourier transformation to the integral equation (4.28) we obtain

$$\begin{aligned}\widehat{w}(t,\xi)&=\theta e^{-tL_0(\xi)}\\&-\int_0^t d\tau e^{-(t-\tau)L_0(\xi)}\int_{\mathbf{R}}a_0(\xi,y)\,\widehat{w}(\tau,\xi-y)\,\widehat{w}(\tau,y)\,dy.\end{aligned} \tag{4.29}$$

Note that for the Fourier transform of $w(t,x)=t^{-\frac{1}{\delta}}f\left(xt^{-\frac{1}{\delta}}\right)$ we have

$$\begin{aligned}\widehat{w}(t,\xi)&=(2\pi)^{-\frac{1}{2}}t^{-\frac{1}{\delta}}\int_{\mathbf{R}}e^{i\xi x}f\left(xt^{-\frac{1}{\delta}}\right)dx\\&=(2\pi)^{-\frac{1}{2}}\int_{\mathbf{R}}e^{-i\xi t^{\frac{1}{\delta}}y}f(y)\,dy=\widehat{f}\left(\xi t^{\frac{1}{\delta}}\right);\end{aligned}$$

therefore, from (4.29) we get

$$\begin{aligned}\widehat{f}\left(\xi t^{\frac{1}{\delta}}\right)&=\theta e^{-tL_0(\xi)}\\&-\int_0^t d\tau e^{-(t-\tau)L_0(\xi)}\int_{\mathbf{R}}a_0(\xi,y)\,\widehat{f}\left((\xi-y)\,\tau^{\frac{1}{\delta}}\right)\widehat{f}\left(y\tau^{\frac{1}{\delta}}\right)dy.\end{aligned}$$

Hence by changing $\eta=\xi t^{\frac{1}{\delta}}$, $\eta'=yt^{\frac{1}{\delta}}$, $\tau=tz$ and by recognizing both that the symbol $a_0(\eta,\eta')$ is homogeneous of order $\omega+\alpha$, that is $a_0\left(\eta t^{-\frac{1}{\delta}},\eta' t^{-\frac{1}{\delta}}\right)=t^{-\frac{\omega+\alpha}{\delta}}a_0(\eta,\eta')$, and that $\delta=\omega+\alpha+1$ is critical, we find

$$\widehat{f}(\eta) = \theta e^{-L_0(\eta)}$$
$$- \int_0^1 dz e^{-(1-z)L_0(\eta)} \int_{\mathbf{R}} a_0(\eta, \eta') \widehat{f}\left((\eta - \eta') z^{\frac{1}{\delta}}\right) \widehat{f}\left(\eta' z^{\frac{1}{\delta}}\right) d\eta'. \quad (4.30)$$

Define $\mathbf{Z} = \left\{\phi \in \mathbf{L}^1(\mathbf{R}) \cap \mathbf{L}^\infty(\mathbf{R}) : \|\phi\|_{\mathbf{Z}} < \infty\right\}$ and the norm

$$\|\phi\|_{\mathbf{Z}} = \sup_{\rho \in [0,\alpha]} \left\| |\cdot|^\rho \phi \right\|_{\mathbf{L}^1} + \sup_{\rho \in [0,\alpha+\gamma]} \left\| |\cdot|^\rho \phi \right\|_{\mathbf{L}^\infty}.$$

The existence of a unique self-similar solution $\widehat{f} \in \mathbf{Z}$ follows from estimate (4.24) and Lemma 4.3.

In the following theorem we state the main term of the large time asymptotic behavior of solutions $u(t,x)$ to the Cauchy problem (4.14) in terms of the self-similar solution $t^{-\frac{1}{\delta}} f\left(xt^{-\frac{1}{\delta}}\right)$.

Theorem 4.13. *Suppose that the nonlinear operator $\mathcal{N}$ satisfies asymptotic relationship (4.27) and estimates (4.20) with $\omega > 0$, $\tilde{\theta} + \sigma \in [0, \nu)$ if $\nu > 0$ or $\sigma = 0 = \tilde{\theta}$ if $\nu = 0$. Let the linear operator $\mathcal{L}$ satisfy conditions (4.21) and (4.26) with $\delta = 1 + \alpha + \omega$, $\alpha \geq 0$. Let the initial data u_0 be such that*

$$\|u_0\|_{\mathbf{A}^{0,\infty}} + \|u_0\|_{\mathbf{B}^{0,\frac{1}{1-b}}} \leq \varepsilon,$$

where $\varepsilon > 0$ is sufficiently small. The value $b \in [0,1]$ is such that $b = 1$ if $\nu \geq 1 + \sigma + \tilde{\theta}$, $b < 1 - \sqrt{1 + \sigma + \theta - \nu}$ if $\nu \in (0, 1 + \sigma + \theta)$ and $b = 0$ if $\nu = 0$. Also we suppose that

$$\|u_0 - \theta\delta_0\|_{\mathbf{A}^{-\gamma,\infty}} \leq \varepsilon,$$

where $\delta_0(x)$ is the Dirac delta-function, $\gamma \in (0, \min(1, \omega))$. Then for large time the solution $u(t,x)$ to the Cauchy problem (4.14) tends to the self-similar solution $t^{-\frac{1}{\delta}} f_\theta\left(xt^{-\frac{1}{\delta}}\right)$ of equation (4.25), that is

$$\left\| u(t) - t^{-\frac{1}{\delta}} f_\theta\left((\cdot) t^{-\frac{1}{\delta}}\right) \right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1+\gamma}{\delta}}$$

for all $t \geq 1$.

Remark 4.14. The conditions of the theorems on the initial data u_0 can also be expressed in terms of the standard weighted Sobolev spaces as follows

$$\|u_0\|_{\mathbf{H}^{\beta - \frac{b}{2},0}} + \|u_0\|_{\mathbf{H}^{0,\beta+2\gamma}} \leq \varepsilon,$$

where $\beta > \frac{1}{2}$. However, the conditions on the initial data u_0 are expressed more precisely in the norms $\mathbf{A}^{0,p}$ and $\mathbf{B}^{0,p}$.

Remark 4.15. We give two examples of the application of Theorem 4.13: 1) In the case of KdVB equation (4.17) we have $a(t,\xi,y) = a_0(\xi,y) = \frac{i\xi}{2}$, $L(\xi) = |\xi|^2 - i\xi^3$, $L_0(\xi) = |\xi|^2$. The conditions (4.20), (4.21) and (4.26), (4.27) are fulfilled with $\theta = \omega = 1$, $\sigma = \alpha = 0$, $\delta = \nu = 2$; hence we can take $\gamma \in (0,1)$, $b = 1$. Then for small initial data u_0 such that

$$\|u_0\|_{\mathbf{A}^{0,\infty}} + \|u_0\|_{\mathbf{B}^{0,\infty}} \le \varepsilon, \text{ and } \|u_0 - \theta\delta_0\|_{\mathbf{A}^{-\gamma,\infty}} \le \varepsilon,$$

the asymptotics

$$\left\| u(t) - t^{-\frac{1}{2}} f_\theta\left((\cdot)\, t^{-\frac{1}{2}}\right) \right\|_{\mathbf{L}^\infty} \le C t^{-\frac{1+\gamma}{2}} \tag{4.31}$$

is true for large t, where f_θ is the self-similar solution for the Burgers equation (4.16) with the initial data $u_0(x) = \delta_0(x)$. 2) In the case of BBMPB equation (4.18) we have $a(t,\xi,y) = \frac{i\xi}{2}\left(1+|\xi|^2\right)^{-1}$, $L(\xi) = \left(1+|\xi|^2\right)^{-1}\left(|\xi|^2 - i\xi^3\right)$, $a_0(\xi,y) = \frac{i\xi}{2}$, $L_0(\xi) = |\xi|^2$. The conditions (4.20), (4.21) and (4.26), (4.27) are fulfilled with $\theta = 0$, $\omega = 1$, $\sigma = \alpha = 0$, $\delta = 2$, $\nu = 0$; hence we can take $\gamma \in (0,1)$, $b = 0$. Then for small initial data u_0 such that

$$\|u_0\|_{\mathbf{A}^{0,\infty}} + \|u_0\|_{\mathbf{B}^{0,1}} \le \varepsilon, \text{ and } \|u_0 - \theta\delta_0\|_{\mathbf{A}^{-\gamma,\infty}} \le \varepsilon,$$

the asymptotics (4.31) is valid.

4.2.4 Proof of Theorem 4.13

Before the proof of Theorem 4.13 we prepare some preliminary estimates in the following two lemmas. First we estimate the difference $\mathcal{G}(t) - \mathcal{G}_0(t)$ where

$$\mathcal{G}_0(t)\,\phi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-tL_0(\xi)}\hat{\phi}(\xi)\right),$$

and $L_0(\xi)$ is a homogeneous function of order δ, that is $L_0(t\xi) = t^\delta L_0(\xi)$ for all $\xi \in \mathbf{R}$, $t > 0$. From Lemma 1.39 we obtain the following estimate.

Lemma 4.16. *Let the linear operator $\mathcal{L}$ satisfy the dissipation condition (4.21) and the asymptotic representation (4.26). Then the estimate*

$$\begin{aligned} &\left\| \int_0^t \left(\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau)\right)\psi(\tau)\,d\tau \right\|_{\mathbf{A}^{\rho,p}} \\ &\le C\langle t\rangle^{1-\lambda-\frac{\rho+\omega+\gamma}{\delta}-\frac{1}{\delta p}} \sup_{p\le q\le\infty}\sup_{\tau>0}\left(\{\tau\}^\kappa \langle\tau\rangle^{\lambda+\frac{1}{\delta q}} \|\psi(\tau)\|_{\mathbf{A}^{-\omega,q}}\right) \end{aligned}$$

is valid for $\rho \ge 0$, $1 \le p \le \infty$, where $\kappa < 1$, $\lambda < 1$, $\rho + \omega + \gamma < \delta$.

Now we estimate the nonlinearities

$$\mathcal{N}^{(1)}(\varphi,\phi) = \overline{\mathcal{F}}_{\xi\to x}\int_{|y|\le 2} a(t,\xi,y)\,\widehat{\varphi}(t,\xi-y)\,\widehat{\phi}(t,y)\,dy,$$

$$\mathcal{N}^{(2)}(\varphi,\phi) = \overline{\mathcal{F}}_{\xi\to x}\int_{|y|\ge 2} a(t,\xi,y)\,\widehat{\varphi}(t,\xi-y)\,\widehat{\phi}(t,y)\,dy$$

in the norms $\mathbf{A}^{-\omega,p}$.

Lemma 4.17. *Let the symbol $a(t,\xi,y)$ of the nonlinear operators $\mathcal{N}^{(1)}(\varphi,\phi)$ and $\mathcal{N}^{(2)}(\varphi,\phi)$ satisfy condition (4.20). Then the inequalities*

$$\left\|\mathcal{N}^{(1)}(\varphi,\phi)\right\|_{\mathbf{A}^{-\omega,p}} \leq C\left(\|\varphi\|_{\mathbf{A}^{\alpha,1}}+\|\varphi\|_{\mathbf{B}^{0,\infty}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}}+\|\phi\|_{\mathbf{B}^{0,\infty}}\right)$$
$$+C\left(\|\phi\|_{\mathbf{A}^{\alpha,1}}+\|\phi\|_{\mathbf{B}^{0,\infty}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}}+\|\varphi\|_{\mathbf{B}^{0,\infty}}\right)$$

and

$$\left\|\mathcal{N}^{(2)}(\varphi,\phi)\right\|_{\mathbf{A}^{-\omega,p}} \leq C\|\varphi\|_{\mathbf{B}^{\sigma,1}}\|\phi\|_{\mathbf{B}^{0,\infty}}+C\|\phi\|_{\mathbf{B}^{\sigma,1}}\|\varphi\|_{\mathbf{B}^{0,\infty}}$$

are valid for $1 \leq p \leq \infty$, provided that the right-hand sides are bounded.

Proof. By virtue of condition (4.20) and by the Young inequality we obtain

$$\begin{aligned}
&\left\|\mathcal{N}^{(1)}(\varphi,\phi)\right\|_{\mathbf{A}^{-\omega,p}} \\
&\leq \left\|\int_{|y|\leq 2}|\cdot|^{-\omega}|a(t,\cdot,y)|\left|\widehat{\varphi}(t,\cdot-y)\widehat{\phi}(t,y)\right|dy\right\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)} \\
&\leq C\left\|\int_{|y|\leq 2}\left(\{\cdot-y\}^{\alpha}+\{y\}^{\alpha}\right)\left|\widehat{\varphi}(t,\cdot-y)\widehat{\phi}(t,y)\right|dy\right\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)} \\
&\leq C\left(\left\||\cdot|^{\alpha}\widehat{\varphi}\right\|_{\mathbf{L}^1_\xi(|\xi|\leq 1)}+\|\widehat{\varphi}\|_{\mathbf{L}^\infty_\xi(|\xi|\geq 1)}\right) \\
&\times\left(\left\|\widehat{\phi}\right\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)}+\left\|\widehat{\phi}\right\|_{\mathbf{L}^\infty_\xi(|\xi|\geq 1)}\right) \\
&+C\left(\left\||\cdot|^{\alpha}\widehat{\phi}\right\|_{\mathbf{L}^1_\xi(|\xi|\leq 1)}+\left\|\widehat{\phi}\right\|_{\mathbf{L}^\infty_\xi(|\xi|\geq 1)}\right) \\
&\times\left(\|\widehat{\varphi}\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)}+\|\widehat{\varphi}\|_{\mathbf{L}^\infty_\xi(|\xi|>1)}\right) \\
&\leq C\left(\|\varphi\|_{\mathbf{A}^{\alpha,1}}+\|\varphi\|_{\mathbf{B}^{0,\infty}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}}+\|\phi\|_{\mathbf{B}^{0,\infty}}\right) \\
&+C\left(\|\phi\|_{\mathbf{A}^{\alpha,1}}+\|\phi\|_{\mathbf{B}^{0,\infty}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}}+\|\varphi\|_{\mathbf{B}^{0,\infty}}\right)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&\left\|\mathcal{N}^{(2)}(\varphi,\phi)\right\|_{\mathbf{A}^{-\omega,p}} \\
&\leq \left\|\int_{|y|\geq 2}|\cdot|^{-\omega}|a(t,\cdot,y)|\left|\widehat{\varphi}(t,\cdot-y)\widehat{\phi}(t,y)\right|dy\right\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)} \\
&\leq C\left\|\int_{|y|\geq 2}\left(|\cdot-y|^{\sigma}+|y|^{\sigma}\right)\left|\widehat{\varphi}(t,\cdot-y)\widehat{\phi}(t,y)\right|dy\right\|_{\mathbf{L}^\infty_\xi(|\xi|\leq 1)} \\
&\leq C\left\||\cdot|^{\sigma}\widehat{\varphi}\right\|_{\mathbf{L}^1_\xi(|\xi|\geq 1)}\left\|\widehat{\phi}\right\|_{\mathbf{L}^\infty_\xi(|\xi|\geq 1)}+C\left\||\cdot|^{\sigma}\widehat{\phi}\right\|_{\mathbf{L}^1_\xi(|\xi|\geq 1)}\|\widehat{\varphi}\|_{\mathbf{L}^\infty_\xi(|\xi|>1)} \\
&\leq C\|\varphi\|_{\mathbf{B}^{\sigma,1}}\|\phi\|_{\mathbf{B}^{0,\infty}}+C\|\phi\|_{\mathbf{B}^{\sigma,1}}\|\varphi\|_{\mathbf{B}^{0,\infty}}.
\end{aligned}$$

Finally, the estimates of the lemma follow, and Lemma 4.17 is proved.

We write the nonlinear Cauchy problem (4.25) as the integral equation (4.28). First we note that Theorem 4.9 is also applicable to equation (4.28) if we take $b = 1$, $\sigma = \alpha$, $\nu = \delta$, $\theta = \omega$, so we have estimates

$$\|u\|_{\mathbf{X}} \leq C\varepsilon \text{ and } \|w\|_{\mathbf{X}_0} \leq C\varepsilon,$$

where

$$\|\phi\|_{\mathbf{X}_0} = \sup_{\rho\in[0,\alpha+\gamma]} \sup_{1\leq p\leq\infty} \sup_{t>0} \langle t\rangle^{\frac{\rho}{\delta}+\frac{1}{\delta p}} \|\phi(t)\|_{\mathbf{A}^{\rho,p}} + \sup_{s\in[0,\alpha]} \sup_{1\leq p\leq\infty} \sup_{t>0} \{t\}^{\frac{s}{\delta}+\frac{1}{\delta p}} \langle t\rangle^{\frac{1+\alpha+\gamma}{\delta}+\frac{1}{\delta p}} \|\phi(t)\|_{\mathbf{B}^{s,p}}.$$

Therefore, in particular, the estimates are valid

$$\|u(t)\|_{\mathbf{B}^{s,p}} \leq C\varepsilon \{t\}^{-\frac{s}{\nu}-\frac{b}{\nu p}} \langle t\rangle^{-\frac{1+\alpha+\gamma}{\delta}-\frac{1}{\delta p}}$$

for all $t > 0$, $s \in [0, \sigma]$, $1 \leq p \leq \infty$, and

$$\|w(t)\|_{\mathbf{B}^{s,p}} \leq C\varepsilon \{t\}^{-\frac{s}{\delta}-\frac{1}{\delta p}} \langle t\rangle^{-\frac{1+\alpha+\gamma}{\delta}-\frac{1}{\delta p}}$$

for all $t > 0$, $s \in [0, \alpha]$, $1 \leq p \leq \infty$. These estimates are sufficient for evaluating the large time decay of the remainder $v(t) = u(t) - w(t)$ in the $\mathbf{B}$ norms

$$\begin{aligned} \|v(t)\|_{\mathbf{B}^{0,1}} &\leq C \|u(t)\|_{\mathbf{B}^{0,1}} + \|w(t)\|_{\mathbf{B}^{0,1}} \\ &\leq C\varepsilon \left(\{t\}^{-\frac{b}{\nu}} + \{t\}^{-\frac{1}{\delta}}\right) \langle t\rangle^{-\frac{2+\alpha+\gamma}{\delta}}. \end{aligned} \tag{4.32}$$

We now estimate the difference $v(t) = u(t) - w(t)$ in the norms $\mathbf{A}^{\rho,p}$ and show that

$$\|v(t)\|_{\mathbf{A}^{\rho,p}} \leq C\varepsilon \langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{1}{\delta p}} \tag{4.33}$$

for all $t > 0$, $\rho \in [0, \alpha]$, $1 \leq p \leq \infty$.

We represent $\mathcal{N}(u) = \mathcal{N}^{(1)}(u) + \mathcal{N}^{(2)}(u)$ and $\mathcal{N}_0(w) = \mathcal{N}_0^{(1)}(w) + \mathcal{N}_0^{(2)}(w)$, where

$$\mathcal{N}^{(1)}(u) = \overline{\mathcal{F}}_{\xi\to x} \int_{|y|\leq 2} a(t,\xi,y)\, \widehat{u}(t,\xi-y)\, \widehat{u}(t,y)\, dy$$
$$\mathcal{N}^{(2)}(u) = \overline{\mathcal{F}}_{\xi\to x} \int_{|y|\geq 2} a(t,\xi,y)\, \widehat{u}(t,\xi-y)\, \widehat{u}(t,y)\, dy,$$

the terms $\mathcal{N}_0^{(1)}(w,w)$, $\mathcal{N}_0^{(2)}(w,w)$ are defined similarly.

Then for the difference $v(t) = u(t) - w(t)$ we get from (4.28)

$$
\begin{aligned}
v(t) = & \left(\mathcal{G}(t) - \mathcal{G}_0(t)\right) u_0 \\
& - \int_0^t \left(\mathcal{G}(t-\tau)\mathcal{N}^{(2)}(u) - \mathcal{G}_0(t-\tau)\mathcal{N}_0^{(2)}(w)\right) d\tau \\
& - \int_0^t \left(\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau)\right)\mathcal{N}^{(1)}(u)(\tau)\, d\tau \\
& - \int_0^t \mathcal{G}_0(t-\tau)\left(\mathcal{N}^{(1)}(u)(\tau) - \mathcal{N}_0^{(1)}(u)(\tau)\right) d\tau \\
& - \int_0^t \mathcal{G}_0(t-\tau)\left(\mathcal{N}_0^{(1)}(u)(\tau) - \mathcal{N}_0^{(1)}(w)(\tau)\right) d\tau.
\end{aligned}
$$

For the first summand we have

$$
\left\|\left(\mathcal{G}(t) - \mathcal{G}_0(t)\right) u_0\right\|_{\mathbf{A}^{\rho,p}} \le C\langle t\rangle^{-\frac{\rho+\gamma}{\delta} - \frac{1}{\delta p}} \|u_0\|_{\mathbf{A}^{0,\infty}}.
$$

The second summand we estimate by Lemmas 4.11 and 4.17

$$
\begin{aligned}
& \left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}^{(2)}(u)\, d\tau\right\|_{\mathbf{A}^{\rho,p}} \\
\le & C\langle t\rangle^{1-\widetilde{\lambda} - \frac{\rho+\omega}{\delta} - \frac{1}{\delta p}} \sup_{p\le q\le\infty}\sup_{\tau>0}\left(\{\tau\}^{\kappa}\langle\tau\rangle^{\widetilde{\lambda}+\frac{1}{\delta q}}\left\|\mathcal{N}^{(2)}(u)\right\|_{\mathbf{A}^{-\omega,q}}\right) \\
\le & C\langle t\rangle^{1-\widetilde{\lambda} - \frac{\rho+\omega}{\delta} - \frac{1}{\delta p}} \sup_{p\le q\le\infty}\sup_{\tau>0}\left(\{\tau\}^{\kappa}\langle\tau\rangle^{\widetilde{\lambda}+\frac{1}{\delta q}}\|u\|_{\mathbf{B}^{\sigma,1}}\|u\|_{\mathbf{B}^{0,\infty}}\right) \\
\le & \, C\langle t\rangle^{-\frac{\rho+\gamma}{\delta} - \frac{1}{\delta p}}\|u\|_{\mathbf{X}}^2 \le C\varepsilon^2\langle t\rangle^{-\frac{\rho+\gamma}{\delta} - \frac{1}{\delta p}}.
\end{aligned}
$$

where $\kappa = \frac{\sigma+b}{\nu}$, $\widetilde{\lambda} = 1 - \frac{\omega-\gamma}{\delta} = \frac{1+\alpha+\gamma}{\delta}$, $\omega > \gamma > 0$. In the same manner we have

$$
\begin{aligned}
& \left\|\int_0^t \mathcal{G}_0(t-\tau)\mathcal{N}_0^{(2)}(w)\, d\tau\right\|_{\mathbf{A}^{\rho,p}} \\
\le & C\langle t\rangle^{1-\widetilde{\lambda} - \frac{\rho+\omega}{\delta} - \frac{1}{\delta p}} \sup_{p\le q\le\infty}\sup_{\tau>0}\left(\{\tau\}^{\kappa_0}\langle\tau\rangle^{\widetilde{\lambda}+\frac{1}{\delta q}}\left\|\mathcal{N}_0^{(2)}(w)\right\|_{\mathbf{A}^{-\omega,q}}\right) \\
\le & C\langle t\rangle^{1-\widetilde{\lambda} - \frac{\rho+\omega}{\delta} - \frac{1}{\delta p}} \sup_{p\le q\le\infty}\sup_{\tau>0}\left(\{\tau\}^{\kappa_0}\langle\tau\rangle^{\widetilde{\lambda}+\frac{1}{\delta q}}\|w\|_{\mathbf{B}^{\alpha,1}}\|w\|_{\mathbf{B}^{0,\infty}}\right) \\
\le & \, C\langle t\rangle^{-\frac{\rho+\gamma}{\delta} - \frac{1}{\delta p}}\|w\|_{\mathbf{X}_0}^2 \le C\varepsilon^2\langle t\rangle^{-\frac{\rho+\gamma}{\delta} - \frac{1}{\delta p}},
\end{aligned}
$$

where $\kappa_0 = \frac{\alpha+1}{\delta}$. By Lemma 4.16 we have with $\lambda = 1 - \frac{\omega}{\delta} = \frac{1+\alpha}{\delta}$

$$
\begin{aligned}
& \left\|\int_0^t \left(\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau)\right)\mathcal{N}^{(1)}(u)(\tau)\, d\tau\right\|_{\mathbf{A}^{\rho,p}} \\
\le & C\langle t\rangle^{1-\lambda - \frac{\rho+\omega+\gamma}{\delta} - \frac{1}{\delta p}} \sup_{p\le q\le\infty}\sup_{\tau>0}\left(\langle\tau\rangle^{\lambda+\frac{1}{\delta q}}\left\|\mathcal{N}^{(1)}(u)(\tau)\right\|_{\mathbf{A}^{-\omega,q}}\right) \\
\le & \, C\varepsilon^2\langle t\rangle^{-\frac{\rho+\gamma}{\delta} - \frac{1}{\delta p}}
\end{aligned}
$$

since via Lemma 4.17

$$\begin{aligned}
&\langle\tau\rangle^{\lambda+\frac{1}{\delta p}}\left\|\mathcal{N}^{(1)}(u)(\tau)\right\|_{\mathbf{A}^{-\omega,p}}\\
&\leq C\left(\langle\tau\rangle^{\frac{1+\alpha}{\delta}}\|u(\tau)\|_{\mathbf{A}^{\alpha,1}}+\langle\tau\rangle^{\frac{1+\alpha+\gamma}{\delta}}\|u(\tau)\|_{\mathbf{B}^{0,\infty}}\right)\\
&\times\left(\langle\tau\rangle^{\frac{1}{\delta p}}\|u(\tau)\|_{\mathbf{A}^{0,p}}+\langle\tau\rangle^{\frac{1+\alpha+\gamma}{\delta}}\|u(\tau)\|_{\mathbf{B}^{0,\infty}}\right)\leq C\|u\|_{\mathbf{X}}^{2}\leq C\varepsilon^{2}.
\end{aligned}$$

Similarly, via Lemma 4.11, we obtain

$$\begin{aligned}
&\left\|\int_0^t\mathcal{G}_0(t-\tau)\left(\mathcal{N}^{(1)}(u)(\tau)-\mathcal{N}_0^{(1)}(u)(\tau)\right)d\tau\right\|_{\mathbf{A}^{\rho,p}}\\
&\leq C\langle t\rangle^{1-\widetilde{\lambda}-\frac{\rho+\omega}{\delta}-\frac{1}{\delta p}}\\
&\times\sup_{p\leq q\leq\infty}\sup_{\tau>0}\left(\langle\tau\rangle^{\widetilde{\lambda}+\frac{1}{\delta q}}\left\|\mathcal{N}^{(1)}(u)(\tau)-\mathcal{N}_0^{(1)}(u)(\tau)\right\|_{\mathbf{A}^{-\omega,q}}\right)\\
&\leq C\varepsilon^2\langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{1}{\delta p}},
\end{aligned}$$

where $\widetilde{\lambda}=1-\frac{\omega-\gamma}{\delta}=\frac{1+\alpha+\gamma}{\delta}$, $\omega>\gamma>0$; since by Lemma 4.17 and condition (4.27) we have

$$\begin{aligned}
&\langle t\rangle^{\widetilde{\lambda}+\frac{1}{\delta p}}\left\|\mathcal{N}^{(1)}(u)(t)-\mathcal{N}_0^{(1)}(u)(t)\right\|_{\mathbf{A}^{-\omega,p}}\\
&\leq C\left(\langle t\rangle^{\frac{\alpha+\gamma+1}{\delta}}\|u(t)\|_{\mathbf{A}^{\alpha+\gamma,1}}+\langle t\rangle^{\frac{1+\alpha+\gamma}{\delta}}\|u(t)\|_{\mathbf{B}^{0,\infty}}\right)\\
&\times\left(\langle t\rangle^{\frac{1}{\delta p}}\|u(t)\|_{\mathbf{A}^{0,p}}+\langle t\rangle^{\frac{1+\alpha+\gamma}{\delta}}\|u(t)\|_{\mathbf{B}^{0,\infty}}\right)\\
&+C\left(\langle t\rangle^{\frac{\alpha+1}{\delta}}\|u(t)\|_{\mathbf{A}^{\alpha,1}}+\langle t\rangle^{\frac{1+\alpha+\gamma}{\delta}}\|u(t)\|_{\mathbf{B}^{0,\infty}}\right)\\
&\times\left(\langle t\rangle^{\frac{1}{\delta p}}\|u(t)\|_{\mathbf{A}^{0,p}}+\langle t\rangle^{\frac{1+\alpha+\gamma}{\delta}}\|u(t)\|_{\mathbf{B}^{0,\infty}}\right)\leq C\|u\|_{\mathbf{X}}^{2}.
\end{aligned}$$

Finally, by Lemma 4.11 we find for $\rho\in[0,\alpha]$, $1\leq p\leq\infty$

$$\begin{aligned}
&\langle t\rangle^{\frac{\rho+\gamma}{\delta}+\frac{1}{\delta p}}\left\|\int_0^t\mathcal{G}_0(t-\tau)\left(\mathcal{N}_0^{(1)}(u)(\tau)-\mathcal{N}_0^{(1)}(w)(\tau)\right)d\tau\right\|_{\mathbf{A}^{\rho,p}}\\
&\leq C\sup_{p\leq q\leq\infty}\sup_{\tau>0}\left(\langle\tau\rangle^{\widetilde{\lambda}+\frac{1}{\delta q}}\left\|\left(\mathcal{N}_0^{(1)}(u)(\tau)-\mathcal{N}_0^{(1)}(w)(\tau)\right)\right\|_{\mathbf{A}^{-\omega,q}}\right)\\
&\leq C\varepsilon^2+C\varepsilon\sup_{p\leq q\leq\infty}\sup_{\tau>0}\langle\tau\rangle^{\frac{\alpha+\gamma}{\delta}+\frac{1}{\delta q}}\|v(\tau)\|_{\mathbf{A}^{\alpha,q}},
\end{aligned}$$

where $\widetilde{\lambda}=1-\frac{\omega-\gamma}{\delta}=\frac{1+\alpha+\gamma}{\delta}$, $\omega>0$, $\gamma\in(0,\min(1,\omega))$, since in view of Lemma 4.17 we have

$$\begin{aligned}
&\langle\tau\rangle^{\widetilde{\lambda}+\frac{1}{\delta q}}\left\|\left(\mathcal{N}_0^{(1)}(u)(\tau)-\mathcal{N}_0^{(1)}(w)(\tau)\right)\right\|_{\mathbf{A}^{-\omega,q}}\\
&\le C\langle\tau\rangle^{\frac{1+\alpha+\gamma}{\delta}}\left(\|v(\tau)\|_{\mathbf{A}^{\alpha,1}}+\|u\|_{\mathbf{B}^{0,\infty}}+\|w\|_{\mathbf{B}^{0,\infty}}\right)\\
&\times\langle\tau\rangle^{\frac{1}{\delta q}}\left(\|u\|_{\mathbf{A}^{0,q}}+\|w\|_{\mathbf{A}^{0,q}}+\|u\|_{\mathbf{B}^{0,\infty}}+\|w\|_{\mathbf{B}^{0,\infty}}\right)\\
&+C\langle\tau\rangle^{\frac{\alpha+1}{\delta}}\left(\|u\|_{\mathbf{A}^{\alpha,1}}+\|w\|_{\mathbf{A}^{\alpha,1}}+\|u\|_{\mathbf{B}^{0,\infty}}+\|w\|_{\mathbf{B}^{0,\infty}}\right)\\
&\times\langle\tau\rangle^{\frac{\gamma}{\delta}+\frac{1}{\delta q}}\left(\|v(\tau)\|_{\mathbf{A}^{0,q}}+\|u\|_{\mathbf{B}^{0,\infty}}+\|w\|_{\mathbf{B}^{0,\infty}}\right)\\
&\le\left(\|u\|_{\mathbf{X}}+\|w\|_{\mathbf{X}_0}\right)^2\\
&+C\sup_{\rho\in[0,\alpha]}\sup_{1\le p\le q}\langle\tau\rangle^{\frac{\rho+\gamma}{\delta}+\frac{1}{\delta p}}\|v(\tau)\|_{\mathbf{A}^{\rho,p}}\left(\|u\|_{\mathbf{X}}+\|w\|_{\mathbf{X}_0}\right)\\
&\le C\varepsilon^2+C\varepsilon\sup_{\rho\in[0,\alpha]}\sup_{1\le p\le q}\langle\tau\rangle^{\frac{\rho+\gamma}{\delta}+\frac{1}{\delta p}}\|v(\tau)\|_{\mathbf{A}^{\rho,p}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sup_{\rho\in[0,\alpha]}\sup_{1\le p\le\infty}\sup_{t>0}\langle t\rangle^{\frac{\rho+\gamma}{\delta}+\frac{1}{\delta p}}\|v(t)\|_{\mathbf{A}^{\rho,p}}\\
&\le C\varepsilon^2+C\varepsilon\sup_{\rho\in[0,\alpha]}\sup_{1\le p\le\infty}\sup_{t>0}\langle t\rangle^{\frac{\rho+\gamma}{\delta}+\frac{1}{\delta p}}\|v(t)\|_{\mathbf{A}^{\rho,p}},
\end{aligned}$$

and (4.33) follows.

Consider a self-similar solution $h(t,x)$ of (4.25) constructed in Section 4.2.3. Then for the difference $h(t)-w(t)$ we get from (4.28)

$$\begin{aligned}
h(t)-w(t)&=\mathcal{G}_0(t)\widetilde{u_0}\\
&-\int_0^t\mathcal{G}_0(t-\tau)\left(\mathcal{N}_0(h)(\tau)-\mathcal{N}_0(w)(\tau)\right)d\tau,
\end{aligned}\tag{4.34}$$

where $\widetilde{u_0}(x)=u_0(x)-\theta\delta(x)$. We suppose that the initial data u_0 have the asymptotic representation $\widehat{u_0}(\xi)=\theta+O(|\xi|^\gamma)$ as $\xi\to0$, so

$$\|\widetilde{u_0}\|_{\mathbf{A}^{-\gamma,\infty}}+\|\widetilde{u_0}\|_{\mathbf{B}^{0,\infty}}\le\varepsilon.$$

By estimates (4.22) and (4.23) we get

$$\begin{aligned}
\|\mathcal{G}_0(t)\widetilde{u_0}\|_{\mathbf{A}^{\rho,p}}&\le C\langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{1}{\delta p}}\|\widetilde{u_0}\|_{\mathbf{A}^{-\gamma,\infty}},\\
\|\mathcal{G}_0(t)\widetilde{u_0}\|_{\mathbf{B}^{s,p}}&\le Ce^{-\frac{\mu}{2}t}\{t\}^{-\frac{s}{\delta}-\frac{1}{\delta p}}\|\widetilde{u_0}\|_{\mathbf{B}^{0,\infty}};
\end{aligned}$$

hence

$$\|\mathcal{G}_0(t)\widetilde{u_0}\|_{\mathbf{X}_\gamma}\le C\left(\|\widetilde{u_0}\|_{\mathbf{A}^{-\gamma,\infty}}+\|\widetilde{u_0}\|_{\mathbf{B}^{0,\infty}}\right),$$

where we denote

$$\begin{aligned}
\|\phi\|_{\mathbf{X}_\gamma}&=\sup_{\rho\in[0,\alpha]}\sup_{1\le p\le\infty}\sup_{t>0}\langle t\rangle^{\frac{\rho+\gamma}{\delta}+\frac{1}{\delta p}}\|\phi(t)\|_{\mathbf{A}^{\rho,p}}\\
&+\sup_{s\in[0,\alpha]}\sup_{1\le p\le\infty}\sup_{t>0}\{t\}^{\frac{s}{\delta}+\frac{1}{\delta p}}\langle t\rangle^{\frac{1+\alpha+\gamma}{\delta}+\frac{1}{\delta p}}\|\phi(t)\|_{\mathbf{B}^{s,p}}.
\end{aligned}$$

The second summand in equation (4.34) we estimate by virtue of Lemma 4.11 taking $\kappa_0 = \frac{\alpha+1}{\delta} = 1 - \frac{\omega}{\delta}$, $\widetilde{\lambda} = 1 - \frac{\omega-\gamma}{\delta}$, $r = \infty$, $b = 1$

$$\begin{aligned}&\left\| \int_0^t \mathcal{G}_0 (t-\tau) (\mathcal{N}_0 (h) (\tau) - \mathcal{N}_0 (w) (\tau)) \, d\tau \right\|_{\mathbf{A}^{\rho,p}} \\ &\leq C \langle t \rangle^{1-\widetilde{\lambda}-\frac{\rho+\omega}{\delta}-\frac{1}{\delta p}} \\ &\times \sup_{p \leq q \leq \infty} \sup_{\tau > 0} \left(\{\tau\}^{\kappa_0} \langle \tau \rangle^{\widetilde{\lambda}+\frac{1}{\delta q}} \| (\mathcal{N}_0 (h) (\tau) - \mathcal{N}_0 (w) (\tau)) \|_{\mathbf{A}^{-\omega,q}} \right)\end{aligned}$$

and

$$\begin{aligned}&\left\| \int_0^t \mathcal{G}_0 (t-\tau) (\mathcal{N}_0 (h) (\tau) - \mathcal{N}_0 (w) (\tau)) \, d\tau \right\|_{\mathbf{B}^{s,p}} \\ &\leq C \{t\}^{1-\kappa_0-\frac{s+\omega}{\delta}-\frac{1}{\delta p}} \langle t \rangle^{-\widetilde{\lambda}-\frac{1}{\delta p}} \\ &\times \sup_{p \leq q \leq \infty} \sup_{\tau > 0} \left(\{\tau\}^{\kappa_0+\frac{1}{\delta q}} \langle \tau \rangle^{\widetilde{\lambda}+\frac{1}{\delta q}} \| (\mathcal{N}_0 (h) (\tau) - \mathcal{N}_0 (w) (\tau)) \|_{\mathbf{B}^{-\omega,q}} \right),\end{aligned}$$

for all $t > 0$, $s, \rho \in [0, \alpha]$, $1 \leq p \leq \infty$. Therefore,

$$\begin{aligned}&\left\| \int_0^t \mathcal{G}_0 (t-\tau) (\mathcal{N}_0 (h) (\tau) - \mathcal{N}_0 (w) (\tau)) \, d\tau \right\|_{\mathbf{X}_\gamma} \\ &\leq C\varepsilon^2 + C\varepsilon \|h - v\|_{\mathbf{X}_\gamma},\end{aligned}$$

since by Lemma 4.12 we have

$$\begin{aligned}&\{\tau\}^{\kappa_0+\frac{1}{\delta q}} \langle \tau \rangle^{\widetilde{\lambda}+\frac{1}{\delta q}} \| (\mathcal{N}_0 (h) (\tau) - \mathcal{N}_0 (w) (\tau)) \|_{\mathbf{A}^{-\omega,q}} \\ &\leq C \{\tau\}^{\frac{1+\alpha}{\delta}} \langle \tau \rangle^{\frac{1+\alpha+\gamma}{\delta}} \left(\|h - w\|_{\mathbf{A}^{\alpha,1}} + \|h\|_{\mathbf{B}^{\alpha,1}} + \|w\|_{\mathbf{B}^{\alpha,1}} \right) \\ &\times \langle \tau \rangle^{\frac{1}{\delta q}} \left(\|h\|_{\mathbf{A}^{0,q}} + \|w\|_{\mathbf{A}^{0,q}} + \|h\|_{\mathbf{B}^{0,\infty}} + \|w\|_{\mathbf{B}^{0,\infty}} \right) \\ &+ C \{\tau\}^{\frac{1+\alpha}{\delta}} \langle \tau \rangle^{\frac{1+\alpha}{\delta}} \left(\|h\|_{\mathbf{A}^{\alpha,1}} + \|w\|_{\mathbf{A}^{\alpha,1}} + \|h\|_{\mathbf{B}^{\alpha,1}} + \|w\|_{\mathbf{B}^{\alpha,1}} \right) \\ &\times \langle \tau \rangle^{\frac{\gamma}{\delta}+\frac{1}{\delta q}} \left(\|h - w\|_{\mathbf{A}^{0,q}} + \|h\|_{\mathbf{B}^{0,\infty}} + \|w\|_{\mathbf{B}^{0,\infty}} \right) \\ &\leq C \|h - v\|_{\mathbf{X}_\gamma} \left(\|h\|_{\mathbf{X}_0} + \|w\|_{\mathbf{X}_0} \right) + C \left(\|h\|_{\mathbf{X}_0} + \|w\|_{\mathbf{X}_0} \right)^2\end{aligned}$$

and

$$\begin{aligned}&\{\tau\}^{\kappa_0+\frac{1}{\delta q}} \langle \tau \rangle^{\widetilde{\lambda}+\frac{1}{\delta q}} \| (\mathcal{N}_0 (h) (\tau) - \mathcal{N}_0 (w) (\tau)) \|_{\mathbf{B}^{-\omega,q}} \\ &\leq C \{\tau\}^{\frac{1+\alpha}{\delta}} \langle \tau \rangle^{\frac{1+\alpha+\gamma}{\delta}} \left(\|h\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|w\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|h\|_{\mathbf{B}^{\alpha,1}} + \|w\|_{\mathbf{B}^{\alpha,1}} \right) \\ &\times \tau^{\frac{1}{\delta q}} \left(\|h\|_{\mathbf{A}^{0,q}} + \|w\|_{\mathbf{A}^{0,q}} + \|h\|_{\mathbf{B}^{0,q}} + \|w\|_{\mathbf{B}^{0,q}} \right) \\ &\leq C \left(\|h\|_{\mathbf{X}_0} + \|w\|_{\mathbf{X}_0} \right)^2.\end{aligned}$$

Thus we obtain

$$\|h - w\|_{\mathbf{X}_\gamma} \leq C\varepsilon. \tag{4.35}$$

Since

$$\|u(t) - h(t)\|_{\mathbf{L}^\infty} \leq \|u(t) - h(t)\|_{\mathbf{A}^{0,1}} + \|u(t) - h(t)\|_{\mathbf{B}^{0,1}},$$

via (4.32), (4.33) and (4.35), we have the estimate

$$\|u(t) - h(t)\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1+\gamma}{\delta}}$$

for all $t > 0$. Theorem 4.13 is proved.

4.3 Korteweg-de Vries-Burgers equation

This section is devoted to the famous Korteweg-de Vries-Burgers equation

$$u_t + uu_x - u_{xx} + u_{xxx} = 0, \; x \in \mathbf{R}, \; t > 0. \tag{4.36}$$

Equation (4.36) arises in many fields of physics and technology (see Naumkin and Shishmarev [1994b], and references cited therein) as a simple nonlinear model taking into account the simplest dispersion (described by the term with the third derivative) and dissipation processes (similar to the heat equation) as well as the simplest nonlinear convection effects.

The aim of this section is to remove the smallness condition for the initial data which appears when applying the results of Theorem 4.13 from the previous section. Suppose that the total mass of the initial data $\theta = \int_{\mathbf{R}} u_0(x)\, dx \neq 0$. In the case of $\theta = 0$ the nonlinearity uu_x in equation (4.36) is supercritical, so the large time asymptotics was studied in Chapter 2. Denote by

$$f_\theta(x) = -2\partial_x \log\left(\cosh\frac{\theta}{4} - \sinh\left(\frac{\theta}{4}\right)\operatorname{Erf}\left(\frac{x}{2}\right)\right)$$

the self-similar solution for the Burgers equation (see Burgers [1948])

$$u_t + u\,u_x - u_{xx} = 0, \; x \in \mathbf{R}, \; t > 0, \tag{4.37}$$

defined by the total mass $\theta = \int_{\mathbf{R}} u_0(x)\, dx$ of the initial data. Here $\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$ is the error function.

Our aim is to prove the following result, where we find the large time asymptotic behavior of solutions to the Cauchy problem for equation (4.36) in the case of initial data of arbitrary size.

Theorem 4.18. *Let the initial data* $u_0 \in \mathbf{H}^s(\mathbf{R}) \cap \mathbf{L}^1(\mathbf{R})$, *where* $s > -\frac{1}{2}$. *Then there exists a unique solution* $u(t,x) \in \mathbf{C}^\infty((0,\infty); \mathbf{H}^\infty(\mathbf{R}))$ *to the Cauchy problem for the Korteweg-de Vries-Burgers equation (4.36), which has asymptotics*

$$u(t) = t^{-\frac{1}{2}} f_\theta \left((\cdot)\, t^{-\frac{1}{2}} \right) + o\left(t^{-\frac{1}{2}} \right)$$

as $t \to \infty$, where f_θ is the self-similar solution for the Burgers equation (4.37). Moreover, if additionally the initial data $u_0 \in \mathbf{L}^{1,1}(\mathbf{R})$, then the asymptotics is true

$$u(t) = t^{-\frac{1}{2}} f_\theta \left((\cdot)\, t^{-\frac{1}{2}} \right) + O\left(t^{-\frac{1}{2}-\gamma} \right)$$

as $t \to \infty$, where $\gamma \in \left(0, \frac{1}{2}\right)$.

For the convenience of the reader we now give the idea of the proof. Multiplying equation (4.36) by $2u$ and integrating with respect to $x \in \mathbf{R}$ we get

$$\frac{d}{dt} \|u(t)\|_{\mathbf{L}^2}^2 = -2 \|u_x(t)\|_{\mathbf{L}^2}^2 ;$$

hence, by integrating we see that

$$\|u(t)\|_{\mathbf{L}^2}^2 + 2 \int_0^t \|u_x(\tau)\|_{\mathbf{L}^2}^2 \, d\tau \le \|u_0\|_{\mathbf{L}^2}^2$$

for all $t \ge 0$. In particular we have

$$\|u\|_{\infty,2} \equiv \sup_{t \ge 0} \|u(t)\|_{\mathbf{L}^2} \le \|u_0\|_{\mathbf{L}^2} \tag{4.38}$$

and

$$\|u_x\|_{2,2} \equiv \left\| \|u_x(t,x)\|_{\mathbf{L}_x^2} \right\|_{\mathbf{L}_t^2(0,\infty)} \le \|u_0\|_{\mathbf{L}^2} . \tag{4.39}$$

Note that time decay estimates (4.38) and (4.39) are not optimal. To get optimal time decay we need to show that the $\mathbf{L}^1(\mathbf{R})$ norm of the solution does not grow with time. We multiply equation (4.36) by $S(t,x) \equiv \operatorname{sign} u(t,x) \equiv \frac{u(t,x)}{|u(t,x)|}$ and integrate with respect to x over $\mathbf{R}$ to get

$$\begin{aligned} &\int_{\mathbf{R}} u_t(t,x) S(t,x)\, dx + \int_{\mathbf{R}} u(t,x) u_x(t,x) S(t,x)\, dx \\ &= \int_{\mathbf{R}} u_{xx}(t,x) S(t,x)\, dx - \int_{\mathbf{R}} u_{xxx}(t,x) S(t,x)\, dx. \end{aligned}$$

We have

$$\int_{\mathbf{R}} u_t(t,x) S(t,x)\, dx = \int_{\mathbf{R}} \frac{\partial}{\partial t} |u(t,x)|\, dx = \frac{d}{dt} \|u(t)\|_{\mathbf{L}^1} ,$$

$$\int_{\mathbf{R}} u(t,x) u_x(t,x) S(t,x)\, dx = \int_{\mathbf{R}} \frac{\partial}{\partial x} \left(|u(t,x)|\, u(t,x) \right) dx = 0,$$

and

$$\int_{\mathbf{R}} u_{xx}(t,x) S(t,x)\, dx \le 0.$$

Therefore, we find

$$\frac{d}{dt}\left\|u\left(t\right)\right\|_{\mathbf{L}^{1}} \leq -\int_{\mathbf{R}} u_{xxx}\left(t,x\right) S\left(t,x\right) dx. \quad (4.40)$$

Now the main difficulty is to attain an appropriate estimate of the right-hand side of (4.40). Lemma 4.23 below gives us a rough estimate, being however enough to improve the time decay of the $\mathbf{L}^{2}(\mathbf{R})$ norm of the solution. Then applying Lemma 4.22 and Lemma 4.24 we successively improve the time decay estimates until we obtain the optimal one.

Before proving Theorem 4.18 we yield in the next subsection some preliminary estimates of the Green operator $\mathcal{G}(t)$ solving the linearized Cauchy problem corresponding to (4.36).

4.3.1 Lemmas

Consider estimates of the Green function

$$G\left(t,x\right) = \left(2\pi\right)^{-\frac{1}{2}} \int_{\mathbf{R}} e^{ix\xi - t\xi^{2} + it\xi^{3}} d\xi.$$

By the method of the proof of Lemma 1.28 we obtain the following result.

Lemma 4.19. *The estimates are true*

$$\left\|x^{n}\partial_{x}^{\omega}G\left(t\right)\right\|_{\mathbf{L}^{p}} \leq C\left\{t\right\}^{-\frac{\omega}{2}-\frac{1}{2}\left(1-\frac{1}{p}\right)} \left\langle t\right\rangle^{\frac{n-\omega}{2}-\frac{1}{2}\left(1-\frac{1}{p}\right)}$$

for all $t>0$, $\omega, n \geq 0$, $2 \leq p \leq \infty$ *and*

$$\left\|x^{n}\partial_{x}^{\omega}G\left(t\right)\right\|_{\mathbf{L}^{p}} \leq C\left\{t\right\}^{-\frac{\omega}{2}-\frac{1}{4}} \left\langle t\right\rangle^{\frac{n-\omega}{2}-\frac{1}{2}\left(1-\frac{1}{p}\right)}$$

for all $t>0$, $\omega, n \geq 0$, $1 \leq p \leq 2$.

Denote the Green operator

$$\mathcal{G}\left(t-\tau\right)\phi\left(\tau\right) = \int_{\mathbf{R}} G\left(t-\tau, x-y\right)\phi\left(\tau,y\right) dy,$$

where the Green function $G\left(t,x\right) = \overline{\mathcal{F}}_{\xi\to x} e^{-t\xi^{2}+it\xi^{3}}$. Consider the integral equation associated with the Cauchy problem for the Korteweg - de Vries - Burgers equation (4.36)

$$u\left(t,x\right) = \mathcal{G}\left(t\right)u_{0} - \frac{1}{2}\int_{0}^{t}\partial_{x}\mathcal{G}\left(t-\tau\right)u^{2}\left(\tau\right) d\tau. \quad (4.41)$$

Define the norms

$$\left\|\phi\right\|_{p,q} \equiv \left\|\left\|\phi\left(t,x\right)\right\|_{\mathbf{L}^{q}\left(\mathbf{R}_{x}\right)}\right\|_{\mathbf{L}^{p}\left(\mathbf{R}_{t}^{+}\right)}.$$

We now estimate the first derivative of the solution.

Lemma 4.20. *Let the initial data* $u_0 \in \mathbf{W}_1^1(\mathbf{R}) \cap \mathbf{H}^1(\mathbf{R})$, *and the norms be bounded*

$$\|u_0\|_{\mathbf{W}_1^1} + \|u_0\|_{\mathbf{H}^1} + \|u\|_{\infty,2} + \|u_x\|_{2,2} \le C.$$

Then the estimate is true

$$\|u_x\|_{p,2} \le C \tag{4.42}$$

for $2 \le p \le \infty$. *Moreover if in addition*

$$\|u\|_{\alpha,2} \le C,$$

where $4 < \alpha < 6$, *then the time-decay estimate (4.42) is valid for all* $\frac{\alpha}{3} < p \le \infty$.

Proof. By virtue of the Young inequality and estimates of Lemma 4.19 we obtain from the integral equation (4.41)

$$\begin{aligned}
\|u_x(t)\|_{\mathbf{L}^2} &\le \|\partial_x \mathcal{G}(t) u_0\|_{\mathbf{L}^2} + \int_{t-\varepsilon}^{t} \|\partial_x \mathcal{G}(t-\tau) u(\tau) u_x(\tau)\|_{\mathbf{L}^2} d\tau \\
&+ \frac{1}{2}\int_0^{t-\varepsilon} \left\|\partial_x^2 \mathcal{G}(t-\tau) u^2(\tau)\right\|_{\mathbf{L}^2} d\tau \\
&\le \|u_0\|_{\mathbf{H}^1} + \int_{t-\varepsilon}^{t} \|\partial_x G(t-\tau)\|_{\mathbf{L}^2} \|u(\tau)\|_{\mathbf{L}^2} \|u_x(\tau)\|_{\mathbf{L}^2} d\tau \\
&+ C\int_0^{t-\varepsilon} \left\|\partial_x^2 G(t-\tau)\right\|_{\mathbf{L}^2} \|u(\tau)\|_{\mathbf{L}^2}^2 d\tau \\
&\le C + C\int_{t-\varepsilon}^{t} (t-\tau)^{-\frac{3}{4}} \|u_x(\tau)\|_{\mathbf{L}^2} d\tau + C\int_0^{t-\varepsilon} (t-\tau)^{-\frac{5}{4}} d\tau \\
&\le C + C\varepsilon^{\frac{1}{4}} \|u_x\|_{\infty,2} + C\varepsilon^{-\frac{1}{4}};
\end{aligned}$$

hence, by choosing $\varepsilon > 0$ so that $C\varepsilon^{\frac{1}{4}} = \frac{1}{2}$ in view of the Hölder inequality, we get estimate (4.42) for $2 \le p \le \infty$.

Now we assume that $\|u\|_{\alpha,2} \le C$, where $4 < \alpha < 6$. By the integral equation (4.41) we get

$$\begin{aligned}
\|u_x(t)\|_{\mathbf{L}^2} &\le C\langle t\rangle^{-\frac{3}{4}} \left(\|u_0\|_{\mathbf{H}^1} + \|u_0\|_{\mathbf{W}_1^1}\right) + C\langle t\rangle^{-\frac{5}{4}} \int_0^{\frac{t}{2}} \|u(\tau)\|_{\mathbf{L}^2}^2 d\tau \\
&+ C\int_{\varepsilon}^{\frac{t}{2}} \{\tau\}^{-\frac{5}{4}} \langle\tau\rangle^{-1} \|u_x(t-\tau)\|_{\mathbf{L}^2}^{\frac{1}{2}} \|u(t-\tau)\|_{\mathbf{L}^2}^{\frac{3}{2}} d\tau \\
&+ C\int_0^{\varepsilon} \tau^{-\frac{3}{4}} \|u(t-\tau)\|_{\mathbf{L}^2} \|u_x(t-\tau)\|_{\mathbf{L}^2} d\tau.
\end{aligned}$$

Since by the Hölder inequality

$$\int_0^{\frac{t}{2}} \|u(\tau)\|_{\mathbf{L}^2}^2 d\tau \le t^{1-\frac{2}{\alpha}} \left(\int_0^{\frac{t}{2}} \|u(\tau)\|_{\mathbf{L}^2}^{\alpha} d\tau\right)^{\frac{2}{\alpha}} \le C\langle t\rangle^{1-\frac{2}{\alpha}} \tag{4.43}$$

and by the Young inequality

$$\left\| \int_a^b \phi(t-\tau)\,\psi(t-\tau)\,f(\tau)\,d\tau \right\|_{\mathbf{L}^p(\mathbf{R}_t^+)} \le \|f\|_{\mathbf{L}^q(a,b)} \|\phi\|_{\mathbf{L}^r(\mathbf{R}_t^+)} \|\psi\|_{\mathbf{L}^s(\mathbf{R}_t^+)}$$

with $\frac{1}{p}+1=\frac{1}{q}+\frac{1}{r}+\frac{1}{s}$, we then obtain

$$\begin{aligned}
\|u_x\|_{p,2} &\le C \left\| \langle t \rangle^{-\frac{3}{4}} \right\|_{\mathbf{L}^p(\mathbf{R}_t^+)} + C \left\| \langle t \rangle^{-\frac{1}{4}-\frac{2}{\alpha}} \right\|_{\mathbf{L}^p(\mathbf{R}_t^+)} \\
&\quad + C \left\| \{t\}^{-\frac{5}{4}} \langle t \rangle^{-1} \right\|_{\mathbf{L}^q\left(\varepsilon,\frac{t}{2}\right)} \|u_x\|_{p,2}^{\frac{1}{2}} \|u\|_{\alpha,2}^{\frac{3}{2}} \\
&\quad + C \left\| t^{-\frac{3}{4}} \right\|_{\mathbf{L}^1(0,\varepsilon)} \|u_x\|_{p,2} \\
&\le C + C\varepsilon^{-1} \|u_x\|_{p,2}^{\frac{1}{2}} + C\varepsilon^{\frac{1}{4}} \|u_x\|_{p,2} \\
&\le C + C\varepsilon^{-\frac{9}{4}} + C\varepsilon^{\frac{1}{4}} \|u_x\|_{p,2}\,,
\end{aligned}$$

where $q = \left(1 - \frac{3}{2\alpha} + \frac{1}{2p}\right)^{-1} > 1$ if $p > \frac{\alpha}{3}$. We now choose $\varepsilon > 0$ such that $C\varepsilon^{\frac{1}{4}} = \frac{1}{2}$, and then get estimate (4.42) for $\frac{\alpha}{3} < p \le \infty$. Lemma 4.20 is proved.

Lemma 4.21. *Let the initial data* $u_0 \in \mathbf{W}_1^2(\mathbf{R}) \cap \mathbf{H}^2(\mathbf{R})$ *and the norms*

$$\|u_0\|_{\mathbf{W}_1^2} + \|u_0\|_{\mathbf{H}^2} + \|u\|_{\infty,2} + \|u_x\|_{2,2} \le C.$$

Then the estimate is true

$$\|u_{xx}\|_{p,2} \le C, \tag{4.44}$$

where $\frac{4}{3} < p \le \infty$. *Suppose in addition that*

$$\|u\|_{\alpha,2} \le C \tag{4.45}$$

where $6 \le \alpha \le \infty$, *then*

$$\|u_{xx}\|_{p,1} \le C, \tag{4.46}$$

for $\left(\frac{1}{2}+\frac{1}{\alpha}\right)^{-1} < p \le \infty$. *Finally if estimate (4.45) is true for* $4 < \alpha < 6$, *then inequality (4.46) is fulfilled with* $\frac{\alpha}{4} < p \le \infty$.

Proof. As above in view of the integral equation (4.41) we find

$$\begin{aligned}
&\|u_{xx}(t)\|_{\mathbf{L}^2} \le \left\| \partial_x^2 \mathcal{G}(t)\, u_0 \right\|_{\mathbf{L}^2} \\
&+ C \int_0^{t-\varepsilon} d\tau \left\| \partial_x^2 G(t-\tau) \right\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^2}^{\frac{1}{2}} \|u_x(\tau)\|_{\mathbf{L}^2}^{\frac{3}{2}} \\
&+ C \int_{t-\varepsilon}^{t} \|\partial_x G(t-\tau)\|_{\mathbf{L}^2} \left(\|u(\tau)\|_{\mathbf{L}^2} \|u_{xx}(\tau)\|_{\mathbf{L}^2} + \|u_x(\tau)\|_{\mathbf{L}^2}^2 \right) d\tau.
\end{aligned}$$

First we note that

$$\left\|\partial_x^2 \mathcal{G}(t)\, u_0\right\|_{\mathbf{L}^2} \leq C \langle t\rangle^{-\frac{5}{4}} \left(\|u_0\|_{\mathbf{H}^2} + \|u_0\|_{\mathbf{W}_1^2}\right).$$

By the Young inequality we obtain for $\frac{4}{3} < p \leq \infty$

$$\left\| \int_0^{t-\varepsilon} d\tau \left\|\partial_x^2 G(t-\tau)\right\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^2}^{\frac{1}{2}} \|u_x(\tau)\|_{\mathbf{L}^2}^{\frac{3}{2}} \right\|_{\mathbf{L}_t^p}$$

$$\leq C \left(\int_\varepsilon^t \{\tau\}^{-\frac{5p}{p+4}} \langle \tau\rangle^{-\frac{4p}{p+4}} d\tau \right)^{\frac{1}{p}+\frac{1}{4}} \|u\|_{\infty,2}^{\frac{1}{2}} \|u_x\|_{2,2}^{\frac{3}{2}} \leq C\varepsilon^{\frac{1}{p}-1}$$

and, similarly,

$$\left\| \int_{t-\varepsilon}^t \|\partial_x G(t-\tau)\|_{\mathbf{L}^2} \left(\|u(\tau)\|_{\mathbf{L}^2} \|u_{xx}(\tau)\|_{\mathbf{L}^2} + \|u_x(\tau)\|_{\mathbf{L}^2}^2 \right) d\tau \right\|_{\mathbf{L}_t^p}$$

$$\leq C \left\| \int_0^\varepsilon \tau^{-\frac{3}{4}} \left(\|u(t-\tau)\|_{\mathbf{L}^2} \|u_{xx}(t-\tau)\|_{\mathbf{L}^2} + \|u_x(t-\tau)\|_{\mathbf{L}^2}^2 \right) d\tau \right\|_{\mathbf{L}_t^p}$$

$$\leq C\varepsilon^{\frac{1}{4}} \left(\|u_x\|_{2p,2}^2 + \|u\|_{\infty,2} \|u_{xx}\|_{p,2} \right) \leq C + C\varepsilon^{\frac{1}{4}} \|u_{xx}\|_{p,2}.$$

Collecting these estimates we get

$$\|u_{xx}\|_{p,2} \leq C\varepsilon^{-1} + C\varepsilon^{\frac{1}{4}} \|u_{xx}\|_{p,2};$$

hence by choosing $\varepsilon > 0$ such that $C\varepsilon^{\frac{1}{4}} = \frac{1}{2}$, we obtain the first estimate (4.44).

We now prove the second estimate of the lemma. By integral equation (4.41) we find

$$\|u_{xx}(t)\|_{\mathbf{L}^1} \leq \left\|\partial_x^2 \mathcal{G}(t)\, u_0\right\|_{\mathbf{L}^1}$$

$$+ C \left\| \int_0^{\frac{t}{2}} \partial_x^3 \mathcal{G}(t-\tau)\, u^2(\tau)\, d\tau \right\|_{\mathbf{L}^1} + C \left\| \int_{\frac{t}{2}}^{t-1} \partial_x^3 \mathcal{G}(t-\tau)\, u^2(\tau)\, d\tau \right\|_{\mathbf{L}^1}$$

$$+ C \int_{t-1}^t \|\partial_x G(t-\tau)\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^2} \|u_{xx}(\tau)\|_{\mathbf{L}^2} d\tau,$$

where we have used the identity

$$\|u_x\|_{\mathbf{L}^2}^2 = -\int_{\mathbf{R}} u u_{xx} dx.$$

By the Cauchy inequality in view of (4.43) we obtain

$$
\begin{aligned}
&\left\|\int_0^{\frac{t}{2}} \partial_x^3 \mathcal{G}(t-\tau)\, u^2(\tau)\, d\tau\right\|_{\mathbf{L}^1} \\
&\le C \int_0^{\frac{t}{2}} \left\|\partial_x^2 G(t-\tau)\right\|_{\mathbf{L}^1} \|u_x(\tau)\|_{\mathbf{L}^2} \|u(\tau)\|_{\mathbf{L}^2}\, d\tau \\
&\le C \langle t\rangle^{-1} \|u_x\|_{2,2} \left(\int_0^{\frac{t}{2}} \|u(\tau)\|_{\mathbf{L}^2}^2\, d\tau\right)^{\frac{1}{2}} \le C \langle t\rangle^{-\frac{1}{2}-\frac{1}{\alpha}};
\end{aligned}
$$

hence we yield

$$
\left\|\int_0^{\frac{t}{2}} \partial_x^3 \mathcal{G}(t-\tau)\, u^2(\tau)\, d\tau\right\|_{p,1} \le C
$$

for $\left(\frac{1}{2}+\frac{1}{\alpha}\right)^{-1} < p \le \infty$. Now we estimate the next term by the Young inequality with $\frac{1}{p} = \frac{1}{q} + \frac{1}{\alpha} - \frac{1}{2}$ (that is $q > 1$)

$$
\begin{aligned}
&\left\|\int_{\frac{t}{2}}^{t-1} \partial_x^3 \mathcal{G}(t-\tau)\, u^2(\tau)\, d\tau\right\|_{p,1} \\
&\le \left\|\int_1^{\frac{t}{2}} d\tau \left\|\partial_x^2 G(\tau)\right\|_{\mathbf{L}^1} \|u(t-\tau)\|_{\mathbf{L}^2} \|u_x(t-\tau)\|_{\mathbf{L}^2}\right\|_{\mathbf{L}^p\left(\mathbf{R}_t^+\right)} \\
&\le C \|u\|_{\alpha,2} \|u_x\|_{2,2} \int_1^{\frac{t}{2}} \tau^{-q} d\tau \le C.
\end{aligned}
$$

Finally we obtain

$$
\begin{aligned}
&\left\|\int_{t-1}^{t} \|\partial_x G(t-\tau)\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^2} \|u_{xx}(\tau)\|_{\mathbf{L}^2}\, d\tau\right\|_{p,1} \\
&\le C \|u\|_{\infty,2} \|u_{xx}\|_{p,2} \int_0^{\varepsilon} \tau^{-\frac{3}{4}} d\tau \le C
\end{aligned}
$$

for $\left(\frac{1}{2}+\frac{1}{\alpha}\right)^{-1} < p \le \infty$. We also have the estimate

$$
\left\|\partial_x^2 \mathcal{G}(t)\, u_0\right\|_{\mathbf{L}^1} \le C \langle t\rangle^{-1} \left(\|u_0\|_{\mathbf{H}^2} + \|u_0\|_{\mathbf{W}_1^2}\right).
$$

Collecting these inequalities we get the second estimate of the lemma.

Now we suppose that estimate (4.45) is true for $4 < \alpha < 6$. We have from (4.41)

$$
\begin{aligned}
&\|u_{xx}(t)\|_{\mathbf{L}^1} \le C \langle t\rangle^{-1} \left(\|u_0\|_{\mathbf{H}^2} + \|u_0\|_{\mathbf{W}_1^2}\right) \\
&+ C \left\|\int_0^{\frac{t}{2}} \partial_x^3 \mathcal{G}(t-\tau)\, u^2(\tau)\, d\tau\right\|_{\mathbf{L}^1} + C \left\|\int_{\frac{t}{2}}^{t-\varepsilon} \partial_x^3 \mathcal{G}(t-\tau)\, u^2(\tau)\, d\tau\right\|_{\mathbf{L}^1} \\
&+ C \int_{t-\varepsilon}^{t} \|\partial_x G(t-\tau)\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^\infty} \|u_{xx}(\tau)\|_{\mathbf{L}^1}\, d\tau.
\end{aligned}
$$

Then by applying estimate (4.42) we obtain with $\frac{1}{p} - \frac{1}{\alpha} < \frac{1}{s} < \frac{3}{\alpha}$

$$\begin{aligned}
&\left\| \int_0^{\frac{t}{2}} \partial_x^3 \mathcal{G}(t-\tau) u^2(\tau) d\tau \right\|_{p,1} \\
&\le C \left\| \int_0^{\frac{t}{2}} d\tau \left\| \partial_x^2 G(t-\tau) \right\|_{\mathbf{L}^1} \| u(\tau) \|_{\mathbf{L}^2} \| u_x(\tau) \|_{\mathbf{L}^2} \right\|_{\mathbf{L}^p(\mathbf{R}_t^+)} \\
&\le C \left\| \langle t \rangle^{-1} \int_0^{\frac{t}{2}} \| u(\tau) \|_{\mathbf{L}^2} \| u_x(\tau) \|_{\mathbf{L}^2} d\tau \right\|_{\mathbf{L}^p(\mathbf{R}_t^+)} \\
&\le C \| u \|_{\alpha,2} \| u_x \|_{s,2} \left\| \langle t \rangle^{-\frac{1}{\alpha} - \frac{1}{s}} \right\|_{\mathbf{L}^p(\mathbf{R}_t^+)} \le C,
\end{aligned}$$

for $\frac{\alpha}{4} < p \le \infty$. Likewise by estimate (4.42) and via the Young inequality with $\frac{1}{p} - \frac{1}{\alpha} < \frac{1}{\beta} < \frac{3}{\alpha}$ so that $\frac{1}{q} = 1 + \frac{1}{p} - \frac{1}{\alpha} - \frac{1}{\beta} < 1$

$$\begin{aligned}
&\left\| \int_{\frac{t}{2}}^{t-\varepsilon} \partial_x^3 \mathcal{G}(t-\tau) u^2(\tau) d\tau \right\|_{p,1} \\
&\le C \left\| \int_\varepsilon^{\frac{t}{2}} \left\| \partial_x^2 G(\tau) \right\|_{\mathbf{L}^1} \| u(t-\tau) \|_{\mathbf{L}^2} \| u_x(t-\tau) \|_{\mathbf{L}^2} \right\|_{\mathbf{L}^p(\mathbf{R}_t^+)} \\
&\le C \left\| \{t\}^{-\frac{5}{4}} \langle t \rangle^{-1} \right\|_{\mathbf{L}^q\left(\varepsilon, \frac{t}{2}\right)} \| u \|_{\alpha,2} \| u_x \|_{\beta,2} \le C\varepsilon^{-1}.
\end{aligned}$$

Finally we attain

$$\begin{aligned}
&\left\| \int_{t-\varepsilon}^{t} \| \partial_x G(t-\tau) \|_{\mathbf{L}^1} \| u(\tau) \|_{\mathbf{L}^\infty} \| u_{xx}(\tau) \|_{\mathbf{L}^1} d\tau \right\|_{\mathbf{L}^p(\mathbf{R}_t^+)} \\
&\le C \| u \|_{\infty,2}^{\frac{1}{2}} \| u_x \|_{\infty,2}^{\frac{1}{2}} \| u_{xx} \|_{p,1} \int_0^\varepsilon \tau^{-\frac{3}{4}} d\tau \le C\varepsilon^{\frac{1}{4}}.
\end{aligned}$$

Gathering these estimates we get

$$\| u_{xx} \|_{p,1} \le C\varepsilon^{-1} + C\varepsilon^{\frac{1}{4}} \| u_{xx} \|_{p,1};$$

thus by choosing $\varepsilon > 0$ such that $C\varepsilon^{\frac{1}{4}} = \frac{1}{2}$, we obtain estimate (4.46) with $\frac{\alpha}{4} < p \le \infty$. Lemma 4.21 is proved.

Now we give estimates for the third derivative of the solution.

Lemma 4.22. *Let the initial data* $u_0 \in \mathbf{W}_1^3(\mathbf{R}) \cap \mathbf{H}^2(\mathbf{R})$ *and the norms*

$$\| u_0 \|_{\mathbf{W}_1^3} + \| u_0 \|_{\mathbf{H}^2} + \| u \|_{\infty,2} + \| u \|_{\alpha,2} + \| u_x \|_{2,2} \le C$$

where $4 \le \alpha \le \infty$. *Then the estimate is true*

$$\|u_{xxx}\|_{p,1} \leq C, \tag{4.47}$$

with $\left(\frac{3}{4}+\frac{3}{2\alpha}\right)^{-1} < p \leq \infty$ *if* $6 \leq \alpha \leq \infty$ *and with* $1 \leq p \leq \infty$ *if* $4 < \alpha < 6$.

Proof. By the integral equation (4.41) we have

$$\begin{aligned}
&\|u_{xxx}(t)\|_{\mathbf{L}^1} \leq \left\|\partial_x^3 \mathcal{G}(t) u_0\right\|_{\mathbf{L}^1} \\
&+ C\int_0^{\frac{t}{2}} \left\|\partial_x^4 G(t-\tau)\right\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^2}^2 \, d\tau \\
&+ C\left\|\int_{\frac{t}{2}}^{t-\varepsilon} \partial_x^4 \mathcal{G}(t-\tau) u^2(\tau)\, d\tau\right\|_{\mathbf{L}^1} \\
&+ C\int_{t-\varepsilon}^{t} \|\partial_x G(t-\tau)\|_{\mathbf{L}^1} \left(\|u(\tau)\|_{\mathbf{L}^\infty} \|u_{xxx}(\tau)\|_{\mathbf{L}^1}\right. \\
&\left.+ \|u_x(\tau)\|_{\mathbf{L}^2} \|u_{xx}(\tau)\|_{\mathbf{L}^2}\right) d\tau.
\end{aligned} \tag{4.48}$$

By estimate (4.43) we get

$$\begin{aligned}
&\int_0^{\frac{t}{2}} \left\|\partial_x^4 G(t-\tau)\right\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^2}^2 \, d\tau \\
&\leq Ct^{-2}\int_0^{\frac{t}{2}} \|u(\tau)\|_{\mathbf{L}^2}^2 \, d\tau \leq Ct^{-1-\frac{2}{\alpha}};
\end{aligned}$$

therefore,

$$\begin{aligned}
&\left\|\int_0^{\frac{t}{2}} \left\|\partial_x^4 G(t-\tau)\right\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^2}^2 \, d\tau\right\|_{\mathbf{L}^p(\varepsilon,\infty)} \\
&\leq C\left\|t^{-1-\frac{2}{\alpha}}\right\|_{\mathbf{L}^p(\varepsilon,\infty)} \leq C\varepsilon^{\frac{1}{p}-1-\frac{2}{\alpha}}
\end{aligned} \tag{4.49}$$

for all $1 \leq p \leq \infty$. The third summand in the right-hand side of (4.48) we represent as follows

$$\begin{aligned}
&\left\|\int_{\frac{t}{2}}^{t-\varepsilon} \partial_x^4 \mathcal{G}(t-\tau) u^2(\tau)\, d\tau\right\|_{\mathbf{L}^1} \\
&\leq C\int_{\frac{t}{2}}^{t-\varepsilon} \left\|\partial_x^2 G(t-\tau)\right\|_{\mathbf{L}^1} \left(\|u_x(\tau)\|_{\mathbf{L}^2}^2 + \|u(\tau)\|_{\mathbf{L}^\infty} \|u_{xx}(\tau)\|_{\mathbf{L}^1}\right) d\tau \\
&\leq C\int_{\frac{t}{2}}^{t-\varepsilon} \left\|\partial_x^2 G(t-\tau)\right\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^\infty} \|u_{xx}(\tau)\|_{\mathbf{L}^1}\, d\tau.
\end{aligned}$$

By the estimate of Lemma 4.19 we obtain

$$\int_{\frac{t}{2}}^{t-\varepsilon} \left\|\partial_x^2 G\left(t-\tau\right)\right\|_{\mathbf{L}^1} \left\|u\left(\tau\right)\right\|_{\mathbf{L}^\infty} \left\|u_{xx}\left(\tau\right)\right\|_{\mathbf{L}^1} d\tau$$
$$\leq C \int_{\varepsilon}^{\frac{t}{2}} \{\tau\}^{-\frac{5}{4}} \langle\tau\rangle^{-1} \left\|u\left(t-\tau\right)\right\|_{\mathbf{L}^2}^{\frac{1}{2}} \left\|u_x\left(t-\tau\right)\right\|_{\mathbf{L}^2}^{\frac{1}{2}} \left\|u_{xx}\left(t-\tau\right)\right\|_{\mathbf{L}^1} d\tau.$$

By the Young inequality with $\frac{1}{p} = \frac{1}{q} - 1 + \frac{1}{4} + \frac{1}{2\alpha} + \frac{1}{r} < \frac{3}{4} + \frac{3}{2\alpha}$, so that $\frac{1}{r} < \frac{1}{2} + \frac{1}{\alpha}$ and $q > 1$ we get

$$\left\| \int_{\frac{t}{2}}^{t-\varepsilon} \left\|\partial_x^2 G\left(t-\tau\right)\right\|_{\mathbf{L}^1} \left\|u\left(\tau\right)\right\|_{\mathbf{L}^\infty} \left\|u_{xx}\left(\tau\right)\right\|_{\mathbf{L}^1} d\tau \right\|_{\mathbf{L}^p\left(\mathbf{R}_t^+\right)}$$
$$\leq C \left\|u\right\|_{\alpha,2}^{\frac{1}{2}} \left\|u_x\right\|_{2,2}^{\frac{1}{2}} \left\|u_{xx}\right\|_{r,1} \left(\int_{\varepsilon}^{\frac{t}{2}} \{\tau\}^{-\frac{5}{4}q} \langle\tau\rangle^{-q} d\tau \right)^{\frac{1}{q}} \leq C\varepsilon^{-\frac{5}{4}}.$$

Finally we estimate

$$\left\| \int_{t-\varepsilon}^{t} \left\|\partial_x G\left(t-\tau\right)\right\|_{\mathbf{L}^1} \left(\left\|u\left(\tau\right)\right\|_{\mathbf{L}^\infty} \left\|u_{xxx}\left(\tau\right)\right\|_{\mathbf{L}^1} \right.\right.$$
$$\left.\left. + \left\|u_x\left(\tau\right)\right\|_{\mathbf{L}^2} \left\|u_{xx}\left(\tau\right)\right\|_{\mathbf{L}^2} \right) d\tau \right\|_{\mathbf{L}^p\left(\mathbf{R}_t^+\right)}$$
$$\leq C \left(\left\|u_{xxx}\right\|_{p,1} + \left\|u_x\right\|_{2p,2} \left\|u_{xx}\right\|_{2p,2} \right) \int_0^{\varepsilon} \tau^{-\frac{3}{4}} d\tau \leq C + C\varepsilon^{\frac{1}{4}} \left\|u_{xxx}\right\|_{p,1}.$$

Thus we get

$$\left\|u_{xxx}\right\|_{p,1} \leq C\varepsilon^{\frac{1}{p}-1-\frac{2}{\alpha}} + C\varepsilon^{-\frac{5}{4}} + C\varepsilon^{\frac{1}{4}} \left\|u_{xxx}\right\|_{p,1},$$

and by choosing $C\varepsilon^{\frac{1}{4}} = \frac{1}{2}$ we attain estimate (4.47) for the case $6 \leq \alpha \leq \infty$.

For the case $4 < \alpha < 6$ we need to improve the estimate of the third term in the right-hand side of (4.48). We write it as follows with small $\gamma \in (0, 1)$

$$\left\| \int_{\frac{t}{2}}^{t-\varepsilon} \partial_x^4 \mathcal{G}\left(t-\tau\right) u^2\left(\tau\right) d\tau \right\|_{\mathbf{L}^1} = \left\| \int_{\frac{t}{2}}^{t-\varepsilon} \partial_x^{2+\gamma} \mathcal{G}\left(t-\tau\right) \partial_x^{2-\gamma} u^2\left(\tau\right) d\tau \right\|_{\mathbf{L}^1}$$
$$\leq C \int_{\frac{t}{2}}^{t-\varepsilon} \left\|\partial_x^{2+\gamma} G\left(t-\tau\right)\right\|_{\mathbf{L}^1} \left\|\partial_x^{1-\gamma} u\left(\tau\right) u_x\left(\tau\right)\right\|_{\mathbf{L}^1} d\tau.$$

Note that by the Hölder inequality

$$\left\|\phi\left(\cdot+y\right) - \phi\left(\cdot\right)\right\|_{\mathbf{L}^1}$$
$$\leq \left\| \left(\int_0^y \left|\phi'\left(\cdot+z\right)\right| dz \right)^{\delta} \left(\left|\phi\left(\cdot+y\right)\right| + \left|\phi\left(\cdot\right)\right| \right)^{1-\delta} \right\|_{\mathbf{L}^1}$$
$$\leq C \left\| \int_0^y \left|\phi'\left(\cdot+z\right)\right| dz \right\|_{\mathbf{L}^1}^{\delta} \left\|\phi\right\|_{\mathbf{L}^1}^{1-\delta} \leq Cy^{\delta} \left\|\partial_x \phi\right\|_{\mathbf{L}^1}^{\delta} \left\|\phi\right\|_{\mathbf{L}^1}^{1-\delta}.$$

Hence by taking $0 < \delta_1 < 1-\gamma < \delta_2 < 1$ we get (we can take δ_j close to 1)

$$\begin{aligned}
\left\|\partial_x^{1-\gamma}\phi\right\|_{\mathbf{L}^1} &= C\left\|\int_0^\infty (\phi(\cdot+y)-\phi(\cdot))\frac{dy}{y^{2-\gamma}}\right\|_{\mathbf{L}^1} \\
&\le C\int_1^\infty \frac{dy}{y^{2-\gamma}}\|\phi(\cdot+y)-\phi(\cdot)\|_{\mathbf{L}^1} \\
&+ C\int_0^1 \frac{dy}{y^{2-\gamma}}\|\phi(\cdot+y)-\phi(\cdot)\|_{\mathbf{L}^1} \\
&\le C\|\partial_x\phi\|_{\mathbf{L}^1}^{\delta_1}\|\phi\|_{\mathbf{L}^1}^{1-\delta_1} + C\|\partial_x\phi\|_{\mathbf{L}^1}^{\delta_2}\|\phi\|_{\mathbf{L}^1}^{1-\delta_2}.
\end{aligned}$$

By Lemma 4.23 we have

$$\left\|\partial_x^{2+\gamma}G(t-\tau)\right\|_{\mathbf{L}^1} \le C\{t\}^{-\frac{5}{4}-\frac{\gamma}{4}}\langle t\rangle^{-1-\frac{\gamma}{2}}.$$

Therefore, we obtain

$$\begin{aligned}
&\left\|\int_{\frac{t}{2}}^{t-\varepsilon}\partial_x^4\mathcal{G}(t-\tau)u^2(\tau)\,d\tau\right\|_{\mathbf{L}^1} \le C\int_{\frac{t}{2}}^{t-\varepsilon} d\tau\,\{t-\tau\}^{-\frac{5}{4}-\frac{\gamma}{4}}\langle t-\tau\rangle^{-1-\frac{\gamma}{2}} \\
&\times\sum_{j=1}^2\left(\|u(\tau)\|_{\mathbf{L}^2}^{1-\delta_j}\|u_x(\tau)\|_{\mathbf{L}^2}^{1+\delta_j} + \|u(\tau)\|_{\mathbf{L}^2}^{1-\frac{\delta_j}{2}}\|u_x(\tau)\|_{\mathbf{L}^2}^{1-\frac{\delta_j}{2}}\|u_{xx}(\tau)\|_{\mathbf{L}^1}^{\delta_j}\right).
\end{aligned}$$

By the Hölder inequality with $q = \frac{\alpha}{\alpha-1+\delta_j}$ so that $(1+\delta_j)q > \frac{\alpha}{3}$ and $\frac{2-\delta_j}{2\alpha} + \frac{1}{r} + \frac{1}{s} = 1$ so that $\left(1-\frac{\delta_j}{2}\right)r > \frac{\alpha}{3}$ and $\delta_j s > \frac{\alpha}{4}$, we get

$$\begin{aligned}
&\left\|\int_{\frac{t}{2}}^{t-\varepsilon}\partial_x^4\mathcal{G}(t-\tau)u^2(\tau)\,d\tau\right\|_{1,1} \\
&\le C\varepsilon^{-\frac{1}{4}-\frac{\gamma}{4}}\sum_{j=1}^2\left(\left\|\|u(t)\|_{\mathbf{L}^2}^{1-\delta_j}\|u_x(t)\|_{\mathbf{L}^2}^{1+\delta_j}\right\|_{\mathbf{L}^1(\mathbf{R}_t^+)}\right. \\
&\left.+\left\|\|u(t)\|_{\mathbf{L}^2}^{1-\frac{\delta_j}{2}}\|u_x(t)\|_{\mathbf{L}^2}^{1-\frac{\delta_j}{2}}\|u_{xx}(t)\|_{\mathbf{L}^1}^{\delta_j}\right\|_{\mathbf{L}^1(\mathbf{R}_t^+)}\right) \\
&\le C\varepsilon^{-\frac{1}{4}-\frac{\gamma}{4}}\sum_{j=1}^2\left(\|u\|_{\alpha,2}^{\frac{1-\delta_j}{\alpha}}\|u_x\|_{(1+\delta_j)q,2}^{\frac{1}{q}} + \|u\|_{\alpha,2}^{\frac{2-\delta_j}{2\alpha}}\|u_x\|_{\left(1-\frac{\delta_j}{2}\right)r,2}^{\frac{1}{r}}\|u_{xx}\|_{\delta_j s,1}^{\frac{1}{s}}\right) \\
&\le C\varepsilon^{-\frac{1}{4}-\frac{\gamma}{4}}.
\end{aligned}$$

Thus in the case of $4 < \alpha < 6$ estimate (4.47) is true for all $1 \le p \le \infty$. Lemma 4.22 is proved.

Now we offer estimates for the right-hand side of (4.40). Denote $S(t,x) = 1$ for all $u(t,x) > 0$ and $S(t,x) = -1$ for all $u(t,x) < 0$; $S(t,x) = 0$. for $u(t,x) = 0$.

Lemma 4.23. *Let the initial data* $u_0 \in \mathbf{W}_1^3(\mathbf{R}) \cap \mathbf{H}^3(\mathbf{R})$ *and the norms*

$$\|u_0\|_{\mathbf{W}_1^3} + \|u_0\|_{\mathbf{H}^2} + \|u\|_{\infty,2} + \|u_x\|_{2,2} \leq C.$$

Then the estimate is true

$$\left| \int_0^T dt \int_{\mathbf{R}} u_{xxx}(t,x) S(t,x) \, dx \right| \leq C \langle T \rangle^{\frac{1}{6}+\gamma}$$

for all $T > 0,$ *where* $\gamma > 0$ *is small.*

Proof. Note that by Lemma 4.22 we have

$$\left| \int_0^1 dt \int_{\mathbf{R}} u_{xxx}(t,x) S(t,x) \, dx \right| \leq \int_0^1 \|u_{xxx}(t)\|_{\mathbf{L}^1} dt \leq \|u_{xxx}\|_{\infty,1} \leq C.$$

Therefore we need to estimate the time growth of the integral

$$\int_1^T dt \int_{\mathbf{R}} u_{xxx}(t,x) S(t,x) \, dx.$$

By the integral equation (4.41) we have

$$\begin{aligned} u_{xxx}(t,x) &= \partial_x^3 \mathcal{G}(t) u_0 + \frac{1}{2} \int_0^{\frac{t}{2}} \partial_x^4 \mathcal{G}(t-\tau) u^2(\tau) \, d\tau \\ &+ \int_{\frac{t}{2}}^{t-\nu t} \partial_x^3 \mathcal{G}(t-\tau) u(\tau) u_x(\tau) \, d\tau \\ &+ \int_{t-\nu t}^{t-1} \partial_x^3 \mathcal{G}(t-\tau) u(\tau) u_x(\tau) \, d\tau \\ &+ \int_{t-1}^{t} \partial_x \mathcal{G}(t-\tau) \partial_x^2 (u(\tau) u_x(\tau)) \, d\tau \end{aligned} \tag{4.50}$$

for all $t > 1$, where $\nu = T^{-\frac{1}{3}}$. The first summand in the right-hand side of (4.50) can be estimated as

$$\left\| \partial_x^3 \mathcal{G}(t) u_0 \right\|_{\mathbf{L}^1} \leq C \langle t \rangle^{-\frac{3}{2}} \left(\|u_0\|_{\mathbf{W}_1^3} + \|u_0\|_{\mathbf{H}^3} \right).$$

For the second term we have

$$\left\| \int_0^{\frac{t}{2}} \partial_x^4 \mathcal{G}(t-\tau) u^2(\tau) \, d\tau \right\|_{\mathbf{L}^1} \leq \int_0^{\frac{t}{2}} \left\| \partial_x^4 G(t-\tau) \right\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^2}^2 \, d\tau \leq C t^{-1}$$

for $t > 1$; hence

$$\int_1^T dt \int_{\mathbf{R}} S(t,x) \int_0^{\frac{t}{2}} \partial_x^4 \mathcal{G}(t-\tau) u^2(\tau) \, d\tau dx$$
$$\leq \int_1^T dt \left\| \int_0^{\frac{t}{2}} \partial_x^4 \mathcal{G}(t-\tau) u^2(\tau) \, d\tau \right\|_{\mathbf{L}^1}$$
$$\leq C \int_1^T \frac{dt}{t} \leq C \log(T+1).$$

Consider the third summand in the right-hand side of (4.50)

$$\int_1^T dt \left\| \int_{\frac{t}{2}}^{t-\nu t} \partial_x^3 \mathcal{G}(t-\tau) u(\tau) u_x(\tau) \, d\tau \right\|_{\mathbf{L}^1}$$
$$\leq \int_1^T dt \int_{\nu t}^{\frac{t}{2}} \left\| \partial_x^3 G(\tau) \right\|_{\mathbf{L}^1} \|u(t-\tau)\|_{\mathbf{L}^2} \|u_x(t-\tau)\|_{\mathbf{L}^2} \, d\tau$$
$$\leq C \int_1^T dt \int_{\nu t}^{\frac{t}{2}} \tau^{-\frac{3}{2}} \|u\|_{\infty,2} \|u_x(t-\tau)\|_{\mathbf{L}^2} \, d\tau$$
$$\leq \frac{C}{\sqrt{\nu}} \int_1^T dt \int_{\nu t}^{\frac{t}{2}} \tau^{-1} t^{-\frac{1}{2}} \|u_x(t-\tau)\|_{\mathbf{L}^2} \, d\tau.$$

Changing the order of integration and applying the Schwartz inequality we obtain

$$\int_1^T dt \int_{\nu t}^{\frac{t}{2}} \tau^{-1} t^{-\frac{1}{2}} \|u_x(t-\tau)\|_{\mathbf{L}^2} \, d\tau$$
$$= \int_\nu^{T/2} \frac{d\tau}{\tau} \int_{\max(1,2\tau)}^{\min(T,\tau/\nu)} t^{-\frac{1}{2}} \|u_x(t-\tau)\|_{\mathbf{L}^2} \, d\tau$$
$$\leq C \sqrt{\log(T+1)} \|u_x\|_{2,2} \int_\nu^{T/2} \frac{d\tau}{\tau} \leq C (\log(T+1))^{\frac{3}{2}}.$$

Thus

$$\int_1^T dt \left\| \int_{\frac{t}{2}}^{t-\nu t} \partial_x^3 \mathcal{G}(t-\tau) u(\tau) u_x(\tau) \, d\tau \right\|_{\mathbf{L}^1} \leq C T^{\frac{1}{6}} (\log(T+1))^{\frac{3}{2}}.$$

We now estimate the forth term in the right-hand side of (4.50)

$$\int_{\mathbf{R}} S(t,x) \int_{t-\nu t}^{t-1} \partial_x^3 \mathcal{G}(t-\tau) u(\tau) u_x(\tau) d\tau dx$$
$$= \int_{\mathbf{R}} dx |u(t,x)| \int_{t-\nu t}^{t-1} d\tau \int_{\mathbf{R}} \partial_x^3 G(t-\tau, x-y) u_y(\tau, y) dy$$
$$+ \int_{\mathbf{R}} dx S(t,x) \int_{t-\nu t}^{t-1} d\tau$$
$$\times \int_{\mathbf{R}} \partial_x^3 G(t-\tau, x-y)(u(\tau,y) - u(\tau,x)) u_y(\tau,y) dy$$
$$+ \int_{\mathbf{R}} dx S(t,x) \int_{t-\nu t}^{t-1} d\tau (u(\tau,x) - u(t,x))$$
$$\times \int_{\mathbf{R}} \partial_x^3 G(t-\tau, x-y) u_y(\tau,y) dy = I_1 + I_2 + I_3.$$

In the integral I_1 we integrate by parts to get

$$I_1 = \int_{\mathbf{R}} dx |u(t,x)| \int_{t-\nu t}^{t-1} d\tau \int_{\mathbf{R}} \partial_x^3 G(t-\tau, x-y) u_y(\tau,y) dy$$
$$= -\int_{\mathbf{R}} dx u_x(t,x) S(t,x) \int_{t-\nu t}^{t-1} d\tau \int_{\mathbf{R}} G_{yy}(t-\tau, y) u_x(\tau, x-y) dy;$$

hence

$$\int_1^T |I_1(t)| dt \le \int_1^T dt \|u_x(t)\|_{\mathbf{L}^2} \int_{t-\nu t}^{t-1} \frac{d\tau}{t-\tau} \|u_x(\tau)\|_{\mathbf{L}^2}$$
$$\le C \int_1^T dt \|u_x(t)\|_{\mathbf{L}^2} \int_1^{\nu t} \frac{d\tau}{\tau} \|u_x(t-\tau)\|_{\mathbf{L}^2}$$
$$\le C \|u_x\|_{2,2}^2 \log(T+1) \le C \log(T+1).$$

In the integral I_2 using the identity $u(\tau, x-y) - u(\tau, x) = \int_0^y u_x(t, x-z) dz$ we have

$$I_2 = \int_{\mathbf{R}} dx S(t,x) \int_{t-\nu t}^{t-1} d\tau$$
$$\times \int_{\mathbf{R}} \partial_y^3 G(t-\tau, y)(u(\tau, x-y) - u(\tau,x)) u_x(\tau, x-y) dy$$
$$= \int_{\mathbf{R}} dx S(t,x) \int_{t-\nu t}^{t-1} d\tau$$
$$\times \int_{\mathbf{R}} \partial_y^3 G(t-\tau, y) \int_0^y u_x(t, x-z) dz u_x(\tau, x-y) dy$$
$$= \int_{t-\nu t}^{t-1} d\tau \int_{\mathbf{R}} dy \partial_y^3 G(t-\tau, y)$$
$$\times \int_{\mathbf{R}} dx u_x(\tau, x-y) S(t,x) \int_0^y u_x(t, x-z) dz;$$

hence by the Cauchy inequality we estimate

$$\left|\int_{\mathbf{R}} dx u_x(\tau, x-y) S(t,x) \int_0^y u_x(t, x-z)\, dz\right|$$
$$\leq \|u_x(\tau,x)\|_{\mathbf{L}^2_x} \left\|\int_0^y u_x(t,x-z)\,dz\right\|_{\mathbf{L}^2_x} \leq |y| \, \|u_x(\tau)\|_{\mathbf{L}^2} \|u_x(t)\|_{\mathbf{L}^2}.$$

Therefore,

$$|I_2(t)| \leq \int_{t-\nu t}^{t-1} d\tau \, \|u_x(\tau,x)\|_{\mathbf{L}^2_x} \int_{\mathbf{R}} \left|\partial_y^3 G(t-\tau, y)\right|$$
$$\times \left\|\int_0^y u_x(t,x-z)\,dz\right\|_{\mathbf{L}^2_x} dy$$
$$\leq \|u_x(t)\|_{\mathbf{L}^2} \int_{t-\nu t}^{t-1} \|u_x(\tau)\|_{\mathbf{L}^2} \left\| |y| \, \partial_y^3 G(t-\tau,y)\right\|_{\mathbf{L}^1} d\tau$$
$$\leq C \|u_x(t)\|_{\mathbf{L}^2} \int_{t-\nu t}^{t-1} \frac{d\tau}{t-\tau} \|u_x(\tau)\|_{\mathbf{L}^2}.$$

Finally by the Young inequality,

$$\int_1^T |I_2(t)|\,dt \leq C \int_1^T dt \, \|u_x(t)\|_{\mathbf{L}^2} \int_1^{t\nu} \frac{d\tau}{\tau} \|u_x(t-\tau)\|_{\mathbf{L}^2}$$
$$\leq C \|u_x\|_{2,2}^2 \log(T+1) \leq C \log(T+1).$$

To estimate I_3 we use equation (4.36)

$$u(t,x) - u(t-\tau,x)$$
$$= \int_0^\tau u_t(t-t',x)\,dt' = -\int_0^\tau u(t-t',x)\, u_x(t-t',x)\,dt'$$
$$+ \int_0^\tau u_{xx}(t-t',x)\,dt' - \int_0^\tau u_{xxx}(t-t',x)\,dt',$$

then we have

$$
\begin{aligned}
&|I_3(t)| \le \int_1^{\nu t} d\tau \int_0^{\tau} \|u(t-t')\,u_x(t-t')\|_{\mathbf{L}^2}\,dt' \\
&\times \left\|\partial_x^3 \mathcal{G}(\tau)\,u_x(t-\tau)\right\|_{\mathbf{L}^2} \\
&+ \int_1^{\nu t} d\tau \int_0^{\tau} \|u_{xx}(t-t')\|_{\mathbf{L}^2}\,dt' \left\|\partial_x^3 \mathcal{G}(\tau)\,u_x(t-\tau)\right\|_{\mathbf{L}^2} \\
&+ \int_1^{\nu t} d\tau \int_0^{\tau} \|u_{xxx}(t-t')\|_{\mathbf{L}^1}\,dt' \left\|\partial_x^3 \mathcal{G}(\tau)\,u_x(t-\tau)\right\|_{\mathbf{L}^\infty} \\
&\le C \int_1^{\nu t} \left\|\partial_x^3 G(\tau)\right\|_{\mathbf{L}^1} d\tau\, \|u_x(t-\tau)\|_{\mathbf{L}^2} \int_0^{\tau} \|u_x(t-t')\|_{\mathbf{L}^2}^{\frac{3}{2}}\,dt' \\
&C \int_1^{\nu t} \left\|\partial_x^3 G(\tau)\right\|_{\mathbf{L}^1} d\tau\, \|u_x(t-\tau)\|_{\mathbf{L}^2} \int_0^{\tau} \|u_{xx}(t-t')\|_{\mathbf{L}^2}\,dt' \\
&+ C \int_1^{\nu t} \left\|\partial_x^3 G(\tau)\right\|_{\mathbf{L}^1} d\tau \int_0^{\tau} \|u_x(t-\tau)\|_{\mathbf{L}^\infty} \|u_{xxx}(t-t')\|_{\mathbf{L}^1}\,dt'.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_1^T |I_3(t)|\,dt \\
&\le C \int_1^T dt \int_1^{\nu t} \tau^{-\frac{3}{2}} d\tau\, \|u_x(t-\tau)\|_{\mathbf{L}^2} \int_0^{\tau} \|u_x(t-t')\|_{\mathbf{L}^2}^{\frac{3}{2}}\,dt' \\
&+ C \int_1^T dt \int_1^{\nu t} \tau^{-\frac{3}{2}} d\tau\, \|u_x(t-\tau)\|_{\mathbf{L}^2} \int_0^{\tau} \|u_{xx}(t-t')\|_{\mathbf{L}^2}\,dt' \\
&+ C \int_1^T dt \int_1^{\nu t} \tau^{-\frac{3}{2}} d\tau\, \|u_x(t-\tau)\|_{\mathbf{L}^\infty} \int_0^{\tau} \|u_{xxx}(t-t')\|_{\mathbf{L}^1}\,dt' \\
&= I_4 + I_5 + I_6.
\end{aligned}
$$

Using the Young inequality $ab^{\frac{3}{2}} \le a^2 b^{\frac{1}{2}} + a^{\frac{1}{2}} b^2$ we obtain

$$
\begin{aligned}
I_4 &= \int_1^T dt \int_1^{\nu t} \tau^{-\frac{3}{2}} d\tau\, \|u_x(t-\tau)\|_{\mathbf{L}^2} \int_0^{\tau} \|u_x(t-t')\|_{\mathbf{L}^2}^{\frac{3}{2}}\,dt' \\
&\le \int_1^T dt \int_1^{\nu t} \tau^{-\frac{3}{2}} d\tau\, \|u_x(t-\tau)\|_{\mathbf{L}^2}^{\frac{1}{2}} \int_0^{\tau} \|u_x(t-t')\|_{\mathbf{L}^2}^{2}\,dt' \\
&+ \int_1^T dt \int_1^{\nu t} \tau^{-\frac{3}{2}} d\tau\, \|u_x(t-\tau)\|_{\mathbf{L}^2}^{2} \int_0^{\tau} \|u_x(t-t')\|_{\mathbf{L}^2}^{\frac{1}{2}}\,dt'.
\end{aligned}
$$

Via the Hölder inequality

$$
\int_{t'}^{\nu t} \tau^{-\frac{3}{2}} \|u_x(t-\tau)\|_{\mathbf{L}^2}^{\frac{1}{2}}\,d\tau \le C\,(t')^{-\frac{3}{4}} \|u_x\|_{2,2}^{\frac{1}{2}} \le C\,(t')^{-\frac{3}{4}} \tag{4.51}
$$

and

$$
\int_0^{\tau} \|u_x(t-t')\|_{\mathbf{L}^2}^{\frac{1}{2}}\,dt' \le \tau^{\frac{3}{4}} \|u_x\|_{2,2}^{\frac{1}{2}} \le C\tau^{\frac{3}{4}}.
$$

Therefore, by changing the order of integration we obtain

$$\begin{aligned}
I_4 &= \int_1^{T\nu} dt' \int_{t'/\nu}^{T} dt \left\| u_x (t-t') \right\|_{\mathbf{L}^2}^2 \int_{t'}^{\nu t} \tau^{-\frac{3}{2}} \left\| u_x (t-\tau) \right\|_{\mathbf{L}^2}^{\frac{1}{2}} d\tau \\
&+ \int_1^{T\nu} \tau^{-\frac{3}{2}} d\tau \int_{\tau/\nu}^{T} dt \left\| u_x (t-\tau) \right\|_{\mathbf{L}^2}^2 \int_0^{\tau} \left\| u_x (t-t') \right\|_{\mathbf{L}^2}^{\frac{1}{2}} dt' \\
&\le C \int_1^{T\nu} (t')^{-\frac{3}{4}} dt' \le C (T\nu)^{\frac{1}{4}} \le C T^{\frac{1}{6}}
\end{aligned}$$

since $\nu = T^{-\frac{1}{3}}$. We now estimate I_5. By the Young inequality $ab \le a^{\frac{1}{2}} b^p + a^2 b^{3-2p}$ with $p > \frac{4}{3}$ we have

$$\begin{aligned}
I_5 &= C \int_1^{T} dt \int_1^{\nu t} \tau^{-\frac{3}{2}} d\tau \left\| u_x (t-\tau) \right\|_{\mathbf{L}^2} \int_0^{\tau} \left\| u_{xx} (t-t') \right\|_{\mathbf{L}^2} dt' \\
&\le \int_1^{T} dt \int_1^{\nu t} \tau^{-\frac{3}{2}} d\tau \left\| u_x (t-\tau) \right\|_{\mathbf{L}^2}^{\frac{1}{2}} \int_0^{\tau} \left\| u_{xx} (t-t') \right\|_{\mathbf{L}^2}^{p} dt' \\
&+ \int_1^{T} dt \int_1^{\nu t} \tau^{-\frac{3}{2}} d\tau \left\| u_x (t-\tau) \right\|_{\mathbf{L}^2}^{2} \int_0^{\tau} \left\| u_{xx} (t-t') \right\|_{\mathbf{L}^2}^{3-2p} dt'.
\end{aligned}$$

Via the Hölder inequality we get by Lemma 4.21 with $p > \frac{4}{3}$

$$\int_0^{\tau} \left\| u_{xx} (t-t') \right\|_{\mathbf{L}^2}^{3-2p} dt' \le C \tau^{3-\frac{3}{p}} \left\| u_{xx} \right\|_{p,2}^{3-2p} \le C \tau^{3-\frac{3}{p}}.$$

Therefore, by changing the order of integration by virtue of (4.51) we obtain

$$\begin{aligned}
I_5 &= \int_0^{T\nu} dt' \int_{t'/\nu}^{T} dt \left\| u_{xx} (t-t') \right\|_{\mathbf{L}^2}^{p} \int_{t'}^{\nu t} \tau^{-\frac{3}{2}} \left\| u_x (t-\tau) \right\|_{\mathbf{L}^2}^{\frac{1}{2}} d\tau \\
&+ C \int_0^{T\nu} d\tau \tau^{-\frac{3}{2}} \int_{\frac{\tau}{\nu}}^{T} dt \left\| u_x (t-\tau) \right\|_{\mathbf{L}^2}^{2} \int_0^{\tau} \left\| u_{xx} (t-t') \right\|_{\mathbf{L}^2}^{3-2p} dt' \\
&\le C \int_0^{T\nu} (t')^{-\frac{3}{4}} dt' + C \int_0^{T\nu} \tau^{\frac{3}{2}-\frac{3}{p}} d\tau \int_{\frac{\tau}{\nu}}^{T} dt \left\| u_x (t-\tau) \right\|_{\mathbf{L}^2}^{2} \\
&\le C (T\nu)^{\frac{1}{4}} + C (T\nu)^{\frac{5}{2}-\frac{3}{p}} \le C T^{\frac{1}{6}+\gamma}
\end{aligned}$$

where $\gamma > 0$ is small. The integral I_6 is estimated similarly since

$$\left\| u_x \right\|_{\mathbf{L}^\infty} \le \left\| u_x \right\|_{\mathbf{L}^2}^{\frac{1}{2}} \left\| u_{xx} \right\|_{\mathbf{L}^2}^{\frac{1}{2}}.$$

Then for the last summand in (4.50) we use estimates of Lemma 4.21 and Lemma 4.22. Thus Lemma 4.23 is proved.

Now we estimate the decay rate of the $\mathbf{L}^2 (\mathbf{R})$ norm of the solutions.

Lemma 4.24. *Let*

$$\left| \int_0^t d\tau \int_{\mathbf{R}} u_{xxx}(\tau, x) S(\tau, x) \, dx \right| \leq C \langle t \rangle^{\beta}$$

for all $t > 0$, *where* $\beta \in \left[0, \frac{1}{4}\right)$. *Then the estimates are valid*

$$\|u(t)\|_{\mathbf{L}^1} \leq C(1+t)^{\beta}$$

and

$$\|u(t)\|_{\mathbf{L}^2} \leq C(1+t)^{\beta - \frac{1}{4}}$$

for all $t > 0$.

Proof. First let us estimate the $\mathbf{L}^1(\mathbf{R})$ norm of the solution. We multiply equation (4.36) by $S(t,x)$ and integrate with respect to x over $\mathbf{R}$ to get

$$\begin{aligned}
&\int_{\mathbf{R}} u_t(t,x) S(t,x) \, dx + \int_{\mathbf{R}} u(t,x) u_x(t,x) S(t,x) \, dx \\
&= \int_{\mathbf{R}} u_{xx}(t,x) S(t,x) \, dx - \int_{\mathbf{R}} u_{xxx}(t,x) S(t,x) \, dx.
\end{aligned}$$

We have by (1.31)

$$\begin{aligned}
\int_{\mathbf{R}} u_t(t,x) S(t,x) \, dx &= \frac{d}{dt} \|u(t)\|_{\mathbf{L}^1}, \\
\int_{\mathbf{R}} u(t,x) u_x(t,x) S(t,x) \, dx &= 0, \\
\int_{\mathbf{R}} u_{xx}(t,x) S(t,x) \, dx &\leq 0.
\end{aligned}$$

Therefore we get

$$\frac{d}{dt} \|u(t)\|_{\mathbf{L}^1} \leq - \int_{\mathbf{R}} u_{xxx}(t,x) S(t,x) \, dx. \tag{4.52}$$

Integration of inequality (4.52) yields

$$\|u(t)\|_{\mathbf{L}^1} \leq \|u_0\|_{\mathbf{L}^1} + C \langle t \rangle^{\beta};$$

hence we have

$$\sup_{\xi \in \mathbf{R}} |\widehat{u}(t,\xi)| \leq (2\pi)^{-\frac{1}{2}} \|u(t)\|_{\mathbf{L}^1} \leq C \langle t \rangle^{\beta}. \tag{4.53}$$

Thus the first estimate of the lemma is fulfilled. To prove the second estimate we multiply equation (4.36) by $2u$ and integrate with respect to $x \in \mathbf{R}$ to get

$$\frac{d}{dt} \|u(t)\|_{\mathbf{L}^2}^2 = -2 \|u_x(t)\|_{\mathbf{L}^2}^2. \tag{4.54}$$

By the Plancherel theorem using the Fourier splitting method due to Schonbek [1995], we have

$$\begin{aligned} \|u_x(t)\|_{\mathbf{L}^2}^2 = \|\xi\widehat{u}(t)\|_{\mathbf{L}^2}^2 &= \int_{|\xi|\le\delta} |\widehat{u}(t,\xi)|^2 \xi^2 d\xi + \int_{|\xi|\ge\delta} |\widehat{u}(t,\xi)|^2 \xi^2 d\xi \\ &\ge \delta^2 \|u(t)\|_{\mathbf{L}^2}^2 - 2\delta^3 \sup_{|\xi|\le\delta} |\widehat{u}(t,\xi)|^2, \end{aligned}$$

where $\delta > 0$. Thus from (4.54) we have the inequality

$$\frac{d}{dt} \|u(t)\|_{\mathbf{L}^2}^2 \le -2\delta^2 \|u(t)\|_{\mathbf{L}^2}^2 + 4\delta^3 \sup_{|\xi|\le\delta} |\widehat{u}(t,\xi)|^2. \tag{4.55}$$

We choose $\delta = (1+t)^{-\frac{1}{2}}$ and change $\|u(t)\|_{\mathbf{L}^2}^2 = (1+t)^{-2} W(t)$. Then via (4.53) we get from (4.55)

$$\frac{d}{dt} W(t) \le 4C(1+t)^{2\beta+\frac{1}{2}}. \tag{4.56}$$

Integration of (4.56) with respect to time yields

$$W(t) \le \|u_0\|_{\mathbf{L}^2}^2 + C\left((1+t)^{2\beta+\frac{3}{2}} - 1\right).$$

Therefore we obtain the second estimate of the lemma, and Lemma 4.24 is proved.

4.3.2 Proof of Theorem 4.18

By the local smoothing property of the parabolic type equations we see that the solutions to the Cauchy problem for the Korteweg-de Vries-Burgers equation (4.36) become smooth

$$u(t,x) \in \mathbf{C}^1\left((0,t_0];\mathbf{H}^\infty(\mathbf{R})\right)$$

for $t_0 > 0$ (see Theorem 2.49 from Section 2.5, Chapter 2). Changing the initial time to $t_0 > 0$, we can suppose that the initial data $u_0 \in \mathbf{H}^3(\mathbf{R}) \cap \mathbf{W}_1^3(\mathbf{R})$. Now we can revise the above time decay estimate (4.38) for the $\mathbf{L}^2(\mathbf{R})$ norm of the solution. We apply Lemma 4.23 to get

$$\left|\int_0^t d\tau \int_{\mathbf{R}} u_{xxx}(\tau,x) S(\tau,x) dx\right| \le C\langle t\rangle^{\beta_0} \tag{4.57}$$

for all $t > 0$, where $\beta_0 = \frac{1}{6} + \gamma$, $\gamma > 0$ is small. Then by Lemma 4.24 we find the time decay of the $\mathbf{L}^2(\mathbf{R})$ norm

$$\|u\|_{\alpha_0,2} \le C \tag{4.58}$$

where $\alpha_0 > \left(\frac{1}{4} - \beta_0\right)^{-1}$. We can choose $\alpha_0 = 12 + O(\gamma)$. Now we apply Lemma 4.22 with $\alpha_0 = 12 + O(\gamma)$. Then by the Hölder inequality we obtain

$$\left| \int_0^t d\tau \int_{\mathbf{R}} u_{xxx}(\tau, x) S(\tau, x) dx \right| \leq C t^{1-\frac{1}{p_0}} \|u_{xxx}\|_{p_0,1} \leq C t^{1-\frac{1}{p_0}},$$

for all $t > 0$, where $p_0 > \left(\frac{3}{4} + \frac{3}{2\alpha_0}\right)^{-1}$. Hence we arrive at the estimate (4.57) with β_0 replaced by $\beta_1 = 1 - \frac{1}{p_0} = \frac{1}{8} + O(\gamma)$. We again apply Lemma 4.24 to get better time decay of the $\mathbf{L}^2(\mathbf{R})$ norm (4.58) with α_0 replaced by $\alpha_1 = 8 + O(\gamma)$. Namely

$$\|u\|_{\alpha_1,2} \leq C.$$

Then Lemma 4.22 yields estimate (4.57) with β_0 replaced by $\beta_2 = 1 - \frac{3}{4} - \frac{3}{2\alpha_1} + O(\gamma) = \frac{1}{16} + O(\gamma)$. Now by Lemma 4.24 we attain time decay estimate (4.58) with α_0 replaced by $\alpha_2 = \frac{4}{1-4\beta_2} + O(\gamma) = \frac{16}{3} + O(\gamma) < 6$. Lemma 4.22 now gives us estimate (4.57) with $p = 1$, that is $\beta = 0$. Therefore, by virtue of Lemma 4.24, we obtain an optimal time decay estimate of the $\mathbf{L}^2(\mathbf{R})$ norm

$$\|u(t)\|_{\mathbf{L}^2} \leq C(1+t)^{-\frac{1}{4}} \tag{4.59}$$

for all $t > 0$. Using (4.59) we can prove the following optimal time decay estimates

$$\|u(t)\|_{\mathbf{L}^p} \leq C \langle t \rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \tag{4.60}$$

for all $t > 0$, where $1 \leq p \leq \infty$. For $1 \leq p \leq 2$ estimate (4.60) follows from (4.59), Lemma 4.24 and the Hölder inequality. Let us prove (4.60) for $p = \infty$. By the integral equation (4.41) and by Hölder inequality we get

$$\begin{aligned}
\|u(t)\|_{\mathbf{L}^\infty} &\leq \|\mathcal{G}(t) u_0\|_{\mathbf{L}^\infty} + \frac{1}{2} \int_0^{\frac{t}{2}} \|\partial_x G(t-\tau)\|_{\mathbf{L}^\infty} \|u(\tau)\|_{\mathbf{L}^2}^2 d\tau \\
&+ \frac{1}{2} \int_{\frac{t}{2}}^t \|\partial_x G(t-\tau)\|_{\mathbf{L}^4} \left\|u^2(\tau)\right\|_{\mathbf{L}^{\frac{4}{3}}} d\tau \\
&\leq C t^{-\frac{1}{2}} + C t^{-1} \int_0^{\frac{t}{2}} \langle \tau \rangle^{-\frac{1}{2}} d\tau \\
&+ C \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{7}{8}} \|u(\tau)\|_{\mathbf{L}^\infty}^{\frac{1}{2}} \|u(\tau)\|_{\mathbf{L}^2}^{\frac{3}{2}} d\tau;
\end{aligned}$$

hence,

$$\begin{aligned}
\|u(t)\|_{\mathbf{L}^\infty} &\leq C t^{-\frac{1}{2}} + C t^{-\frac{3}{8}} \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{7}{8}} \|u(\tau)\|_{\mathbf{L}^\infty}^{\frac{1}{2}} d\tau \\
&\leq C t^{-\frac{1}{2}} + C\varepsilon t^{-\frac{1}{8}} \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{7}{8}} \|u(\tau)\|_{\mathbf{L}^\infty} d\tau \\
&+ \frac{C}{\varepsilon} t^{-\frac{5}{8}} \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{7}{8}} d\tau.
\end{aligned}$$

Therefore by the Gronwall lemma it follows that

$$\|u(t)\|_{\mathbf{L}^\infty} \le \frac{C}{\varepsilon} t^{-\frac{1}{2}}.$$

We find (4.60) for all $2 \le p \le \infty$ via the Hölder inequality. In the same manner we get the estimates

$$\|u(t)\|_{\mathbf{L}^p} + \langle t\rangle \left\|\partial_x^2 u(t)\right\|_{\mathbf{L}^p} \le C \langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \tag{4.61}$$

for all $t > 0$, $1 \le p \le \infty$.

Now we compute the large time asymptotics of the solutions. Now we take the initial time $T > 0$ to be sufficiently large and define $v(t,x)$ as a solution to the Cauchy problem for the Burgers equation with $u(T,x)$ as the initial data

$$\begin{cases} v_t + v v_x - v_{xx} = 0, \ t > T, \ x \in \mathbf{R}, \\ v(T,x) = u(T,x), \ x \in \mathbf{R}. \end{cases} \tag{4.62}$$

By the Hopf-Cole Hopf [1950] transformation $v(t,x) = -2\frac{\partial}{\partial x}\log Z(t,x)$ it is converted to the heat equation $Z_t = Z_{xx}$, so we have

$$Z(t,x) = \int_{\mathbf{R}} dy G_0(t, x-y) \exp\left(-\frac{1}{2}\int_{-\infty}^{y} u(T,\xi)\,d\xi\right).$$

Here $G_0(t,x) = (4\pi t)^{-\frac{1}{2}} e^{-x^2/4t}$ is the Green function for the heat equation. Note that the following estimates are true

$$\|v(t)\|_{\mathbf{L}^p} + \langle t\rangle \left\|\partial_x^2 v(t)\right\|_{\mathbf{L}^p} \le C \langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \tag{4.63}$$

for all $t > T$, $1 \le p \le \infty$.

Consider now the difference $w(t,x) = u(t,x) - v(t,x)$ for $t > T$. By (4.36) and (4.62) we get the Cauchy problem

$$\begin{cases} w_t + \frac{\partial}{\partial x}(vw) + \frac{1}{2}\frac{\partial}{\partial x}w^2 - w_{xx} + w_{xxx} + v_{xxx} = 0, \ t > T, \ x \in \mathbf{R}, \\ w(T,x) = 0, \ x \in \mathbf{R}. \end{cases} \tag{4.64}$$

In this section we consider the large data. Therefore we need to eliminate the linear term $\frac{\partial}{\partial x}(vw)$. We change $g(t,x) = Z(t,x)\int_{-\infty}^{x} w(t,y)\,dy$, then we get

$$Zv = -2Z_x, \ Zw = g_x + \frac{1}{2}gv,$$

$$Zw_x = g_{xx} + g_x v + \frac{1}{2}g v_x + \frac{1}{4}g v^2$$

and

$$Zw_{xx} = g_{xxx} + \frac{3}{2}g_{xx}v + \frac{3}{2}g_x v_x + \frac{3}{4}g_x v^2 + \frac{1}{2}g v_{xx} + \frac{3}{4}g v v_x + \frac{1}{8}g v^3.$$

From (4.64) we obtain the Cauchy problem

$$\begin{cases} g_t - g_{xx} + g_{xxx} + F = 0, \ t > T, \ x \in \mathbf{R}, \\ g(T,x) = 0, \ x \in \mathbf{R}, \end{cases} \tag{4.65}$$

where

$$F = \frac{1}{2Z}\left(g_x + \frac{1}{2}gv\right)^2 + \frac{3}{2}g_{xx}v + \frac{3}{2}g_x v_x + \frac{3}{4}g_x v^2 + \frac{1}{2}gv_{xx} + \frac{3}{4}gvv_x + \frac{1}{8}gv^3 + Zv_{xx}.$$

By virtue of estimates (4.61) and (4.63) we have

$$\|Z(t)\|_{\mathbf{L}^\infty} + \left\|Z^{-1}(t)\right\|_{\mathbf{L}^\infty} \le C \tag{4.66}$$

for all $t \ge T$ and a rough time decay estimate

$$\|w(t)\|_{\mathbf{L}^p} + \langle t\rangle \left\|\partial_x^2 w(t)\right\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \tag{4.67}$$

for all $t \ge T$, $1 \le p \le \infty$. Let us prove the estimate

$$\|g(t)\|_{\mathbf{L}^p} + \langle t\rangle^{\frac{1}{2}} \|g_x(t)\|_{\mathbf{L}^p} < C\langle t\rangle^{-\gamma+\frac{1}{2p}} \tag{4.68}$$

for all $t \ge T$, $2 \le p \le \infty$, where $\gamma \in \left(0, \frac{1}{2}\right)$. On the contrary, suppose that for some $t = T_1$ estimate (4.68) is violated, that is we have

$$\|g(t)\|_{\mathbf{L}^p} + \langle t\rangle^{\frac{1}{2}} \|g_x(t)\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\gamma+\frac{1}{2p}} \tag{4.69}$$

for all $t \in [T, T_1]$, $1 \le p \le \infty$. In view of (4.63), (4.67), (4.66) and (4.69) we find

$$\begin{aligned} \|F(t)\|_{\mathbf{L}^p} &\le C\|g_x\|_{\mathbf{L}^\infty}\|g_x\|_{\mathbf{L}^p} + C\|g\|^2_{\mathbf{L}^\infty}\left\|v^2\right\|_{\mathbf{L}^p} + C\|g_{xx}v\|_{\mathbf{L}^p} \\ &+ C\|g_x v_x\|_{\mathbf{L}^p} + C\left\|g_x v^2\right\|_{\mathbf{L}^p} + C\|gv_{xx}\|_{\mathbf{L}^p} \\ &+ C\|gvv_x\|_{\mathbf{L}^p} + C\left\|gv^3\right\|_{\mathbf{L}^p} + C\|v_{xx}\|_{\mathbf{L}^p} \\ &\le C\left(\langle t\rangle^{-1-2\gamma+\frac{1}{2p}} + \langle t\rangle^{-\frac{3}{2}-\gamma+\frac{1}{2p}} + \langle t\rangle^{-\frac{3}{2}+\frac{1}{2p}}\right) \\ &\le C\max\left\{T^{\gamma-\frac{1}{2}}, T^{-\gamma}\right\}\langle t\rangle^{-\gamma-1+\frac{1}{2p}} \end{aligned} \tag{4.70}$$

for all $t \in [T, T_1]$, $1 \le p \le \infty$, where we have used the identity

$$Zw_x = g_{xx} + g_x v + \frac{1}{2}gv_x + \frac{1}{4}gv^2$$

to treat the estimate of $g_{xx}v$. Using the integral equation associated with (4.65) we obtain

$$\|g(t)\|_{\mathbf{L}^p} \le \int_T^{\frac{t+T}{2}} d\tau \, \|G(t-\tau)\|_{\mathbf{L}^p} \, \|F(\tau)\|_{\mathbf{L}^1}$$
$$+ \int_{\frac{t+T}{2}}^t d\tau \, \|G(t-\tau)\|_{\mathbf{L}^1} \, \|F(\tau)\|_{\mathbf{L}^p} \, ;$$

hence in view of estimate (4.70) we find

$$\|g(t)\|_{\mathbf{L}^p} \le CT^{\gamma-\frac{1}{2}} \int_T^{\frac{t+T}{2}} (t-\tau)^{-\frac{1}{2}+\frac{1}{2p}} \langle \tau \rangle^{-\frac{1}{2}-\gamma} d\tau$$
$$+ CT^{\gamma-\frac{1}{2}} \int_{\frac{t+T}{2}}^t \{t-\tau\}^{-\frac{3}{4}} \langle \tau \rangle^{-1-\gamma+\frac{1}{2p}} d\tau$$
$$\le CT^{\gamma-\frac{1}{2}} \langle t \rangle^{-\gamma+\frac{1}{2p}} < C \langle t \rangle^{-\gamma+\frac{1}{2p}}$$

for all $t \in [T, T_1]$, $1 \le p \le \infty$, since T is sufficiently large. In the same manner we have

$$\|g_x(t)\|_{\mathbf{L}^p} \le \int_T^{\frac{t+T}{2}} d\tau \, \|\partial_x G(t-\tau)\|_{\mathbf{L}^p} \, \|F(\tau)\|_{\mathbf{L}^1}$$
$$+ \int_{\frac{t+T}{2}}^t d\tau \, \|\partial_x G(t-\tau)\|_{\mathbf{L}^1} \, \|F(\tau)\|_{\mathbf{L}^p}$$
$$\le CT^{\gamma-\frac{1}{2}} \int_T^{\frac{t+T}{2}} (t-\tau)^{-\frac{1}{2}+\frac{1}{2p}} \langle \tau \rangle^{-\frac{1}{2}-\gamma} d\tau$$
$$+ CT^{\gamma-\frac{1}{2}} \int_{\frac{t+T}{2}}^t \{t-\tau\}^{-\frac{3}{4}} (t-\tau)^{-\frac{1}{2}} \langle \tau \rangle^{-1-\gamma+\frac{1}{2p}} d\tau$$
$$\le CT^{\gamma-\frac{1}{2}} \langle t \rangle^{-\gamma-\frac{1}{2}+\frac{1}{2p}} < C \langle t \rangle^{-\gamma-\frac{1}{2}+\frac{1}{2p}}$$

for all $t \in [T, T_1]$, $1 \le p \le \infty$. The contradiction obtained proves estimate (4.68) for all $t \ge T$. We have

$$w = Z^{-1} \left(g_x + \frac{1}{2} g v \right)$$

which implies

$$\|u(t) - v(t)\|_{\mathbf{L}^\infty} \le C \langle t \rangle^{-\gamma-\frac{1}{2}} .$$

Hence, the first part of Theorem 4.18 follows. If $u_0 \in \mathbf{L}^{1,1}(\mathbf{R})$, then

$$\left\| v(t) - t^{-\frac{1}{2}} f_\theta \left((\cdot) \, t^{-\frac{1}{2}} \right) \right\|_{\mathbf{L}^\infty} \le C \langle t \rangle^{-\gamma-\frac{1}{2}} .$$

The second estimate of the theorem then follows, and Theorem 4.18 is proved.

4.3.3 Second term of asymptotics

We now obtain the second term of the large time asymptotic behavior of solutions to the Cauchy problem for KdVB equation (4.36) in the case of the initial data of arbitrary size. (This result was published in paper Kaikina and Ruiz-Paredes [2005].)

Theorem 4.25. *Let* $u_0 \in \mathbf{H}^s(\mathbf{R}) \cap \mathbf{L}^{1,1}(\mathbf{R})$, *where* $s > -\frac{1}{2}$, *and* $\theta = \int_{\mathbf{R}} u_0(x)\,dx \neq 0$. *Then the solution* $u(t,x)$ *to the Cauchy problem for the Korteweg-de Vries-Burgers equation (4.36) with the initial condition* $u_0(x)$ *has asymptotics*

$$u(t) = t^{-\frac{1}{2}} f_\theta\left((\cdot)\,t^{-\frac{1}{2}}\right) + \frac{\log t}{t}\widetilde{f}_\theta\left((\cdot)\,t^{-\frac{1}{2}}\right) + O\left(\frac{\sqrt{\log t}}{t}\right) \tag{4.71}$$

as $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}$, *where*

$$\widetilde{f}_\theta(x) = -\frac{\left(f_\theta(x) - \frac{x}{2}\right) e^{-x^2/4}}{2\sqrt{\pi} H(x)} \int_{\mathbf{R}} H(y)\, f_\theta^3(y)\, dy$$

and

$$H(x) = \cosh\frac{\theta}{4} - \sinh\frac{\theta}{4} Erf\left(\frac{x}{2}\right).$$

4.3.4 Proof of Theorem 4.25

In the previous subsections (see Theorem 4.18) it was proved that if the initial data $u_0 \in \mathbf{H}^s(\mathbf{R}) \cap \mathbf{L}^1(\mathbf{R})$, where $s > -\frac{1}{2}$, then there exists a unique solution $u \in \mathbf{C}^1((0,\infty);\mathbf{H}^\infty(\mathbf{R}))$ to the Cauchy problem for the Korteweg-de Vries-Burgers equation (4.36), which has the following optimal time decay estimates (see, in particular estimates (4.61))

$$\left\|\partial_x^k u(t)\right\|_{\mathbf{L}^p} \leq C\langle t\rangle^{-\frac{k}{2}-\frac{1}{2}\left(1-\frac{1}{p}\right)} \tag{4.72}$$

for $t > 0$, where $1 \leq p \leq \infty$, $k = 0,1,2,3$.

We use the integral equation associated with the Cauchy problem for the Korteweg - de Vries - Burgers equation (4.36)

$$u(t,x) = \mathcal{G}(t)\,u_0 - \frac{1}{2}\int_0^t \partial_x \mathcal{G}(t-\tau)\,u^2(\tau)\,d\tau, \tag{4.73}$$

where the Green operator

$$\mathcal{G}(t)\,\phi(\tau) = \int_{\mathbf{R}} G(t, x-y)\,\phi(\tau, y)\,dy$$

and the Green function

$$G(t,x) = (2\pi)^{-\frac{1}{2}} \int_{\mathbf{R}} e^{ix\xi - t\xi^2 + it\xi^3} d\xi.$$

First let us prove the following estimate of the $\mathbf{L}^{1,1}(\mathbf{R})$ norm for solutions of the Cauchy problem (4.36)

$$\|u(t)\|_{\mathbf{L}^{1,1}} \le C \langle t \rangle^{\frac{1}{2}}. \tag{4.74}$$

We multiply equation (4.36) by $|x| S(t,x)$, where $S(t,x) = 1$ for $u(t,x) > 0$ and $S(t,x) = -1$ for $u(t,x) < 0$; $S(t,x) = 0$ for $u(t,x) = 0$, and then we integrate with respect to x over $\mathbf{R}$ to get

$$\begin{aligned} &\int_{\mathbf{R}} u_t(t,x) |x| S(t,x)\, dx + \int_{\mathbf{R}} |x| u(t,x) u_x(t,x) S(t,x)\, dx \\ &= \int_{\mathbf{R}} u_{xx}(t,x) |x| S(t,x)\, dx - \int_{\mathbf{R}} u_{xxx}(t,x) |x| S(t,x)\, dx. \end{aligned}$$

We have

$$\int_{\mathbf{R}} u_t(t,x) |x| S(t,x)\, dx = \int_{\mathbf{R}} \frac{\partial}{\partial t} |u(t,x)| |x|\, dx = \frac{d}{dt} \|u(t)\|_{\mathbf{L}^{1,1}},$$

$$\begin{aligned} &-2 \int_{\mathbf{R}} u(t,x) u_x(t,x) |x| S(t,x)\, dx = -\int_{\mathbf{R}} |x| \frac{\partial}{\partial x} (|u(t,x)| u(t,x))\, dx \\ &= \int_{\mathbf{R}} \operatorname{sign}(x) |u(t,x)| u(t,x)\, dx = \|u(t)\|_{\mathbf{L}^2}^2 \le C \langle t \rangle^{-\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathbf{R}} u_{xx}(t,x) |x| S(t,x)\, dx = -2 \sum_{u(t,\chi_i)=0} |x| |u_x(t,\chi_i)| \\ &- \int_{\mathbf{R}} \operatorname{sign}(x) \frac{\partial}{\partial x} |u(t,x)|\, dx = -2 \sum_{u(t,\chi_i)=0} |x| |u_x(t,\chi_i)| + 2 |u(t,0)| \\ &\le 2 \|u(t)\|_{\mathbf{L}^\infty} \le C \langle t \rangle^{-\frac{1}{2}}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \frac{d}{dt} \|u(t)\|_{\mathbf{L}^{1,1}} &\le C \langle t \rangle^{-\frac{1}{2}} - \int_{\mathbf{R}} u_{xxx}(t,x) |x| S(t,x)\, dx \\ &\le C \langle t \rangle^{-\frac{1}{2}} + \|u_{xxx}(t)\|_{\mathbf{L}^{1,1}}. \end{aligned} \tag{4.75}$$

(We are interested here only in proving the time decay estimates. The question about the existence of the norms, e.g. $\|u_{xxx}(t)\|_{\mathbf{L}^{1,1}}$ locally in time can be easily solved by applying the contraction mapping principle to the corresponding integral equation.) Next we offer estimates for the norm $\|u(t)\|_{\mathbf{L}^{2,1}}$.

Multiplying equation (4.36) by $2x^2u$ and integrating with respect to $x \in \mathbf{R}$ we get

$$\begin{aligned} &\partial_t \int_{\mathbf{R}} x^2 u^2 dx + \frac{2}{3} \int_{\mathbf{R}} x^2 \partial_x u^3 dx \\ &-2 \int_{\mathbf{R}} x^2 u u_{xx} dx + 2 \int_{\mathbf{R}} x^2 u u_{xxx} dx = 0. \end{aligned} \tag{4.76}$$

Since

$$\begin{aligned} \int_{\mathbf{R}} x^2 \partial_x u^3 dx &= -2 \int_{\mathbf{R}} x u^3 dx \leq C \left\| u(t) \right\|_{\mathbf{L}^{2,1}} \left\| u(t) \right\|_{\mathbf{L}^2} \left\| u(t) \right\|_{\mathbf{L}^\infty} \\ &\leq C \langle t \rangle^{-\frac{3}{4}} \left\| u(t) \right\|_{\mathbf{L}^{2,1}}, \end{aligned}$$

$$\begin{aligned} \int_{\mathbf{R}} x^2 u u_{xx} dx &= -\int_{\mathbf{R}} x^2 (u_x)^2 dx + \int_{\mathbf{R}} u^2 dx \\ &= -\left\| u_x(t) \right\|_{\mathbf{L}^{2,1}}^2 + \left\| u(t) \right\|_{\mathbf{L}^2}^2 \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbf{R}} x^2 u u_{xxx} dx &= -\int_{\mathbf{R}} x^2 u_x u_{xx} dx - 2 \int_{\mathbf{R}} x u u_{xx} dx \\ &= 3 \int_{\mathbf{R}} x (u_x)^2 dx \leq 3 \left\| u_x(t) \right\|_{\mathbf{L}^{2,1}} \left\| u_x(t) \right\|_{\mathbf{L}^2} \\ &\leq C \langle t \rangle^{-\frac{3}{4}} \left\| u_x(t) \right\|_{\mathbf{L}^{2,1}} \end{aligned}$$

we get

$$\begin{aligned} \frac{d}{dt} \left\| u(t) \right\|_{\mathbf{L}^{2,1}}^2 &\leq C \langle t \rangle^{-\frac{3}{4}} \left\| u(t) \right\|_{\mathbf{L}^{2,1}} + C \langle t \rangle^{-\frac{1}{2}} \\ &+ C \langle t \rangle^{-\frac{3}{4}} \left\| u_x(t) \right\|_{\mathbf{L}^{2,1}} - 2 \left\| u_x(t) \right\|_{\mathbf{L}^{2,1}}^2 \\ &\leq C \langle t \rangle^{-\frac{3}{4}} \left\| u(t) \right\|_{\mathbf{L}^{2,1}} + C \langle t \rangle^{-\frac{1}{2}}; \end{aligned}$$

hence, by integrating we see that

$$\left\| u(t) \right\|_{\mathbf{L}^{2,1}} \leq C \langle t \rangle^{\frac{1}{4}}. \tag{4.77}$$

In the next lemma we obtain the estimates for the norm $\left\| u_{xxx}(t) \right\|_{\mathbf{L}^{1,1}}$.

Lemma 4.26. *Let the initial data* $u_0 \in \mathbf{H}^2(\mathbf{R}) \cap \mathbf{W}_1^3(\mathbf{R})$ *and estimate (4.72) be valid. Then the estimate is true*

$$\left\| u_{xxx}(t) \right\|_{\mathbf{L}^{1,1}} \leq C \langle t \rangle^{-1} \tag{4.78}$$

for all $t > 0$.

Proof. First we need to estimate the norm $\|u_x(t)\|_{\mathbf{L}^{1,1}}$. By the integral equation (4.73) we have

$$\begin{aligned}
&\|u_x(t)\|_{\mathbf{L}^{1,1}} \le \|\partial_x \mathcal{G}(t) u_0\|_{\mathbf{L}^{1,1}} \\
&+ \int_0^{\frac{t}{2}} \Big(\left\|\partial_x^2 G(t-\tau)\right\|_{\mathbf{L}^{1,1}} \|u(\tau)\|_{\mathbf{L}^2}^2 \\
&+ \left\|\partial_x^2 G(t-\tau)\right\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^2} \|u(\tau)\|_{\mathbf{L}^{2,1}} \Big) d\tau \\
&+ \int_{\frac{t}{2}}^t (\|\partial_x G(t-\tau)\|_{\mathbf{L}^{1,1}} \|u(\tau)\|_{\mathbf{L}^2} \|u_x(\tau)\|_{\mathbf{L}^2} \\
&+ \|\partial_x G(t-\tau)\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^{2,1}} \|u_x(\tau)\|_{\mathbf{L}^2}) \, d\tau;
\end{aligned}$$

hence by estimate

$$\left\|\partial_x^k G(t)\right\|_{\mathbf{L}^{1,1}} \le C \{t\}^{-\frac{1}{4}-\frac{k}{2}} \langle t \rangle^{\frac{1-k}{2}}$$

we get

$$\begin{aligned}
&\|u_x(t)\|_{\mathbf{L}^{1,1}} \le C + C \int_0^{\frac{t}{2}} \left(\langle t-\tau \rangle^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{1}{2}} + \langle t-\tau \rangle^{-1} \right) d\tau \\
&+ \int_{\frac{t}{2}}^t \{t-\tau\}^{-\frac{3}{4}} \left(\langle \tau \rangle^{-1} + \langle t-\tau \rangle^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{1}{2}} \right) d\tau \le C.
\end{aligned}$$

Likewise by the integral equation (4.73) we have

$$\begin{aligned}
&\|u_{xxx}(t)\|_{\mathbf{L}^{1,1}} \le \left\|\partial_x^3 \mathcal{G}(t) u_0\right\|_{\mathbf{L}^{1,1}} \\
&+ C \int_0^{\frac{t}{2}} \Big(\left\|\partial_x^4 G(t-\tau)\right\|_{\mathbf{L}^{1,1}} \|u(\tau)\|_{\mathbf{L}^2}^2 \\
&+ \left\|\partial_x^4 G(t-\tau)\right\|_{\mathbf{L}^1} \|u(\tau)\|_{\mathbf{L}^2} \|u(\tau)\|_{\mathbf{L}^{2,1}} \Big) d\tau \\
&+ C \int_{\frac{t}{2}}^t (\|\partial_x G(t-\tau)\|_{\mathbf{L}^{1,1}} (\|u(\tau)\|_{\mathbf{L}^2} \|u_{xxx}(\tau)\|_{\mathbf{L}^2} \\
&+ \|u_x(\tau)\|_{\mathbf{L}^2} \|u_{xx}(\tau)\|_{\mathbf{L}^2}) \\
&+ \|\partial_x G(t-\tau)\|_{\mathbf{L}^1} (\|u(\tau)\|_{\mathbf{L}^{2,1}} \|u_{xxx}(\tau)\|_{\mathbf{L}^2} \\
&+ \|u_x(\tau)\|_{\mathbf{L}^{1,1}} \|u_{xx}(\tau)\|_{\mathbf{L}^\infty})) \, d\tau;
\end{aligned}$$

hence we obtain

$$\begin{aligned}
&\|u_{xxx}(t)\|_{\mathbf{L}^{1,1}} \le C \langle t \rangle^{-1} + C \int_0^{\frac{t}{2}} \left(\langle t-\tau \rangle^{-\frac{3}{2}} \langle \tau \rangle^{-\frac{1}{2}} + \langle t-\tau \rangle^{-2} \right) d\tau \\
&+ \int_{\frac{t}{2}}^t \{t-\tau\}^{-\frac{3}{4}} \left(\langle \tau \rangle^{-2} + \langle t-\tau \rangle^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{3}{2}} \right) d\tau \le C \langle t \rangle^{-1}.
\end{aligned}$$

Thus the estimate of the lemma is true, and Lemma 4.26 is proved.

Integration of inequality (4.75) yields

$$\|u(t)\|_{\mathbf{L}^{1,1}} \le \|u_0\|_{\mathbf{L}^{1,1}} + C\langle t\rangle^{\frac{1}{2}} \le C\langle t\rangle^{\frac{1}{2}}.$$

Therefore estimate (4.74) is true for all $t \ge 0$.

Now we obtain the second term of the large time asymptotics as $t \to \infty$ of solutions $u(x,t)$ to the Cauchy problem for the Korteweg-de Vries-Burgers equation (4.36). We take the initial time $T > 0$ to be sufficiently large and define $v(t,x)$ as a solution to the Cauchy problem for the Burgers equation with $u(T,x)$ as the initial data

$$\begin{cases} v_t + vv_x - v_{xx} = 0, \ t > T, \ x \in \mathbf{R}, \\ v(T,x) = u(T,x), \ x \in \mathbf{R}. \end{cases} \tag{4.79}$$

By the Hopf-Cole Hopf [1950] transformation $v(t,x) = -2\frac{\partial}{\partial x}\log Z(t,x)$ equation (4.79) is converted to the heat equation $Z_t = Z_{xx}$. Therefore we obtain

$$Z(t,x) = \int_{\mathbf{R}} dy G_0(t,x-y)\exp\left(-\frac{1}{2}\int_{-\infty}^{y} u(T,\xi)\,d\xi\right), \tag{4.80}$$

where $G_0(t,x) = (4\pi t)^{-\frac{1}{2}} e^{-x^2/4t}$ is the Green function for the heat equation. Note that the following estimates are true

$$\|v(t)\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{k}{2}-\frac{1}{2}\left(1-\frac{1}{p}\right)} \tag{4.81}$$

for all $t > T$, $1 \le p \le \infty$, $k = 0,1,2$.

Consider now the difference $w(t,x) = u(t,x) - v(t,x)$ for $t > T$. By (4.36) and (4.79) we get the Cauchy problem

$$\begin{cases} w_t + \frac{\partial}{\partial x}(vw) + \frac{1}{2}\frac{\partial}{\partial x}w^2 - w_{xx} + w_{xxx} + v_{xxx} = 0, \ t > T, \ x \in \mathbf{R}, \\ w(T,x) = 0, \ x \in \mathbf{R}. \end{cases} \tag{4.82}$$

We have the estimates (see (4.66) and (4.68))

$$\|Z(t)\|_{\mathbf{L}^\infty} + \left\|Z^{-1}(t)\right\|_{\mathbf{L}^\infty} \le C \tag{4.83}$$

for all $t \ge T$ and

$$\left\|\partial_x^k w(t)\right\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{k}{2}-\frac{1}{2}\left(1-\frac{1}{p}\right)-\gamma} \tag{4.84}$$

for all $t \ge T$, $2 \le p \le \infty$, $k = 0,1,2$, where $\gamma \in \left(0,\frac{1}{2}\right)$.

Following the heuristic considerations in Section 1 in paper Naumkin and Shishmarev [1994a] we compare the rates of decay of various terms in equation (4.82) and observe that the main term of the asymptotic expansion of $w(x,t)$ as $t \to \infty$ is determined by the linear Cauchy problem

$$\begin{cases} \varphi_t + \frac{\partial}{\partial x}(\varphi v) - \varphi_{xx} + v_{xxx} = 0, \ t > T, \ x \in \mathbf{R}, \\ \varphi(T,x) = 0, \ x \in \mathbf{R}. \end{cases} \tag{4.85}$$

To eliminate the second term from (4.85), let us integrate (4.85) with respect to x and make the substitution

$$\int_{-\infty}^{x} \varphi(y,t)dy = s(x,t)/Z(x,t),$$

where $Z(x,t)$ is defined by (4.80). We obtain

$$\begin{cases} s_t - s_{xx} + Zv_{xx} = 0, \ t > T, \ x \in \mathbf{R}, \\ s\,(T,x) = 0, \ x \in \mathbf{R}. \end{cases} \tag{4.86}$$

It is simple to integrate (4.86)

$$s(x,t) = -\int_{T}^{t} \mathcal{G}_0(t-\tau)Z\,(\tau)\,v_{xx}\,(\tau)\,d\tau. \tag{4.87}$$

In the following lemma we evaluate the large time asymptotics of the solution $\varphi\,(t)$ of linear problem (4.85)

$$\begin{aligned} \varphi\,(t) &= \partial_x \left(\frac{s\,(t)}{Z\,(t)}\right) \\ &= -Z^{-1}\,(t) \int_{T}^{t} (\partial_x \mathcal{G}(t-\tau) + v\,(t)\,\mathcal{G}(t-\tau))Z\,(\tau)\,v_{xx}\,(\tau)\,d\tau. \end{aligned} \tag{4.88}$$

Lemma 4.27. *Let* $u(T,x) \in \mathbf{H}^2(\mathbf{R}) \cap \mathbf{L}^{1,1}(\mathbf{R})$. *Then the asymptotics*

$$\varphi\,(t) = t^{-1} \widetilde{f}_\theta(\chi) \log t + O\left(t^{-1}\sqrt{\log t}\right) \tag{4.89}$$

is valid as $t \to \infty$ *uniformly with respect to* $\chi = x/\sqrt{t} \in \mathbf{R}$, *where*

$$\widetilde{f}_\theta\,(x) = -\frac{1}{4\sqrt{\pi}H\,(x)} e^{-\frac{x^2}{4}} \left(f_\theta\,(x) - \frac{x}{2}\right) \int_{\mathbf{R}} H(y) f_\theta^3(y)dy,$$

$$f_\theta\,(x) = -2\partial_x \log H\,(x)\,,$$

$$H\,(x) = \cosh\frac{\theta}{4} - \sinh\frac{\theta}{4} Erf\left(\frac{x}{2}\right).$$

Proof. Let us represent the integral with respect to τ in (4.88) as the sum of three parts $(t > T + e)$

$$\int_{T}^{t} d\tau = \int_{T}^{T+1} + \int_{t/\log t}^{t} + \int_{T+1}^{t/\log t} \equiv I_1 + I_2 + I_3. \tag{4.90}$$

For all $x \in \mathbf{R}$ and $t > T$ we have

$$0 < C_1 < Z(x,t) < C_2,$$

and for each $t > 0$ the following inequalities hold:

$$\begin{aligned} \left\|\partial_x^l Z(\cdot,t)\right\|_{\mathbf{L}^p} &\le C\left\langle t\right\rangle^{-\frac{l}{2}+\frac{1}{2p}}, \\ \left\|\partial_x^l G(\cdot,t)\right\|_{\mathbf{L}^p} &\le Ct^{-\frac{l}{2}+\frac{1}{2p}}, \\ \left\|\partial_x^l v(\cdot,t)\right\|_{\mathbf{L}^p} &\le C\left\langle t\right\rangle^{-\frac{l}{2}-\frac{1}{2}+\frac{1}{2p}} \end{aligned} \tag{4.91}$$

for all $l = 1, 2, 3$, $1 \le p \le \infty$. By using these inequalities, we readily estimate the first two integrals in representation (4.90)

$$|I_1| \le \frac{C}{t}\int_T^{T+1} \|v_{xx}(\tau)\|_{\mathbf{L}^1} d\tau = O\left(t^{-1}\right) \tag{4.92}$$

as $t \to \infty$ and

$$\begin{aligned} |I_2| &\le \int_{t/\log t}^t d\tau \|v_{xx}(\tau)\|_{\mathbf{L}^\infty}\left(\|\partial_x G(t-\tau)\|_{\mathbf{L}^\infty} + \|v(\tau)\|_{\mathbf{L}^\infty}\|G(t-\tau)\|_{\mathbf{L}^1}\right) \\ &\le C\int_{t/\log t}^t \tau^{-\frac{3}{2}}\left((t-\tau)^{-\frac{1}{2}} + t^{-\frac{1}{2}}\right)d\tau = O\left(t^{-1}\sqrt{\log t}\right), \ t\to\infty. \end{aligned} \tag{4.93}$$

In the third integral I_3 we integrate by parts with respect to y to obtain

$$\begin{aligned} I_3 =& \int_{T+1}^{t/\log t} d\tau \int_0^\infty dy \Lambda_y(x,y,t,\tau)\int_y^\infty F(q,\tau)\,dq \\ &- \int_{T+1}^{t/\log t} d\tau \int_{-\infty}^0 dy \Lambda_y(x,y,t,\tau)\int_{-\infty}^y F(q,\tau)\,dq \\ &+ \int_{T+1}^{t/\log t} d\tau \Lambda(x,0,t,\tau)\int_{\mathbf{R}} F(y,\tau)\,dy \\ \equiv& I_4 + I_5 + I_6, \end{aligned}$$

where

$$\Lambda(x,y,t,\tau) = Z^{-1}(x,t)(\partial_x G(x-y,t-\tau) + v(x,t)G(x-y,t-\tau)).$$

Since

$$\sup_{T+1\le\tau\le t/\log t}\ \sup_{x\in\mathbf{R}}\sup_{y\in\mathbf{R}} |\Lambda_y(x,y,t,\tau)| \le Ct^{-\frac{3}{2}},$$

$$\left\|x\partial_x^l Z(x,t)\right\|_{\mathbf{L}^1(\mathbf{R})} \le C\left\langle t\right\rangle^{1-\frac{l}{2}}, \ l = 1,2,3,$$

and, therefore,

$$\|xv_{xx}(t)\|_{\mathbf{L}^1(\mathbf{R})} \le C\left\langle t\right\rangle^{-\frac{1}{2}},$$

we obtain

$$|I_4| \le Ct^{-\frac{3}{2}}\int_{T+1}^{t/\log t} d\tau \int_0^\infty dy \int_y^\infty |v_{\xi\xi}(\tau,\xi)|\,d\xi = O(t^{-1}). \tag{4.94}$$

The integral I_5 can be estimated similarly. Since

$$\partial_x^l G(x,t-\tau) = \partial_x^l G(x,t) + O\left(t^{-\frac{1}{2}-\frac{l}{2}} \log^{-1} t\right),\ l = 0,1,$$

for $T+1 \le \tau \le t/\log t$, we derive the estimate

$$I_6 = \frac{1}{H(\chi)} \left(\partial_x G(x,t) + \frac{A(\chi)}{\sqrt{t}} G(x,t)\right) \int_{T+1}^{t/\log t} d\tau \int_{\mathbf{R}} F(y,\tau)dy + O(t^{-1}) \tag{4.95}$$

from the estimate

$$\partial_x^l Z(x,t) = t^{-\frac{1}{2}} \left(\frac{d^l H(\chi)}{d\chi^l} + O\left(t^{-\frac{1}{2}}\right)\right),\ l = 0,1,2,3. \tag{4.96}$$

Since

$$\partial_x Z = -2Zv,$$

the integration by parts yields, by virtue of (4.96),

$$\begin{aligned}\int_{\mathbf{R}} F(y,\tau)dy &= -\frac{1}{2}\int_{\mathbf{R}} v^3(y,\tau) Z(y,\tau)dy \\ &= -\frac{1}{2\tau}\int_{\mathbf{R}} f_\theta^3(y) H(y)dy + O(\tau^{-3/2}).\end{aligned}$$

Then from (4.92) - (4.95) we obtain (4.78). Lemma 4.27 is proved.

It follows from (4.73) and (4.85) that the remainder $\psi(x,t) = w(x,t) - \varphi(x,t)$ is the solution to the Cauchy problem

$$\begin{cases} \psi_t + \frac{\partial}{\partial x}(v\psi) - \psi_{xx} + \frac{1}{2}\frac{\partial}{\partial x} w^2 + w_{xxx} = 0,\ t > T,\ x \in \mathbf{R}, \\ \psi(T,x) = 0,\ x \in \mathbf{R}. \end{cases} \tag{4.97}$$

To eliminate the second term from (4.97) as above we integrate this equation with respect to x and introduce the new unknown function

$$r(x,t) = Z(x,t) \int_{-\infty}^{x} \psi(y,t)dy.$$

Then we obtain

$$\begin{cases} r_t - r_{xx} + F = 0,\ t > T,\ x \in \mathbf{R}, \\ r(T,x) = 0,\ x \in \mathbf{R}, \end{cases} \tag{4.98}$$

where

$$F = \frac{1}{2} Z w^2 + Z w_{xx}.$$

In view of (4.84) and (4.83) we find

$$\begin{aligned}\|F(t)\|_{\mathbf{L}^p} &\le C \|w\|_{\mathbf{L}^\infty} \|w\|_{\mathbf{L}^p} + C \|w_{xx}\|_{\mathbf{L}^p} \\ &\le C\langle t\rangle^{-1-2\gamma+\frac{1}{2p}} + C\langle t\rangle^{-\frac{3}{2}-\gamma+\frac{1}{2p}} \le C\langle t\rangle^{-1-2\gamma+\frac{1}{2p}}\end{aligned} \tag{4.99}$$

for all $t \geq T$, $1 \leq p \leq \infty$. Using the integral equation associated with (4.98) we obtain

$$\|r(t)\|_{\mathbf{L}^p} \leq \int_T^{\frac{t+T}{2}} d\tau \, \|G(t-\tau)\|_{\mathbf{L}^p} \|F(\tau)\|_{\mathbf{L}^1} + \int_{\frac{t+T}{2}}^t d\tau \, \|G(t-\tau)\|_{\mathbf{L}^1} \|F(\tau)\|_{\mathbf{L}^p};$$

hence in view of estimate (4.99) we find

$$\|r(t)\|_{\mathbf{L}^p} \leq C \int_T^{\frac{t+T}{2}} (t-\tau)^{-\frac{1}{2}+\frac{1}{2p}} \langle \tau \rangle^{-\frac{1}{2}-2\gamma} d\tau + C \int_{\frac{t+T}{2}}^t \{t-\tau\}^{-\frac{3}{4}} \langle \tau \rangle^{-1-2\gamma+\frac{1}{2p}} d\tau \leq C \langle t \rangle^{-\frac{1}{2}+\frac{1}{2p}}$$

for all $t \geq T$, $1 \leq p \leq \infty$, if we take $\gamma \in \left(\frac{1}{4}, \frac{1}{2}\right)$. In the same manner we have

$$\begin{aligned} \|r_x(t)\|_{\mathbf{L}^p} &\leq \int_T^{\frac{t+T}{2}} d\tau \, \|\partial_x G(t-\tau)\|_{\mathbf{L}^p} \|F(\tau)\|_{\mathbf{L}^1} \\ &+ \int_{\frac{t+T}{2}}^t d\tau \, \|\partial_x G(t-\tau)\|_{\mathbf{L}^1} \|F(\tau)\|_{\mathbf{L}^p} \\ &\leq C \int_T^{\frac{t+T}{2}} (t-\tau)^{-1+\frac{1}{2p}} \langle \tau \rangle^{-\frac{1}{2}-2\gamma} d\tau \\ &+ C \int_{\frac{t+T}{2}}^t \{t-\tau\}^{-\frac{3}{4}} (t-\tau)^{-\frac{1}{2}} \langle \tau \rangle^{-1-2\gamma+\frac{1}{2p}} d\tau \leq C \langle t \rangle^{-1+\frac{1}{2p}} \end{aligned}$$

for all $t \geq T$, $1 \leq p \leq \infty$. Using the identity

$$\psi = Z^{-1} \left(r_x + \frac{1}{2} r v \right),$$

we obtain the estimate

$$\|u(t) - v(t) - \varphi(t)\|_{\mathbf{L}^\infty} \leq C \langle t \rangle^{-1}$$

for all $t \geq T$. When $u_0 \in \mathbf{L}^{1,1}(\mathbf{R})$, then

$$v(t,x) = t^{-\frac{1}{2}} f_\theta \left(x t^{-\frac{1}{2}} \right) + O\left(t^{-1}\right)$$

for $t \to \infty$, and in view of Lemma 4.27, the asymptotics of the theorem follows. Theorem 4.25 is thus proved.

4.4 Benjamin-Bona-Mahony-Burgers equation

This section is devoted to the study of the Cauchy problem for the Benjamin-Bona-Mahony-Burgers (BBM-Burgers) equation

$$\begin{cases} \partial_t (u - u_{xx}) - \mu u_{xx} + \beta u_{xxx} + u u_x = 0, \ x \in \mathbf{R}, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}, \end{cases} \tag{4.100}$$

where $\mu > 0$, $\beta \in \mathbf{R}$. Usually the BBM-Burgers equation is written as follows

$$\partial_t (v - v_{xx}) + \beta v_x - \mu v_{xx} + v v_x = 0,$$

which is equivalent to equation (4.100) in view of the change $u(t,x) = v(t, x + \beta t)$.

In the present section we are interested in the large time asymptotics of solutions to the Cauchy problem for the BBM-Burgers equation (4.100) for the case of the initial data having an arbitrary size. Throughout this section we suppose that the total mass of the initial data $\theta = \int_{\mathbf{R}} u_0(x)\,dx \neq 0$.

Our aim is to prove the following result.

Theorem 4.28. *Let the initial data* $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R})$, *and* $\theta = \int_{\mathbf{R}} u_0(x)\,dx \neq 0$. *Then there exists a unique solution*

$$u(t,x) \in \mathbf{C}\left([0,\infty); \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R})\right)$$

to the Cauchy problem for the BBM-Burgers equation (4.100), which has the asymptotics

$$u(t) = t^{-\frac{1}{2}} f_\theta\left((\cdot)\, t^{-\frac{1}{2}}\right) + o\left(t^{-\frac{1}{2}}\right)$$

as $t \to \infty$ *uniformly with respect to* $x \in R$. *If in addition the initial data* $u_0 \in \mathbf{L}^{1,1}(\mathbf{R})$, *then the asymptotics*

$$u(t) = t^{-\frac{1}{2}} f_\theta\left((\cdot)\, t^{-\frac{1}{2}}\right) + O\left(t^{-\frac{1}{2}-\gamma}\right) \tag{4.101}$$

is true as $t \to \infty$, *where* $\gamma \in \left(0, \frac{1}{2}\right)$ *and*

$$f_\theta(\chi) = -2\sqrt{\mu}\frac{\partial}{\partial \chi} \log\left(\cosh\frac{\theta}{4} - \sinh\left(\frac{\theta}{4}\right) \operatorname{Erf}\left(\frac{\chi}{2\sqrt{\mu}}\right)\right) \tag{4.102}$$

is the self-similar solution for the Burgers (see Burgers [1948]) equation

$$u_t + u u_x - \mu u_{xx} = 0,$$

defined by the total mass $\theta = \int_{\mathbf{R}} u_0(x)\,dx$ *of the initial data. Here*

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$$

is the error function.

Next we obtain the second term of the large time asymptotic behavior of solutions to the Cauchy problem for equation (4.100) in the case of the initial data of arbitrary size. A similar result was shown in Section 4.3 for the KdV-Burgers equation.

Theorem 4.29. *Let* $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R}) \cap \mathbf{L}^{1,1}(\mathbf{R})$*, and* $\theta = \int_{\mathbf{R}} u_0(x)\,dx \neq 0$*. Then the solution* $u(t)$ *to the Cauchy problem for the BBM-Burgers equation (4.100), with the initial condition* u_0 *has the following asymptotics*

$$u(t) = t^{-\frac{1}{2}} f_\theta\left((\cdot)\, t^{-\frac{1}{2}}\right) + \frac{\log t}{t} \widetilde{f}_\theta\left((\cdot)\, t^{-\frac{1}{2}}\right) + O\left(\frac{1}{t}\right) \tag{4.103}$$

as $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}$, *where*

$$\widetilde{f}_\theta(x) = -\frac{\left(f_\theta(x) - \frac{x}{2\sqrt{\mu}}\right) e^{-\frac{x^2}{4\mu}}}{2\sqrt{\pi\mu} H(x)} \int_{\mathbf{R}} f_\theta^3(y)\, H(y)\, dy,$$

with

$$H(\chi) = \cosh\frac{\theta}{4} - \sinh\left(\frac{\theta}{4}\right) \operatorname{Erf}\left(\frac{\chi}{2\sqrt{\mu}}\right).$$

4.4.1 Preliminaries

Consider the linear Cauchy problem

$$\begin{cases} \partial_t(u - u_{xx}) - \mu u_{xx} + \beta u_{xxx} = f(t,x), \ x \in \mathbf{R}, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}. \end{cases} \tag{4.104}$$

Using the Duhamel principle we rewrite problem (4.104) in the form

$$u(t) = \mathcal{G}(t)\, u_0 + \int_0^t \mathcal{G}(t-\tau)\, \mathcal{B} f(\tau)\, d\tau, \tag{4.105}$$

where the Green operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t)\,\phi = \overline{\mathcal{F}}_{\xi\to x} e^{-t\left(\mu\xi^2 - i\beta\xi^3\right)\left(1+\xi^2\right)^{-1}} \hat{\phi}(\xi)$$

and (see Erdélyi et al. [1954])

$$\mathcal{B}\phi = \left(1 - \partial_x^2\right)^{-1} \phi = \overline{\mathcal{F}}_{\xi\to x}\left(1+\xi^2\right)^{-1} \hat{\phi}(\xi) = \frac{1}{2} \int_{\mathbf{R}} e^{-|x-y|} \phi(y)\, dy.$$

Note that

$$\left\|\mathcal{B}^k \phi\right\|_{\mathbf{W}_p^{2k}} \leq C \|\phi\|_{\mathbf{L}^p}$$

for $k \in \mathbf{N}$.

Denote the commutator

$$\left[\partial_x^3 \mathcal{B}^2 \mathcal{G}(t), \psi\right] \phi \equiv \partial_x^3 \mathcal{B}^2 \mathcal{G}(t)(\psi\phi) - \psi \partial_x^3 \mathcal{B}^2 \mathcal{G}(t) \phi.$$

Denote

$$G_0(t,x) = (4\pi\mu t)^{-\frac{1}{2}} e^{-\frac{x^2}{4\mu t}}.$$

From Lemma 1.31 and Lemma 1.33 we get

Lemma 4.30. *The estimates*

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^p} \le e^{-\mu t}\|\phi\|_{\mathbf{L}^p} + C\langle t\rangle^{-\frac{1}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|\phi\|_{\mathbf{L}^r},$$

$$\left\|\partial_x^k \mathcal{G}(t)\phi\right\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{k}{2}}\|\phi\|_{\mathbf{W}_p^k},$$

$$\|\mathcal{G}(t)\phi - \vartheta G_0(t)\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{2}-\frac{a}{2}}\left(\|\phi\|_{\mathbf{L}^\infty} + \|\phi\|_{\mathbf{L}^{1,a}}\right),$$

$$\left\||\cdot|^b\left(\mathcal{G}(t)\phi - \vartheta G_0(t)\right)\right\|_{\mathbf{L}^1} \le Ct^{\frac{b-a}{2}}\|\phi\|_{\mathbf{L}^{1,a}}$$

and

$$\left\|\left[\partial_x^3 \mathcal{B}^2 \mathcal{G}(t), \psi\right]\phi\right\|_{\mathbf{L}^1} \le C\langle t\rangle^{-1}\|\psi_x\|_{\mathbf{L}^2}\|\phi\|_{\mathbf{L}^2}$$

are valid for all $t > 0$ provided that the right-hand sides are finite, where

$$k \in \mathbf{N},\ 1 \le r \le p \le \infty,\ 0 \le b \le a,\ \vartheta = \int_{\mathbf{R}} \phi(x)\,dx.$$

Consider the integral equation associated with the Cauchy problem for the BBM - Burgers equation (4.100)

$$u(t,x) = \mathcal{G}(t)u_0 - \int_0^t \mathcal{G}(t-\tau)\mathcal{B}\left(u(\tau)u_x(\tau)\right)d\tau. \tag{4.106}$$

Define the norms

$$\|\phi\|_{p,q} \equiv \left\|\|\phi(t,x)\|_{\mathbf{L}^q(\mathbf{R}_x)}\right\|_{\mathbf{L}^p\left(\mathbf{R}_t^+\right)}.$$

First let us prove a global existence result for large initial data.

Proposition 4.31. *Suppose that the initial data $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})$, $a \ge 0$. Then there exists a unique global solution*

$$u \in \mathbf{C}\left([0,\infty); \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})\right)$$

to the Cauchy problem (4.100). Moreover the a priori estimates of a solution are valid

$$\|u\|_{\infty,2} + \|u_x\|_{\infty,2} + \|u_x\|_{2,2} \le C\|u_0\|_{\mathbf{H}^1}. \tag{4.107}$$

Proof. By using a standard contraction mapping principle we can easily prove that for some $T > 0$ there exists a unique solution

$$u \in \mathbf{C}\left([0,T]\,;\mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})\right)$$

to the Cauchy problem (4.100). We now multiply equation (4.100) by $2u$ and integrate the resulting equation with respect to x over $\mathbf{R}$ to get

$$\frac{d}{dt}\left(\|u(t)\|_{\mathbf{L}^2}^2 + \|u_x(t)\|_{\mathbf{L}^2}^2\right) + 2\mu\,\|u_x(t)\|_{\mathbf{L}^2}^2 = 0;$$

hence by integrating with respect to time $t > 0$ we see that

$$\|u(t)\|_{\mathbf{L}^2}^2 + \|u_x(t)\|_{\mathbf{L}^2}^2 + 2\mu\int_0^t \|u_x(\tau)\|_{\mathbf{L}^2}^2\,d\tau \le \|u_0\|_{\mathbf{H}^1}^2$$

for all $t \in [0,T]$. Then, by applying estimates of Lemma 4.30, we obtain from integral equation (4.106)

$$\begin{aligned}\|u\|_{\mathbf{W}_1^1} &\le C\,\|u_0\|_{\mathbf{W}_1^1} + C\int_0^t \left\|\partial_x \mathcal{B}u^2(\tau)\right\|_{\mathbf{W}_1^1} d\tau \\ &\le C + C\int_0^t \|u(\tau)\|_{\mathbf{L}^2}^2\,d\tau \le C\,(1+t)\end{aligned}$$

and

$$\begin{aligned}\|u\|_{\mathbf{L}^{1,a}} &\le C\,\|u_0\|_{\mathbf{L}^{1,a}} + C\int_0^t \langle t\rangle^{\frac{a}{2}} \left\|\partial_x \mathcal{B}u^2(\tau)\right\|_{\mathbf{L}^1} d\tau \\ &+ C\int_0^t \left\|\partial_x \mathcal{B}u^2(\tau)\right\|_{\mathbf{L}^{1,a}} d\tau \\ &\le C\,\langle t\rangle^{\frac{a}{2}+1} + C\int_0^t \|u(\tau)\|_{\mathbf{L}^{1,a}}\,d\tau\end{aligned}$$

for all $t \in [0,T]$. The Gronwall lemma yields the estimate

$$e^{-Ct}\left(\|u(t)\|_{\mathbf{L}^{1,a}} + \|u(t)\|_{\mathbf{W}_1^1}\right) \le C$$

for all $t \in [0,T]$, where $C > 0$ does not depend on T. Therefore by a standard continuation argument we can prolong the local solution to the global one, which satisfies the a priori estimate (4.107). Proposition 4.31 is proved.

We now estimate the third derivative of the solution. Denote $S(t,x) = 1$ for $u(t,x) > 0$ and $S(t,x) = -1$ for $u(t,x) < 0$; $S(t,x) = 0$ for $u(t,x) = 0$

Lemma 4.32. *Let the initial data* $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R})$. *Moreover we assume that the norms of the solutions are bounded*

$$\|u\|_{\infty,2} + \|u_x\|_{\infty,2} + \|u_x\|_{2,2} \le C.$$

Then the estimate is true

$$\left| \int_0^T dt \int_{\mathbf{R}} S(t,x) \mathcal{B}u_{xxx}(t,x)\, dx \right| \le C \langle T \rangle^{\frac{1}{6}} \log \langle T \rangle$$

for all $T > 0$.

Proof. By the integral equation (4.106) we have

$$\begin{aligned}\mathcal{B}u_{xxx}(t,x) &= \partial_x^3 \mathcal{B}\mathcal{G}(t) u_0 - \int_0^{t-\nu(t)} \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) u(\tau) u_x(\tau)\, d\tau \\ &\quad - \int_{t-\nu(t)}^t \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) u(\tau) u_x(\tau)\, d\tau, \qquad (4.108)\end{aligned}$$

where $\nu(t) = t^{\frac{2}{3}}$ for $t \ge 1$ and $\nu(t) = 0$ for $t \in (0,1)$. The first summand in the right-hand side of (4.108) can be estimated as

$$\left\| \partial_x^3 \mathcal{B}\mathcal{G}(t) u_0 \right\|_{\mathbf{L}^1} \le C \langle t \rangle^{-\frac{3}{2}} \|u_0\|_{\mathbf{W}_1^1}. \qquad (4.109)$$

For the second term in the right-hand side of (4.108) by changing the order of integration and by applying the Cauchy inequality we find

$$\begin{aligned}&\int_0^T dt \left\| \int_0^{t-\nu(t)} \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) u(\tau) u_x(\tau)\, d\tau \right\|_{\mathbf{L}^1} \\ &\le \int_0^T dt \int_{\nu(t)}^t \langle \tau \rangle^{-\frac{3}{2}} \|u(t-\tau)\|_{\mathbf{L}^2} \|u_x(t-\tau)\|_{\mathbf{L}^2}\, d\tau \\ &\le C \|u\|_{\infty,2} \int_0^T dt \int_{\nu(t)}^t \langle \tau \rangle^{-\frac{3}{2}} \|u_x(t-\tau)\|_{\mathbf{L}^2}\, d\tau \\ &\le C \int_0^T dt \langle t \rangle^{-\frac{1}{3}} \int_0^T \langle \tau \rangle^{-1} \|u_x(t-\tau)\|_{\mathbf{L}^2}\, d\tau \\ &\le C \int_0^T d\tau \langle \tau \rangle^{-1} \int_0^T \langle t \rangle^{-\frac{1}{3}} \|u_x(t-\tau)\|_{\mathbf{L}^2}\, dt \\ &\le C \langle T \rangle^{\frac{1}{6}} \log \langle T \rangle. \qquad (4.110)\end{aligned}$$

We now estimate the third term in the right-hand side of (4.108)

$$
\begin{aligned}
&\int_{\mathbf{R}} S(t,x) \int_{t-\nu(t)}^{t} \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) u(\tau) u_x(\tau) d\tau dx \\
&= \int_{\mathbf{R}} dx \left|u(t,x)\right| \int_{t-\nu(t)}^{t} d\tau \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) u_x(\tau) \\
&+ \int_{\mathbf{R}} dx S(t,x) \int_{t-\nu(t)}^{t} d\tau \left[\partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau), u(\tau)\right] u_x(\tau) \\
&+ \int_{\mathbf{R}} dx S(t,x) \int_{t-\nu(t)}^{t} d\tau \left(u(\tau,x) - u(t,x)\right) \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) u_x(\tau) \\
&= I_1 + I_2 + I_3,
\end{aligned} \tag{4.111}
$$

where the commutator

$$
\begin{aligned}
&\left[\partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau), u(\tau)\right] \phi(\tau) \\
&\equiv \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau)\left(u(\tau)\phi(\tau)\right) - u(\tau) \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) \phi(\tau).
\end{aligned}
$$

In the integral I_1 we integrate by parts to get

$$
\begin{aligned}
I_1 &= \int_{\mathbf{R}} dx \left|u(t,x)\right| \int_{t-\nu(t)}^{t} d\tau \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) u_x(\tau) \\
&= -\int_{\mathbf{R}} dx u_x(t,x) S(t,x) \int_{t-\nu(t)}^{t} d\tau \partial_x^2 \mathcal{B}^2 \mathcal{G}(t-\tau) u_x(\tau);
\end{aligned}
$$

hence by the Young inequality

$$
\begin{aligned}
\int_0^T \left|I_1(t)\right| dt &\le \int_0^T dt \left\|u_x(t)\right\|_{\mathbf{L}^2} \int_{t-\nu(t)}^{t} \frac{d\tau}{\langle t-\tau\rangle} \left\|u_x(\tau)\right\|_{\mathbf{L}^2} \\
&\le C \int_0^T dt \left\|u_x(t)\right\|_{\mathbf{L}^2} \int_0^T \frac{d\tau}{\langle\tau\rangle} \left\|u_x(t-\tau)\right\|_{\mathbf{L}^2} \\
&\le C \left\|u_x\right\|_{2,2}^2 \log(T+1) \le C \log(T+1).
\end{aligned} \tag{4.112}
$$

For the integral I_2 by Lemma 4.30 via the Young inequality, we find

$$
\begin{aligned}
\int_0^T \left|I_2(t)\right| dt &\le \int_0^T dt \int_0^t d\tau \left\|\left[\partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau), u(\tau)\right] u_x(\tau)\right\|_{\mathbf{L}^1} \\
&\le C \int_0^T dt \int_0^t \langle t-\tau\rangle^{-1} \left\|u_x(\tau)\right\|_{\mathbf{L}^2}^2 d\tau \\
&\le C \left\|u_x\right\|_{2,2}^2 \log\langle T\rangle \le C \log\langle T\rangle.
\end{aligned} \tag{4.113}
$$

To estimate I_3 we use integral equation (4.106)

$$
\begin{aligned}
&u(t) - u(t-\tau) = \int_0^\tau u_t(t-t')\,dt' \\
&= \int_0^\tau dt' \partial_t \mathcal{G}(t-t')\,u_0 - \int_0^\tau dt' \mathcal{B}\left(u(t-t')\,u_x(t-t')\right) \\
&- \int_0^\tau dt' \int_0^{t-t'} d\tau' \partial_t \mathcal{G}(t-t'-\tau')\,\mathcal{B}\left(u(\tau')\,u_x(\tau')\right);
\end{aligned}
$$

hence

$$
\begin{aligned}
&\|u(t) - u(t-\tau)\|_{\mathbf{L}^1} \le \int_0^\tau dt' \langle t-t'\rangle^{-1} + \int_0^\tau dt' \|u_x(t-t')\|_{\mathbf{L}^2} \\
&+ \int_0^\tau dt' \int_0^{t-t'} d\tau' \langle \tau'\rangle^{-1} \|u_x(t-t'-\tau')\|_{\mathbf{L}^2}.
\end{aligned}
$$

We then have

$$
\begin{aligned}
&\int_0^T |I_3(t)|\,dt \\
&\le \int_0^T dt \int_0^{\nu(t)} d\tau \|u(t) - u(t-\tau)\|_{\mathbf{L}^1} \left\|\partial_x^3 \mathcal{B}^2 \mathcal{G}(\tau)\,u_x(t-\tau)\right\|_{\mathbf{L}^\infty} \\
&\le C \int_0^T dt \int_0^{\nu(t)} d\tau \langle\tau\rangle^{-\frac{7}{4}} \|u_x(t-\tau)\|_{\mathbf{L}^2} \left(\int_0^\tau dt' \langle t-t'\rangle^{-1}\right. \\
&\left.+ \int_0^\tau dt' \|u_x(t-t')\|_{\mathbf{L}^2} + \int_0^\tau dt' \int_0^{t-t'} d\tau' \langle\tau'\rangle^{-1} \|u_x(t-t'-\tau')\|_{\mathbf{L}^2}\right).
\end{aligned}
$$

Therefore, by changing the order of integration we obtain

$$
\begin{aligned}
&\int_0^T |I_3(t)|\,dt \\
&\le \int_0^{T^{\frac{2}{3}}} dt' \int_{t'}^{T^{\frac{2}{3}}} d\tau \langle\tau\rangle^{-\frac{7}{4}} \int_{\tau^{\frac{3}{2}}}^T dt \|u_x(t-\tau)\|_{\mathbf{L}^2} \left(\langle t-t'\rangle^{-1}\right. \\
&\left.+ \|u_x(t-t')\|_{\mathbf{L}^2} + \int_0^{t-t'} d\tau' \langle\tau'\rangle^{-1} \|u_x(t-t'-\tau')\|_{\mathbf{L}^2}\right) \\
&\le C \log\langle T\rangle \int_0^{T^{\frac{2}{3}}} dt' \int_{t'}^{T^{\frac{2}{3}}} d\tau \langle\tau\rangle^{-\frac{7}{4}} \le C \log\langle T\rangle \int_0^{T^{\frac{2}{3}}} dt' \langle t'\rangle^{-\frac{3}{4}} \\
&\le C \langle T\rangle^{\frac{1}{6}} \log\langle T\rangle. \qquad (4.114)
\end{aligned}
$$

The substitution of estimates (4.112) - (4.114) into (4.111) yields

$$
\left|\int_{\mathbf{R}} S(t,x) \int_{t-\nu(t)}^t \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau)\,u(\tau)\,u_x(\tau)\,d\tau dx\right| \le C \langle T\rangle^{\frac{1}{6}} \log\langle T\rangle. \quad (4.115)
$$

Now from (4.109), (4.110) and (4.115) we get the result of the lemma, and Lemma 4.32 is proved.

Now we give estimates for the third derivative of the solution.

Lemma 4.33. *Let the initial data* $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R})$. *Moreover we assume that the norms of the solutions are bounded*

$$\|u(t)\|_{\mathbf{L}^2} \le C(1+t)^{\sigma-\frac{1}{4}}$$

for all $t > 0$, *where* $\sigma \in \left(0, \frac{1}{4}\right]$. *Then the estimate is true*

$$\|\mathcal{B}u_{xxx}\|_{s,1} \le C,$$

where $s > \max\left(1, \left(\frac{9}{8} - \frac{3}{2}\sigma\right)^{-1}\right)$ *for* $\sigma \in \left[\frac{1}{12}, \frac{1}{4}\right]$ *and* $s = 1$ *for* $\sigma \in \left(0, \frac{1}{12}\right)$. *Moreover we have*

$$\left\|\partial_x^2 \mathcal{B}uu_x\right\|_{s,1} \le C \tag{4.116}$$

where $s > 1$ *for* $\sigma \in \left[\frac{1}{12}, \frac{1}{4}\right]$, *and* $s = 1$ *if* $\sigma \in \left(0, \frac{1}{12}\right)$.

Proof. In view of the integral equation (4.106), we find

$$\begin{aligned}
&\|\mathcal{B}u_{xx}(t)\|_{\mathbf{L}^1} \le \left\|\mathcal{B}\partial_x^2\mathcal{G}(t)u_0\right\|_{\mathbf{L}^1} \\
&+ \left\|\int_0^t \mathcal{B}\partial_x^2\mathcal{G}(t-\tau)\mathcal{B}u(\tau)u_x(\tau)\,d\tau\right\|_{\mathbf{L}^1} \\
&\le \left\|\mathcal{B}\partial_x^2\mathcal{G}(t)u_0\right\|_{\mathbf{L}^1} + C\int_0^t \langle t-\tau\rangle^{-1}\|u(\tau)\|_{\mathbf{L}^2}\|u_x(\tau)\|_{\mathbf{L}^2}\,d\tau.
\end{aligned}$$

First we note that

$$\left\|\mathcal{B}\partial_x^2\mathcal{G}(t)u_0\right\|_{\mathbf{L}^1} \le C\langle t\rangle^{-1}\|u_0\|_{\mathbf{L}^1}.$$

By the Young inequality we obtain

$$\begin{aligned}
&\left\|\int_0^t \langle t-\tau\rangle^{-1}\|u(\tau)\|_{\mathbf{L}^2}\|u_x(\tau)\|_{\mathbf{L}^2}\,d\tau\right\|_{\mathbf{L}_t^s(0,\infty)} \\
&\le C\left\|\int_0^t \langle t-\tau\rangle^{-1}\langle\tau\rangle^{\sigma-\frac{1}{4}}\|u_x(\tau)\|_{\mathbf{L}^2}\,d\tau\right\|_{\mathbf{L}_t^s(0,\infty)} \\
&\le C\left\|\langle t\rangle^{-1}\right\|_{\mathbf{L}_t^{s_1}(0,\infty)}\left\|\langle t\rangle^{\sigma-\frac{1}{4}}\right\|_{\mathbf{L}_t^{s_2}(0,\infty)}\|u_x\|_{2,2}
\end{aligned}$$

for all $t > 0$, where $\frac{1}{s}+1 = \frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{2}$, $s_1 > 1$, $s_2 > \left(\frac{1}{4}-\sigma\right)^{-1}$, $s > \left(\frac{3}{4}-\sigma\right)^{-1}$ for $\sigma \in \left(0, \frac{1}{4}\right)$; in the case of $\sigma = \frac{1}{4}$ we take $s_2 = \infty$. Collecting these estimates we get

$$\|\mathcal{B}u_{xx}\|_{s,1} \le C$$

for all $s > \left(\frac{3}{4} - \sigma\right)^{-1}$.

In the same manner we estimate the third derivative

$$\|\mathcal{B}u_{xxx}(t)\|_{\mathbf{L}^1} \le \left\|\mathcal{B}\partial_x^3 \mathcal{G}(t)\, u_0\right\|_{\mathbf{L}^1} + \left\| \int_0^{\frac{t}{2}} \mathcal{B}^2 \partial_x^4 \mathcal{G}(t-\tau)\, u^2(\tau)\, d\tau \right\|_{\mathbf{L}^1}$$
$$+ \left\| \int_{\frac{t}{2}}^t \mathcal{B}\partial_x^2 \mathcal{G}(t-\tau)\, \partial_x \mathcal{B} u(\tau)\, u_x(\tau)\, d\tau \right\|_{\mathbf{L}^1}.$$

By Lemma 4.30 we find

$$\left\|\mathcal{B}\partial_x^3 \mathcal{G}(t)\, u_0\right\|_{\mathbf{L}^1} \le C \langle t\rangle^{-\frac{3}{2}} \|u_0\|_{\mathbf{W}_1^1}.$$

In addition

$$\|\partial_x \mathcal{B}u(\tau)\, u_x(\tau)\|_{\mathbf{L}^1} \le \left\|\mathcal{B}u_x^2(\tau)\right\|_{\mathbf{L}^1} + \|u(\tau)\, \mathcal{B}u_{xx}(\tau)\|_{\mathbf{L}^1}$$
$$+ \|[u(\tau), \mathcal{B}]\, u_{xx}(\tau)\|_{\mathbf{L}^1},$$

where

$$[u(\tau), \mathcal{B}]\, \partial_x^2 u(\tau) = \frac{1}{2} \int_{\mathbf{R}} e^{-|x-y|} (u(\tau,x) - u(\tau,y))\, u_{yy}(\tau,y)\, dy.$$

By the Cauchy inequality

$$\|[u(\tau), \mathcal{B}]\, u_{xx}(\tau)\|_{\mathbf{L}^1}$$
$$= \frac{1}{2} \left\| \int_{\mathbf{R}} e^{-|x-y|} (u(\tau,x) - u(\tau,y))\, u_{yy}(\tau,y)\, dy \right\|_{\mathbf{L}^1}$$
$$\le \left\| \int_{\mathbf{R}} \operatorname{sign}(x-y)\, e^{-|x-y|} (u(\tau,x) - u(\tau,y))\, u_y(\tau,y)\, dy \right\|_{\mathbf{L}^1}$$
$$+ \left\| \int_{\mathbf{R}} e^{-|x-y|} u_y^2(\tau,y)\, dy \right\|_{\mathbf{L}^1}$$
$$\le C \int_{\mathbf{R}} dx \int_{\mathbf{R}} dy e^{-|x-y|} \operatorname{sign}(x-y)\, |u_y(\tau,y)| \int_0^{x-y} |u_x(\tau, y+z)|\, dz$$
$$+ C \|u_x(\tau)\|_{\mathbf{L}^2}^2$$
$$\le C \int_{\mathbf{R}} d\xi e^{-|\xi|} |\xi| \int_{\mathbf{R}} dy\, |u_y(\tau,y)| \int_0^{\xi} |u_x(\tau, y+z)|\, dz$$
$$\le C \|u_x(\tau)\|_{\mathbf{L}^2}^2.$$

Hence

$$\|\mathcal{B}u_{xxx}(t)\|_{\mathbf{L}^1} \le C \langle t\rangle^{-\frac{3}{2}} \|u_0\|_{\mathbf{W}_1^1} + C \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-2} \langle \tau\rangle^{2\sigma - \frac{1}{2}}\, d\tau$$
$$+ C \int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-1} \left(\|u_x(\tau)\|_{\mathbf{L}^2}^2 + \langle \tau\rangle^{\frac{\sigma}{2} - \frac{1}{8}} \|u_x(\tau)\|_{\mathbf{L}^2}^{\frac{1}{2}} \|\mathcal{B}u_{xx}(\tau)\|_{\mathbf{L}^1} \right) d\tau$$

for all $t > 0$. We have

$$\left\| \int_0^t \langle t-\tau \rangle^{-1} \| u_x(\tau) \|_{\mathbf{L}^2}^2 \, d\tau \right\|_{\mathbf{L}_t^s(0,\infty)} \le C$$

for $s > 1$. Using Lemma 4.30 and the Young inequality we obtain

$$\left\| \int_0^t \langle t-\tau \rangle^{-1} \| u(\tau) \|_{\mathbf{L}^\infty} \| \mathcal{B} u_{xx}(\tau) \|_{\mathbf{L}^1} \, d\tau \right\|_{\mathbf{L}_t^s(0,\infty)}$$
$$\le C \left\| \langle t \rangle^{-1} \right\|_{\mathbf{L}_t^{s_1}(0,\infty)} \| u \|_{\infty,2}^{\frac{1}{2}} \| u_x \|_{2,2}^{\frac{1}{2}} \| \mathcal{B} u_{xx} \|_{s_2,1}$$

where $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{4} - 1 < \frac{3}{4}$, since $s_1 > 1$, $s_2 > 2$.

Collecting these estimates we get

$$\| \mathcal{B} u_{xxx} \|_{s,1} \le C$$

for all $s > \max\left(1, \left(\frac{9}{8} - \frac{3}{2}\sigma\right)^{-1}\right)$ for $\sigma \in \left[\frac{1}{12}, \frac{1}{4}\right]$.

Consider now the case of $\sigma \in \left(0, \frac{1}{12}\right)$. Then we can obtain a better decay estimate for $\| u_x(t) \|_{\mathbf{L}^2}$. In view of the integral equation (4.104) we find

$$\| u_x(t) \|_{\mathbf{L}^2} \le \| \partial_x \mathcal{G}(t) u_0 \|_{\mathbf{L}^2} + \left\| \int_0^t \mathcal{B} \partial_x^2 \mathcal{G}(t-\tau) u^2(\tau) \, d\tau \right\|_{\mathbf{L}^2}$$
$$\le C \langle t \rangle^{-\frac{3}{4}} + C \int_0^t \langle t-\tau \rangle^{-1} \| u(\tau) \|_{\mathbf{L}^2}^{\frac{3}{2}} \| u_x(\tau) \|_{\mathbf{L}^2}^{\frac{1}{2}} \, d\tau$$
$$\le C \langle t \rangle^{-\frac{3}{4}} + C t^{-1} \int_0^{\frac{t}{2}} \langle \tau \rangle^{\frac{3}{2}\sigma - \frac{3}{8}} \| u_x(\tau) \|_{\mathbf{L}^2}^{\frac{1}{2}} \, d\tau$$
$$+ C \langle t \rangle^{\frac{3}{2}\sigma - \frac{3}{8}} \int_{\frac{t}{2}}^t \langle t-\tau \rangle^{-1} \| u_x(\tau) \|_{\mathbf{L}^2}^{\frac{1}{2}} \, d\tau$$
$$\le C \langle t \rangle^{\frac{3}{2}\sigma - \frac{5}{8}} + \frac{C}{\varepsilon} \langle t \rangle^{3\sigma - \frac{3}{4}} \log \langle t \rangle$$
$$+ \frac{C\varepsilon}{\log \langle t \rangle} \int_{\frac{t}{2}}^t \langle t-\tau \rangle^{-1} \| u_x(\tau) \|_{\mathbf{L}^2} \, d\tau.$$

Choosing a sufficiently small $\varepsilon > 0$ and applying the Gronwall inequality we obtain

$$\| u_x \|_{\mathbf{L}^2} \le C \langle t \rangle^{\frac{3}{2}\sigma - \frac{5}{8}}$$

for all $t > 0$. By applying this estimate, we find

$$\| u_x(t) \|_{\mathbf{L}^2} \le C \langle t \rangle^{-\frac{3}{4}} + C \int_0^t \langle t-\tau \rangle^{-1} \| u(\tau) \|_{\mathbf{L}^2}^{\frac{3}{2}} \| u_x(\tau) \|_{\mathbf{L}^2}^{\frac{1}{2}} \, d\tau$$
$$\le C \langle t \rangle^{-\frac{3}{4}} + C t^{-1} \int_0^{\frac{t}{2}} \langle \tau \rangle^{\frac{9}{4}\sigma - \frac{11}{16}} \, d\tau + C \langle t \rangle^{\frac{9}{4}\sigma - \frac{11}{16}} \int_{\frac{t}{2}}^t \langle t-\tau \rangle^{-1} \, d\tau$$
$$\le C \langle t \rangle^{-\frac{3}{4}} + C \langle t \rangle^{\frac{9}{4}\sigma - \frac{11}{16}} \log \langle t \rangle.$$

Iterating this procedure, we gain

$$\|u_x(t)\|_{\mathbf{L}^2} \le C\langle t\rangle^{-\frac{3}{4}} + C\langle t\rangle^{3\sigma-\frac{3}{4}}\log\langle t\rangle .$$

By the Young inequality we obtain

$$\int_0^t \langle t-\tau\rangle^{-1}\|u(\tau)\|_{\mathbf{L}^2}\|u_x(\tau)\|_{\mathbf{L}^2}\,d\tau \le C\int_0^t \langle t-\tau\rangle^{-1}\langle\tau\rangle^{4\sigma-1}\,d\tau$$

for all $t>0$. Collecting these estimates we get

$$\|\mathcal{B}u_{xx}(t)\|_{\mathbf{L}^1} \le C\langle t\rangle^{4\sigma-1}\log^2\langle t\rangle$$

for all $t>0$. In a similar manner we estimate the third derivative

$$\|\mathcal{B}u_{xxx}(t)\|_{\mathbf{L}^1} \le \left\|\mathcal{B}\partial_x^3\mathcal{G}(t)\,u_0\right\|_{\mathbf{L}^1} + \left\|\int_0^{\frac{t}{2}}\mathcal{B}\partial_x^4\mathcal{G}(t-\tau)\,\mathcal{B}u^2(\tau)\,d\tau\right\|_{\mathbf{L}^1}$$
$$+\left\|\int_{\frac{t}{2}}^t \mathcal{B}\partial_x^2\mathcal{G}(t-\tau)\,\partial_x\mathcal{B}u(\tau)\,u_x(\tau)\,d\tau\right\|_{\mathbf{L}^1}.$$

We have

$$\left\|\mathcal{B}\partial_x^3\mathcal{G}(t)\,u_0\right\|_{\mathbf{L}^1} \le C\langle t\rangle^{-\frac{3}{2}};$$

hence,

$$\|\mathcal{B}u_{xxx}(t)\|_{\mathbf{L}^1} \le C\langle t\rangle^{-\frac{3}{2}} + C\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-2}\langle\tau\rangle^{2\sigma-\frac{1}{2}}\,d\tau$$
$$+\,C\int_{\frac{t}{2}}^t\langle t-\tau\rangle^{-1}\langle\tau\rangle^{6\sigma-\frac{3}{2}}\log^2\langle\tau\rangle\,d\tau \le C\langle t\rangle^{6\sigma-\frac{3}{2}}\log^3\langle t\rangle$$

for all $t>0$. By this estimate we get

$$\|\mathcal{B}u_{xxx}\|_{1,1} \le C$$

if $\sigma \in \left(0,\frac{1}{12}\right)$. Estimate (4.116) is proved in an identical fashion since we have

$$\left\|\partial_x^2\mathcal{B}u(\tau)\,u_x(\tau)\right\|_{\mathbf{L}^1} \le C\left\|\partial_x\mathcal{B}u_x^2(\tau)\right\|_{\mathbf{L}^1} + \|u(\tau)\,\mathcal{B}u_{xxx}(\tau)\|_{\mathbf{L}^1}$$
$$+\,\|[u(\tau),\mathcal{B}]\,u_{xxx}(\tau)\|_{\mathbf{L}^1},$$

where by integrating by parts and by using the Cauchy inequality

$$\begin{aligned}
&\|[u(\tau),\mathcal{B}]\,u_{xxx}(\tau)\|_{\mathbf{L}^1} \\
&= \frac{1}{2}\left\|\int_{\mathbf{R}} e^{-|x-y|}\left(u(\tau,x)-u(\tau,y)\right)u_{yyy}(\tau,y)\,dy\right\|_{\mathbf{L}^1} \\
&\le \left\|\int_{\mathbf{R}} \operatorname{sign}(x-y)\,e^{-|x-y|}\left(u(\tau,x)-u(\tau,y)\right)u_{yy}(\tau,y)\,dy\right\|_{\mathbf{L}^1} \\
&+ \left\|\int_{\mathbf{R}} e^{-|x-y|}u_y(\tau,y)\,u_{yy}(\tau,y)\,dy\right\|_{\mathbf{L}^1} \\
&\le C\left\|\int_{\mathbf{R}} e^{-|x-y|}\left(u(\tau,x)-u(\tau,y)\right)u_y(\tau,y)\,dy\right\|_{\mathbf{L}^1} \\
&+C\left\|\int_{\mathbf{R}} e^{-|x-y|}u_y^2(\tau,y)\,dy\right\|_{\mathbf{L}^1} \le C\,\|u_x(\tau)\|_{\mathbf{L}^2}^2
\end{aligned}$$

as in the previous case. Lemma 4.33 is proved.

Now we estimate the decay rate of the $\mathbf{L}^2(\mathbf{R})$ norm of the solutions.

Lemma 4.34. *Let $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R})$. Assume that*

$$\begin{aligned}
&\left|\int_0^t d\tau \int_{\mathbf{R}} S(\tau,x)\,\mathcal{B}u_{xxx}(\tau,x)\,dx\right| \\
&+\int_0^t d\tau\,\left\|\partial_x^2\mathcal{B}\left(u(\tau)\,u_x(\tau)\right)\right\|_{\mathbf{L}^1} \le C\langle t\rangle^{\sigma}
\end{aligned} \tag{4.117}$$

for all $t>0$, where $\sigma \in \left[0,\frac{1}{4}\right)$. Then the estimate is valid

$$\|u(t)\|_{\mathbf{L}^p} \le C\langle t\rangle^{\sigma-\frac{1}{2}\left(1-\frac{1}{p}\right)}$$

for all $t>0$, where $1\le p\le 2$.

Proof. Applying operator $\mathcal{B}=\left(1-\partial_x^2\right)^{-1}$ to (4.100) we get

$$u_t = \mu(1-\mathcal{B})\,u + \beta\partial_x^3\mathcal{B}u + \mathcal{B}uu_x \tag{4.118}$$

since $\mathcal{B} = 1-\partial_x^2\mathcal{B}$. We estimate the $\mathbf{L}^1(\mathbf{R})$ norm. We multiply equation (4.118) by $S(t,x)$ and integrate with respect to x over $\mathbf{R}$ to get

$$\begin{aligned}
&\int_{\mathbf{R}} \partial_t\,|u(t,x)|\,dx = \mu\int_{\mathbf{R}} S(t,x)\,(1-\mathcal{B})\,udx \\
&+\beta\int_{\mathbf{R}} S(t,x)\,\partial_x^3\mathcal{B}udx + \int_{\mathbf{R}} |u(t,x)|\,u_xdx \\
&-\int_{\mathbf{R}} S(t,x)\,\partial_x^2\mathcal{B}uu_xdx.
\end{aligned}$$

We have

$$\int_{\mathbf{R}} \partial_t |u(t,x)| \, dx = \frac{d}{dt} \|u(t)\|_{\mathbf{L}^1},$$
$$\int_{\mathbf{R}} |u(t,x)| \, u_x dx = 0,$$
$$\int_{\mathbf{R}} S(t,x) \mathcal{B} u dx \le \int_{\mathbf{R}} \mathcal{B} |u| \, dx \le \|u(t)\|_{\mathbf{L}^1}.$$

Therefore, we find

$$\|u(t)\|_{\mathbf{L}^1} \le \|u_0\|_{\mathbf{L}^1} + \left| \beta \int_0^t dt \int_{\mathbf{R}} S(t,x) \, \partial_x^3 \mathcal{B} u dx \right| + \left| \int_0^t dt \int_{\mathbf{R}} S(t,x) \, \partial_x^2 \mathcal{B} u u_x dx \right|. \tag{4.119}$$

Then estimate (4.117) yields

$$\sup_{\xi \in \mathbf{R}} |\widehat{u}(t,\xi)| \le C \|u(t)\|_{\mathbf{L}^1} \le \|u_0\|_{\mathbf{L}^1} + C \langle t \rangle^{\sigma} \le C \langle t \rangle^{\sigma} \tag{4.120}$$

for all $t > 0$. Thus the estimate of the lemma with $p = 1$ is fulfilled.

We now multiply equation (4.100) by $2u$, then by integrating with respect to $x \in \mathbf{R}$ we get

$$\frac{d}{dt} \left(\|u(t)\|_{\mathbf{L}^2}^2 + \|u_x(t)\|_{\mathbf{L}^2}^2 \right) = -2\mu \|u_x(t)\|_{\mathbf{L}^2}^2. \tag{4.121}$$

By the Plancherel theorem using the Fourier splitting method from Schonbek [1991], we have

$$\|u_x(t)\|_{\mathbf{L}^2}^2 = \|\xi \widehat{u}(t)\|_{\mathbf{L}^2}^2 = \int_{|\xi| \le \delta} |\widehat{u}(t,\xi)|^2 \xi^2 d\xi + \int_{|\xi| \ge \delta} |\widehat{u}(t,\xi)|^2 \xi^2 d\xi$$
$$\ge \delta^2 \|u(t)\|_{\mathbf{L}^2}^2 - 2\delta^3 \sup_{|\xi| \le \delta} |\widehat{u}(t,\xi)|^2$$

where $\delta > 0$. Thus from (4.121) we have the inequality

$$\frac{d}{dt} \|u(t)\|_{\mathbf{H}^1}^2 \le -\mu \delta^2 \|u(t)\|_{\mathbf{H}^1}^2 + 4\mu \delta^3 \sup_{\xi \le \delta} |\widehat{u}(t,\xi)|^2. \tag{4.122}$$

We choose $\mu \delta^2 = 2(1+t)^{-1}$ and change $\|u(t)\|_{\mathbf{H}^1}^2 = (1+t)^{-2} W(t)$. Then via (4.120) we get from (4.122)

$$\frac{d}{dt} W(t) \le C (1+t)^{2\sigma + \frac{1}{2}}. \tag{4.123}$$

Integration of (4.123) with respect to time yields

$$W(t) \le \|u_0\|_{\mathbf{H}^1}^2 + C \left((1+t)^{\frac{3}{2} + 2\sigma} - 1 \right).$$

Therefore we obtain a time decay estimate of the $\mathbf{L}^2$ norm

$$\|u(t)\|_{\mathbf{L}^2} \le C(1+t)^{\sigma-\frac{1}{4}} \tag{4.124}$$

for all $t > 0$. Lemma 4.34 is proved.

Proposition 4.35. *Suppose that the initial data* $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R})$. *Then the estimates for the solution are valid*

$$\|u(t)\|_{\mathbf{L}^p} + \langle t \rangle \left\|\partial_x^2 \mathcal{B} u(t)\right\|_{\mathbf{L}^p} \le C \langle t \rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$, *where* $1 \le p \le \infty$.

Proof. By proposition 4.31 we have estimate

$$\|u\|_{\infty,2} + \|u_x\|_{\infty,2} + \|u_x\|_{2,2} \le C \|u_0\|_{\mathbf{H}^1}.$$

Now by applying Lemma 4.32 we get

$$\left|\int_0^t d\tau \int_{\mathbf{R}} S(\tau, x) \mathcal{B} u_{xxx}(\tau, x)\, dx\right| \le C \langle t \rangle^{\frac{1}{6}} \log \langle t \rangle \le C \langle t \rangle^{\sigma_0} \tag{4.125}$$

for all $t > 0$, where $\sigma_0 = \frac{1}{6} + \gamma$ and $\gamma > 0$ is small. Also by Lemma 4.33 we have

$$\int_0^t d\tau \left\|\partial_x^2 \mathcal{B}(u(\tau) u_x(\tau))\right\|_{\mathbf{L}^1} \le C \langle t \rangle^{\frac{1}{16}}$$

for all $t > 0$. Then by Lemma 4.34 we find the time decay of the $\mathbf{L}^2(\mathbf{R})$ norm

$$\|u(t)\|_{\mathbf{L}^2} \le C(1+t)^{\sigma_0-\frac{1}{4}} \tag{4.126}$$

for all $t > 0$. Applying Lemma 4.33 and the Hölder inequality we obtain

$$\left|\int_0^t d\tau \int_{\mathbf{R}} S(\tau, x) \mathcal{B} u_{xxx}(\tau, x)\, dx\right| \le C t^{1-\frac{1}{s_0}} \|u_{xxx}\|_{s_0,1} \le C t^{1-\frac{1}{s_0}},$$

for all $t > 0$, where $s_0 > \max\left(1, \left(\frac{9}{8} - \frac{3}{2}\sigma_0\right)^{-1}\right)$. Hence we arrive at estimate (4.125) with σ_0 replaced by $\sigma_1 = 1 - \frac{1}{s_0} = \frac{1}{8} + O(\gamma)$. We again apply Lemma 4.34 to get a better time decay of the $\mathbf{L}^2(\mathbf{R})$ norm (4.126) with σ_0 replaced by $\sigma_1 = \frac{1}{8} + O(\gamma)$:

$$\|u(t)\|_{\mathbf{L}^2} \le C(1+t)^{\sigma_1-\frac{1}{4}}.$$

Then Lemma 4.33 yields estimate (4.125) with σ_0 replaced by $\sigma_2 = \frac{1}{16} + O(\gamma)$. Now by Lemma 4.34 we get time decay estimate (4.126) with σ_0 replaced by $\sigma_2 = \frac{1}{16} + O(\gamma)$. Lemma 4.33 gives us estimate (4.125) with $\sigma_0 = 0$. Therefore by virtue of Lemma 4.34 we obtain an optimal time decay estimate of the $\mathbf{L}^2(\mathbf{R})$ norm

$$\|u(t)\|_{\mathbf{L}^2} \le C(1+t)^{-\frac{1}{4}} \tag{4.127}$$

for all $t > 0$. Using (4.127) we can prove the following optimal time decay estimates

$$\|u(t)\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \tag{4.128}$$

for all $t > 0$, where $1 \le p \le \infty$. For $1 \le p \le 2$, estimate (4.128) follows from (4.127), Lemma 4.34 and the Hölder inequality. Let us prove (4.128) for $p = \infty$. By the integral equation (4.106) and by the Hölder inequality we get

$$\begin{aligned}
\|u(t)\|_{\mathbf{L}^\infty} &\le \|\mathcal{G}(t)\,u_0\|_{\mathbf{L}^\infty} + \frac{1}{2}\int_0^{\frac{t}{2}} \|\partial_x \mathcal{B}G(t-\tau)\|_{\mathbf{L}^\infty} \|u(\tau)\|^2_{\mathbf{L}^2}\, d\tau \\
&+ \frac{1}{2}\int_{\frac{t}{2}}^t \|\partial_x \mathcal{B}G(t-\tau)\|_{\mathbf{L}^4} \left\|u^2(\tau)\right\|_{\mathbf{L}^{\frac{4}{3}}} d\tau \\
&\le Ct^{-\frac{1}{2}} + Ct^{-1}\int_0^{\frac{t}{2}} \langle\tau\rangle^{-\frac{1}{2}}\, d\tau \\
&+ C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{7}{8}} \|u(\tau)\|^{\frac{1}{2}}_{\mathbf{L}^\infty} \|u(\tau)\|^{\frac{3}{2}}_{\mathbf{L}^2}\, d\tau,
\end{aligned}$$

so

$$\begin{aligned}
\|u(t)\|_{\mathbf{L}^\infty} &\le Ct^{-\frac{1}{2}} + Ct^{-\frac{3}{8}}\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{7}{8}} \|u(\tau)\|^{\frac{1}{2}}_{\mathbf{L}^\infty}\, d\tau \\
&\le Ct^{-\frac{1}{2}} + C\varepsilon t^{-\frac{1}{8}}\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{7}{8}} \|u(\tau)\|_{\mathbf{L}^\infty}\, d\tau \\
&+ \frac{C}{\varepsilon} t^{-\frac{5}{8}} \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{7}{8}}\, d\tau.
\end{aligned}$$

Hence by the Gronwall lemma it follows that

$$\|u(t)\|_{\mathbf{L}^\infty} \le \frac{C}{\varepsilon} t^{-\frac{1}{2}}.$$

We find (4.128) for all $2 \le p \le \infty$ via the Hölder inequality. As above we get the estimates

$$\left\|\partial_x^2 \mathcal{B}u(t)\right\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-1}$$

for all $t > 0$, $1 \le p \le \infty$. Proposition 4.35 is thus proved.

4.4.2 Proof of Theorem 4.28

Now we obtain the large time asymptotic formulas for solutions to the Cauchy problem (4.100). Let us take a sufficiently large initial time $T > 0$ and define $v(t,x)$ as a solution to the Cauchy problem for the Burgers equation with $u(T,x)$ as the initial data

$$\begin{cases} v_t + vv_x - \mu v_{xx} = 0, \ t > T, \ x \in \mathbf{R}, \\ v(T,x) = u(T,x), \ x \in \mathbf{R}. \end{cases} \tag{4.129}$$

By the Hopf-Cole Hopf [1950] transformation $v(t,x) = -2\mu \frac{\partial}{\partial x} \log Z(t,x)$ it is converted to the heat equation $Z_t = \mu Z_{xx}$, so we have the solution explicitly

$$Z(t,x) = \int_{\mathbf{R}} dy G_0(t, x-y) \exp\left(-\frac{1}{2\mu} \int_{-\infty}^{y} u(T,\xi) \, d\xi \right), \tag{4.130}$$

where $G_0(t,x) = (4\pi\mu t)^{-\frac{1}{2}} e^{-\frac{x^2}{4\mu t}}$ is the Green function for the heat equation. Note that the following estimates are true

$$\|v(t)\|_{\mathbf{L}^p} + \langle t \rangle \left\| \partial_x^2 \mathcal{B} v(t) \right\|_{\mathbf{L}^p} \le C \langle t \rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \tag{4.131}$$

for all $t > T$, $1 \le p \le \infty$.

Consider now the difference $w(t,x) = u(t,x) - v(t,x)$ for all $t > T$. By (4.100) and (4.129) we get the Cauchy problem

$$\begin{cases} w_t + \partial_x(vw) + \frac{1}{2}\frac{\partial}{\partial x} w^2 - \mu w_{xx} + h_x = 0, \ t > T, \ x \in \mathbf{R}, \\ w(T,x) = 0, \ x \in \mathbf{R}, \end{cases} \tag{4.132}$$

where

$$h = \left(\beta \partial_x^2 + \mu \partial_x^3 \right) \mathcal{B} u - \partial_x \mathcal{B} u u_x.$$

Since we consider the large initial data, we need to eliminate the linear term $\frac{\partial}{\partial x}(vw)$. We change $\partial_x^{-1} w = \int_{-\infty}^{x} w(t,y) \, dy = \mu \frac{g}{Z}$, then from (4.132) we obtain the Cauchy problem

$$\begin{cases} g_t - \mu g_{xx} + F = 0, \ t > T, \ x \in \mathbf{R}, \\ g(T,x) = 0, \ x \in \mathbf{R}, \end{cases} \tag{4.133}$$

where

$$F = \frac{\mu}{2Z} \left(g_x + \frac{1}{2\mu} g v \right)^2 + Zh.$$

By virtue of estimates of Proposition 4.35 and (4.131) we have

$$\|Z(t)\|_{\mathbf{L}^\infty} + \left\| Z^{-1}(t) \right\|_{\mathbf{L}^\infty} \le C \tag{4.134}$$

for all $t \ge T$ and a rough time decay estimate

$$\|w(t)\|_{\mathbf{L}^p} + \langle t \rangle \left\| \partial_x^2 \mathcal{B} w(t) \right\|_{\mathbf{L}^p} \le C \langle t \rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \tag{4.135}$$

for all $t \ge T$, $1 \le p \le \infty$. Let us prove the estimate

$$\|g(t)\|_{\mathbf{L}^p} + \langle t \rangle^{\frac{1}{2}} \|g_x(t)\|_{\mathbf{L}^p} < C \langle t \rangle^{-\gamma + \frac{1}{2p}} \tag{4.136}$$

for all $t \ge T$, $2 \le p \le \infty$, where $\gamma \in \left(0, \frac{1}{2}\right)$. On the contrary we suppose that for some $t = T_1$ estimate (4.136) is violated, that is we have

$$\|g(t)\|_{\mathbf{L}^p} + \langle t\rangle^{\frac{1}{2}} \|g_x(t)\|_{\mathbf{L}^p} \le C \langle t\rangle^{-\gamma+\frac{1}{2p}} \tag{4.137}$$

for all $t \in [T, T_1]$, $1 \le p \le \infty$. In view of (4.131), (4.134), (4.136) and (4.137) we find

$$\begin{aligned} \|F(t)\|_{\mathbf{L}^p} &\le C \|g_x\|_{\mathbf{L}^\infty} \|g_x\|_{\mathbf{L}^p} + C \|g\|^2_{\mathbf{L}^\infty} \left\|v^2\right\|_{\mathbf{L}^p} + C \|Zh\|_{\mathbf{L}^p} \\ &\le C \left(\langle t\rangle^{-1-2\gamma+\frac{1}{2p}} + \langle t\rangle^{-\frac{3}{2}+\frac{1}{2p}}\right) \\ &\le C \max\left\{T^{\gamma-\frac{1}{2}}, T^{-\gamma}\right\} \langle t\rangle^{-\gamma-1+\frac{1}{2p}} \end{aligned} \tag{4.138}$$

for all $t \in [T, T_1]$, $1 \le p \le \infty$. Using the integral equation associated with (4.133) in view of estimate (4.138) we find

$$\begin{aligned} \|g(t)\|_{\mathbf{L}^p} &\le \int_T^t d\tau \|G_0(t-\tau)\|_{\mathbf{L}^p} \|F(\tau)\|_{\mathbf{L}^1} \\ &\le C \max\left\{T^{\gamma-\frac{1}{2}}, T^{-\gamma}\right\} \int_T^t (t-\tau)^{-\frac{1}{2}+\frac{1}{2p}} \langle\tau\rangle^{-\frac{1}{2}-\gamma} d\tau \\ &\le C \max\left\{T^{\gamma-\frac{1}{2}}, T^{-\gamma}\right\} \langle t\rangle^{-\gamma+\frac{1}{2p}} < C \langle t\rangle^{-\gamma+\frac{1}{2p}} \end{aligned}$$

for all $t \in [T, T_1]$, $1 \le p \le \infty$, since T is sufficiently large. In the same manner we have

$$\begin{aligned} \|g_x(t)\|_{\mathbf{L}^p} &\le \int_T^{\frac{t+T}{2}} d\tau \|\partial_x G_0(t-\tau)\|_{\mathbf{L}^p} \|F(\tau)\|_{\mathbf{L}^1} \\ &+ \int_{\frac{t+T}{2}}^t d\tau \|\partial_x G_0(t-\tau)\|_{\mathbf{L}^1} \|F(\tau)\|_{\mathbf{L}^p} \\ &\le C \max\left\{T^{\gamma-\frac{1}{2}}, T^{-\gamma}\right\} \int_T^{\frac{t+T}{2}} (t-\tau)^{-1+\frac{1}{2p}} \langle\tau\rangle^{-\frac{1}{2}-\gamma} d\tau \\ &+ C \max\left\{T^{\gamma-\frac{1}{2}}, T^{-\gamma}\right\} \int_{\frac{t+T}{2}}^t (t-\tau)^{-\frac{1}{2}} \langle\tau\rangle^{-1-\gamma+\frac{1}{2p}} d\tau \\ &\le C \max\left\{T^{\gamma-\frac{1}{2}}, T^{-\gamma}\right\} \langle t\rangle^{-\gamma-\frac{1}{2}+\frac{1}{2p}} < C \langle t\rangle^{-\gamma-\frac{1}{2}+\frac{1}{2p}} \end{aligned}$$

for all $t \in [T, T_1]$, $1 \le p \le \infty$. The contradiction obtained proves estimate (4.136) for all $t \ge T$. Since

$$w = Z^{-1}\left(\mu g_x + \frac{1}{2} g v\right)$$

estimate (4.136) implies

$$\|u(t) - v(t)\|_{\mathbf{L}^\infty} \le C \langle t\rangle^{-\gamma-\frac{1}{2}}$$

for all $t > T$. It is known that if $xu_0 \in \mathbf{L}^1(\mathbf{R})$, then

$$\left\|v(t) - t^{-\frac{1}{2}} f_\theta\left((\cdot)\, t^{-\frac{1}{2}}\right)\right\|_{\mathbf{L}^\infty} \le C \langle t\rangle^{-\gamma-\frac{1}{2}}.$$

Therefore the estimate of the theorem follows, and Theorem 4.28 is proved.

4.4.3 Proof of Theorem 4.29

As in the proof of Proposition 4.35 we can obtain the following estimate of the $\mathbf{L}^{1,1}(\mathbf{R})$ norm of solutions of the Cauchy problem (4.100)

$$\|u(t)\|_{\mathbf{L}^{1,1}} \le C\langle t\rangle^{\frac{1}{2}}. \tag{4.139}$$

Indeed, by applying estimates of Lemma 4.30 to the integral equation (4.106) we get

$$\begin{aligned}
\|u(t)\|_{\mathbf{L}^{1,1}} &\le \|\mathcal{G}(t)u_0\|_{\mathbf{L}^{1,1}} + C\int_0^{\frac{t}{2}} \left\|\partial_x \mathcal{B}\mathcal{G}(t-\tau)u^2(\tau)\right\|_{\mathbf{L}^{1,1}} d\tau \\
&+ C\int_{\frac{t}{2}}^{t} \|\mathcal{G}(t-\tau)\mathcal{B}u(\tau)u_x(\tau)\|_{\mathbf{L}^{1,1}} d\tau \\
&\le C\langle t\rangle^{\frac{1}{2}} + C\int_0^{\frac{t}{2}} \left\|u^2(\tau)\right\|_{\mathbf{L}^1} d\tau \\
&+ C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{1}{2}} \left\|u^2(\tau)\right\|_{\mathbf{L}^{1,1}} d\tau \\
&+ C\int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{\frac{1}{2}} \|\mathcal{B}u(\tau)u_x(\tau)\|_{\mathbf{L}^1} d\tau \\
&+ C\int_{\frac{t}{2}}^{t} \|\mathcal{B}u(\tau)u_x(\tau)\|_{\mathbf{L}^{1,1}} d\tau \\
&\le C\langle t\rangle^{\frac{1}{2}} + C\langle t\rangle^{-\frac{1}{2}}\int_0^t \langle\tau\rangle^{-\frac{1}{2}} \|u(\tau)\|_{\mathbf{L}^{1,1}} d\tau;
\end{aligned}$$

hence, by the Gronwall lemma estimate (4.139) follows for all $t > 0$.

Now we obtain the second term of the large time asymptotics as $t \to \infty$ of solutions $u(t,x)$ to the Cauchy problem (4.100). As in the previous section we take a sufficiently large initial time $T > 0$ and consider the Cauchy problem for the Burgers equation (4.129). Then for the difference $w(t,x) = u(t,x) - v(t,x)$ by (4.100) and (4.129) we get the Cauchy problem (4.132) with estimates (4.136).

Consider now the linear Cauchy problem

$$\begin{cases} \varphi_t + \partial_x(\varphi v) - \mu\varphi_{xx} + \beta v_{xxx} = 0,\ t > T,\ x \in \mathbf{R}, \\ \varphi(T,x) = 0,\ x \in \mathbf{R}. \end{cases} \tag{4.140}$$

To eliminate the second term from (4.140), we integrate (4.140) with respect to x and make the substitution

$$\int_{-\infty}^{x} \varphi(t,y)dy = \mu\frac{s(t,x)}{Z(t,x)},$$

where $Z(t,x)$ is defined by (4.130). We obtain

$$\begin{cases} s_t - \mu s_{xx} + \beta Z v_{xx} = 0, \ t > T, \ x \in \mathbf{R}, \\ s\,(T, x) = 0, \ x \in \mathbf{R}. \end{cases} \tag{4.141}$$

It is simple to integrate (4.141) to get

$$s\,(t, x) = -\beta \int_T^t d\tau \int_{\mathbf{R}} dy G_0\,(t - \tau, x - y)\, Z\,(\tau, y)\, v_{yy}\,(\tau, y)\,. \tag{4.142}$$

Now let us compute the asymptotics of $s\,(t, x)$ as $t \to \infty$. We integrate by parts with respect to y to obtain

$$\begin{aligned} s\,(t, x) &= -\beta \int_T^t d\tau \int_0^\infty dy \partial_x G_0\,(t - \tau, x - y) \int_y^\infty Z\,(\tau, \eta)\, v_{\eta\eta}\,(\tau, \eta)\, d\eta \\ &\quad + \beta \int_T^t d\tau \int_{-\infty}^0 dy \partial_x G_0\,(t - \tau, x - y) \int_{-\infty}^y Z\,(\tau, \eta)\, v_{\eta\eta}\,(\tau, \eta)\, d\eta \\ &\quad - \beta \int_T^t d\tau G_0\,(t - \tau, x) \int_{\mathbf{R}} Z\,(\tau, \eta)\, v_{\eta\eta}\,(\tau, \eta)\, d\eta \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

Since

$$\|x v_{xx}(t)\|_{\mathbf{L}^1(\mathbf{R})} \le C \langle t \rangle^{-\frac{1}{2}}\,,$$

we obtain

$$\begin{aligned} |I_1| &\le |\beta| \int_T^t d\tau \int_0^\infty dy\, |\partial_x G_0\,(t - \tau, x - y)| \int_y^\infty Z\,(\tau, \eta)\, |v_{\eta\eta}\,(\tau, \eta)|\, d\eta \\ &\le C t^{-1} \int_T^{\frac{t+T}{2}} d\tau\, \|x v_{xx}(\tau)\|_{\mathbf{L}^1(\mathbf{R})} + C \int_{\frac{t+T}{2}}^t d\tau\, (t - \tau)^{-\frac{1}{2}}\, \|v_{xx}(\tau)\|_{\mathbf{L}^1(\mathbf{R})} \\ &\le C t^{-1} \int_T^{\frac{t+T}{2}} \langle \tau \rangle^{-\frac{1}{2}}\, d\tau + C \int_{\frac{t+T}{2}}^t (t - \tau)^{-\frac{1}{2}} \langle \tau \rangle^{-1}\, d\tau = O(t^{-\frac{1}{2}}). \end{aligned} \tag{4.143}$$

The integral I_2 is estimated in the same way.

Now we consider I_3. We have the asymptotics

$$Z(t, x) = H\,(\chi) + O\left(t^{-\frac{1}{2}}\right),$$

where

$$H\,(\chi) = \cosh \frac{\theta}{4} - \sinh\left(\frac{\theta}{4}\right) \operatorname{Erf}\left(\frac{\chi}{2\sqrt{\mu}}\right)$$

with $\chi = \frac{x}{\sqrt{t}}$. Then in view of the identity

$$Z_x = -\frac{1}{2\mu} Z v,$$

integration by parts yields

$$\int_{\mathbf{R}} Z(\tau,\eta)\, v_{\eta\eta}(\tau,\eta)\, d\eta = -\frac{1}{2}\int_{\mathbf{R}} v^3(\tau,y) Z(\tau,y) dy$$
$$= -\frac{1}{2\tau}\int_{\mathbf{R}} f_\theta^3(y) H(y) dy + O(\tau^{-3/2}).$$

Therefore

$$\begin{aligned} I_3 &= -\beta \int_T^t d\tau G_0(t-\tau,x) \int_{\mathbf{R}} Z(\tau,\eta)\, v_{\eta\eta}(\tau,\eta)\, d\eta \\ &= \frac{\beta}{2\sqrt{4\pi\mu}} \int_{\mathbf{R}} f_\theta^3(y) H(y) dy \int_T^t \frac{d\tau}{\tau\sqrt{t-\tau}} e^{-\frac{x^2}{4\mu(t-\tau)}} + O\left(t^{-\frac{1}{2}}\right) \\ &= \frac{\beta}{2} G_0(t,x) \log t \int_{\mathbf{R}} f_\theta^3(y) H(y) dy + O\left(t^{-\frac{1}{2}}\right). \end{aligned} \tag{4.144}$$

Hence the asymptotics is true

$$s(t,x) = \frac{\beta}{2\sqrt{4\pi\mu t}} e^{-\frac{\xi^2}{4\mu}} \log t \int_{\mathbf{R}} f_\theta^3(y) H(y) dy + O\left(t^{-\frac{1}{2}}\right)$$

for large $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$. This formula can be differentiated with respect to x. Hence we see that

$$\varphi(t,x) = \mu\partial_x \left(\frac{s(t,x)}{Z(t,x)}\right) = \frac{\log t}{t} \widetilde{f}_\theta(\chi) + O\left(\frac{1}{t}\right)$$

as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $\chi = \frac{x}{\sqrt{t}}$ and

$$\widetilde{f}_\theta(x) = -\frac{\beta\left(f_\theta(x) - \frac{x}{2\sqrt{\mu}}\right) e^{-\frac{x^2}{4\mu}}}{2\sqrt{\pi\mu} H(x)} \int_{\mathbf{R}} f_\theta^3(y) H(y) dy.$$

It follows from (4.132) and (4.140) that the remainder $\psi(t,x) = w(t,x) - \varphi(t,x)$ is the solution to the Cauchy problem

$$\begin{cases} \psi_t + \partial_x(v\psi) - \mu\psi_{xx} + \frac{1}{2}\partial_x w^2 + \beta w_{xxx} + \partial_x h_1 = 0, \ t > T, \ x \in \mathbf{R}, \\ \psi(T,x) = 0, \ x \in \mathbf{R}, \end{cases} \tag{4.145}$$

where

$$h_1 = \left(-\beta\partial_x^4 + \mu\partial_x^3\right) \mathcal{B}u - \partial_x \mathcal{B}uu_x.$$

To eliminate the second term from (4.145) as above we integrate this equation with respect to x and introduce the new unknown function

$$r(t,x) = \frac{Z(t,x)}{\mu} \int_{-\infty}^x \psi(t,y) dy.$$

Then we obtain

$$\begin{cases} r_t - \mu r_{xx} + F_1 = 0, \ t > T, \ x \in \mathbf{R}, \\ r(T,x) = 0, \ x \in \mathbf{R}. \end{cases} \tag{4.146}$$

where

$$F_1 = \frac{1}{2} Z w^2 + Z\beta w_{xx} + Z h_1.$$

In view of (4.131) and (4.136) we find

$$\begin{aligned} \|F_1(t)\|_{\mathbf{L}^p} &\le C\|w\|_{\mathbf{L}^\infty}\|w\|_{\mathbf{L}^p} + C\|w_{xx}\|_{\mathbf{L}^p} + \|h_1\|_{\mathbf{L}^p} \\ &\le C\langle t\rangle^{-1-2\gamma+\frac{1}{2p}} + C\langle t\rangle^{-\frac{3}{2}-\gamma+\frac{1}{2p}} \le C\langle t\rangle^{-\frac{3}{2}-\gamma+\frac{1}{2p}} \end{aligned} \tag{4.147}$$

for all $t \ge T$, $1 \le p \le \infty$ if we choose $\gamma \in \left(\frac{1}{4}, \frac{1}{2}\right)$. Using the integral equation associated with (4.146) we obtain in view of (4.147)

$$\begin{aligned} &\|r(t)\|_{\mathbf{L}^p} \le \int_T^{\frac{t+T}{2}} d\tau \, \|G(t-\tau)\|_{\mathbf{L}^p}\|F_1(\tau)\|_{\mathbf{L}^1} \\ &+ \int_{\frac{t+T}{2}}^t d\tau \, \|G(t-\tau)\|_{\mathbf{L}^1}\|F_1(\tau)\|_{\mathbf{L}^p} \\ &\le C\int_T^{\frac{t+T}{2}} (t-\tau)^{-\frac{1}{2}+\frac{1}{2p}}\langle\tau\rangle^{-1-\gamma} d\tau \\ &+ C\int_{\frac{t+T}{2}}^t \langle\tau\rangle^{-\frac{3}{2}-\gamma+\frac{1}{2p}} d\tau \le C\langle t\rangle^{-\frac{1}{2}+\frac{1}{2p}} \end{aligned}$$

for all $t \ge T$, $1 \le p \le \infty$. In the same manner we have

$$\begin{aligned} &\|r_x(t)\|_{\mathbf{L}^p} \le \int_T^{\frac{t+T}{2}} d\tau \, \|\partial_x G(t-\tau)\|_{\mathbf{L}^p}\|F_1(\tau)\|_{\mathbf{L}^1} \\ &+ \int_{\frac{t+T}{2}}^t d\tau \, \|\partial_x G(t-\tau)\|_{\mathbf{L}^1}\|F_1(\tau)\|_{\mathbf{L}^p} \\ &\le C\int_T^{\frac{t+T}{2}} (t-\tau)^{-1+\frac{1}{2p}}\langle\tau\rangle^{-1-\gamma} d\tau \\ &+ C\int_{\frac{t+T}{2}}^t (t-\tau)^{-\frac{1}{2}}\langle\tau\rangle^{-\frac{3}{2}-\gamma+\frac{1}{2p}} d\tau \le C\langle t\rangle^{-1+\frac{1}{2p}} \end{aligned}$$

for all $t \ge T$, $1 \le p \le \infty$. Then by the identity

$$\psi = Z^{-1}\left(r_x + \frac{1}{2} r v\right)$$

we obtain the estimate

$$\|u(t) - v(t) - \varphi(t)\|_{\mathbf{L}^\infty} \le C\langle t\rangle^{-1}$$

for all $t \ge T$. Theorem 4.29 is proved.

4.5 A system of nonlinear equations

This section is devoted to the study of the Cauchy problem for the system of nonlinear nonlocal evolution equations

$$u_t + \mathcal{N}(u) + \mathcal{L}u = 0,\ x \in \mathbf{R}^n,\ t > 0 \tag{4.148}$$

with initial data $u(0,x) = \widetilde{u}(x),\ x \in \mathbf{R}^n$, where $u(t,x)$ is a vector $u = \{u_j\}|_{j=1,\ldots,m}$. The linear part of system (4.148) is a pseudodifferential operator defined by the Fourier transformation as follows

$$\mathcal{L}u = \overline{\mathcal{F}}_{\xi \to x} L(\xi) \mathcal{F}_{x \to \xi} u,$$

where the symbol $L(\xi)$ is a matrix $L = \{L_{jk}\}|_{j,k=1,\ldots,m}$. The nonlinearity $\mathcal{N}(u)$ is a quadratic pseudodifferential operator

$$\mathcal{N}(u) = \overline{\mathcal{F}}_{\xi \to x} \sum_{k,l=1}^{m} \int_{\mathbf{R}^n} a^{kl}(t,\xi,y)\, \hat{u}_k(t,\xi - y)\, \hat{u}_l(t,y)\, dy;$$

here the symbols $a^{kl}(t,\xi,y)$ are vectors $a^{kl} = \left\{a_j^{kl}\right\}\big|_{j=1,\ldots,m}$. We suppose that the symbols $a^{kl}(t,\xi,y)$ are continuous vector functions with respect to time $t > 0$. Suppose that the operators $\mathcal{N}$ and $\mathcal{L}$ have a finite order, that is the symbols $a^{kl}(t,\xi,y)$ and $L(\xi)$ grow with respect to y and ξ no faster than a power

$$\left|a^{kl}(t,\xi,y)\right| \le C\left(\langle\xi\rangle^{\kappa} + \langle y\rangle^{\kappa}\right), |L(\xi)| \le C\langle\xi\rangle^{\kappa},$$

where $C > 0$. The absolute value of vectors $\left|a^{kl}\right|$ and matrix $|L|$ we understand as a maximum of their components: $\left|a^{kl}\right| = \max_{j=1,..,m}\left|a_j^{kl}\right|$, $|L| = \max_{j,k=1,..,m}|L_{jk}|$.

This section is devoted to the study of the large time asymptotic behavior of solutions to the Cauchy problem for nonlinear evolution equation (4.148) in the critical case. Below we describe our suppositions in more detail.

As in the supercritical case (see Chapter 2 , Section 2.7) we rewrite the Cauchy problem (4.148) in the form of the integral equation

$$u(t) = \mathcal{G}(t)\widetilde{u} - \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\, d\tau, \tag{4.149}$$

where the Green operator $\mathcal{G}(t)\psi = \overline{\mathcal{F}}_{\xi \to x}\left(e^{-tL(\xi)}\hat{\psi}(\xi)\right)$.

Let the linear operator $\mathcal{L}$ satisfy the dissipation condition which in terms of the eigenvalues of the matrix $L(\xi)$ has the form

$$\operatorname{Re}\lambda_j(\xi) \ge \mu\{\xi\}^{\delta}\langle\xi\rangle^{\gamma} \tag{4.150}$$

for all $\xi \in \mathbf{R}^n$, where $\mu > 0$, $\gamma > 0$, $\delta > 0$. Also we suppose that the matrices $P^{(j)}(\xi)$ defined in (2.146) and (2.147) satisfy the estimates

$$\left|\partial_{\xi}^{r} P^{(j)}(\xi)\right| \leq C \tag{4.151}$$

for all $\xi \in \mathbf{R}^n$, $|r| = 0, 1$. Assume that the symbol of the nonlinear operator $\mathcal{N}$ satisfies the estimates

$$\left|a^{k,l}(t, \xi, y)\right| \leq C \langle \xi \rangle^{\tilde{\theta}} \{\xi\}^{\omega} \left(\langle \xi - y \rangle^{\sigma} \{\xi - y\}^{\alpha} + \langle y \rangle^{\sigma} \{y\}^{\alpha}\right) \tag{4.152}$$

for all $\xi, y \in \mathbf{R}^n$, $t > 0$, $k, l = 1, ..., m$, where $\tilde{\theta}$, σ, $\alpha \geq 0$. We consider the case of nonlinearity of the type of the total derivative, that is we suppose that $\omega > 0$. System (4.148) with this type of nonlinearity is called a diffusion-convection type system. We are interested in the case of nonzero total mass of the initial data $\int_{\mathbf{R}^n} \widetilde{u}(x)\, dx \equiv \theta \neq 0$.

Now we define the critical case with respect to the large time asymptotic behavior of solutions by the relation

$$\delta = n + \alpha + \omega.$$

Define $\mathbf{X} = \{\phi \in \mathcal{S}' : \|\phi\|_{\mathbf{X}} < \infty\}$, with the norm

$$\|\phi\|_{\mathbf{X}} = \sup_{\rho \in [0, \alpha+\gamma]} \sup_{1 \leq p \leq \infty} \sup_{t > 0} \langle t \rangle^{\frac{\rho}{\delta} + \frac{n}{\delta p}} \|\phi(t)\|_{\mathbf{A}^{\rho,p}}$$

$$+ \sup_{s \in [0, \sigma]} \sup_{1 \leq p \leq \infty} \sup_{t > 0} \{t\}^{\frac{s}{\nu}} \langle t \rangle^{\frac{n+\alpha+\gamma}{\delta} + \frac{n}{\delta p}} \|\phi(t)\|_{\mathbf{B}^{s,p}},$$

where $\gamma \in \left(0, \frac{1}{\delta} (\min(1, \omega))^2\right)$. Here the norms

$$\|\varphi(t)\|_{\mathbf{A}^{\rho,p}} = \||\cdot|^{\rho} \widehat{\varphi}(t, \cdot)\|_{\mathbf{L}^p_{\xi}(|\xi| \leq 1)},$$

$$\|\varphi(t)\|_{\mathbf{B}^{s,p}} = \||\cdot|^{s} \widehat{\varphi}(t, \cdot)\|_{\mathbf{L}^p_{\xi}(|\xi| \geq 1)}.$$

Note that the norm $\mathbf{A}^{\rho,p}$ is responsible for the large time asymptotic properties of solutions and the norm $\mathbf{B}^{s,p}$ describes the regularity of solutions.

Theorem 4.36. *Let the linear operator $\mathcal{L}$ satisfy conditions (4.150) with $\delta = n + \alpha + \omega$, $\alpha \geq 0$, $\omega > 0$. Suppose that the nonlinear operator $\mathcal{N}$ satisfies estimates (4.152) with $\tilde{\theta} + \sigma < \nu$, $\nu > 0$. Let the initial data $\widetilde{u}$ be such that*

$$\|\widetilde{u}\|_{\mathbf{A}^{0,\infty}} + \|\widetilde{u}\|_{\mathbf{B}^{0,\infty}} + \|\widetilde{u}\|_{\mathbf{B}^{0,1}} \leq \varepsilon,$$

where $\varepsilon > 0$ is sufficiently small. Then there exists a unique solution $u(t,x) \in \mathbf{X}$ of the Cauchy problem (4.148). Moreover, the solutions $u(t,x)$ have the time decay estimate

$$\|u(t)\|_{\mathbf{L}^{\infty}} \leq C \langle t \rangle^{-\frac{n}{\delta}}$$

for all $t > 0$.

Remark 4.37. Note that in the case of zero total mass of the initial data $\theta = \int_{\mathbf{R}^n} \widetilde{u}(x)\,dx = 0$, the solutions of the Cauchy problem for equation (4.148) obtain more rapid time decay rate

$$\|u(t)\|_{\mathbf{L}^\infty} \le Ct^{-\frac{n+\min(1,\omega)}{\delta}}.$$

Thus the critical value is shifted $\delta_c = n + \alpha + \omega + \min(1,\omega)$ in this case.

To find the asymptotic formulas for the solution we assume that the eigenvalues of the symbol $L(\xi)$ has the following asymptotic representation in the origin

$$\lambda_j(\xi) = i\xi b^{(j)} + \mu_j |\xi|^\delta + O\left(|\xi|^{\delta+\gamma}\right) \tag{4.153}$$

for all $|\xi| \le 1$, where $\mu_j > 0$, $\delta > 0$, $\gamma > 0$, $b^{(j)} \in \mathbf{R}^n$. Let $\mathcal{N}_0$ have a symbol $a_0(\xi, y) \in \mathbf{C}^1(\mathbf{R}^n \times \mathbf{R}^n)$, homogeneous with respect to ξ and y of order $\omega+\alpha$, that is $a_0(t\xi, ty) = t^{\alpha+\omega} a_0(\xi, y)$ for all $\xi, y \in \mathbf{R}^n$, $t > 0$. We suppose that the relation

$$\begin{aligned}
&|a(t,\xi,y) - a_0(\xi,y)| \\
&\le C\langle\xi\rangle^\theta \{\xi\}^\omega \left(\langle\xi-y\rangle^\sigma \{\xi-y\}^{\alpha+\gamma} + \langle y\rangle^\sigma \{y\}^{\alpha+\gamma}\right) \\
&+ C\langle t\rangle^{-\frac{\gamma}{\delta}} \langle\xi\rangle^{\tilde{\theta}} \{\xi\}^\omega \left(\langle\xi-y\rangle^\sigma \{\xi-y\}^\alpha + \langle y\rangle^\sigma \{y\}^\alpha\right)
\end{aligned} \tag{4.154}$$

is true for all ξ, $y \in \mathbf{R}^n$, $t > 0$, where θ, σ, $\alpha \ge 0$, ω, $\gamma > 0$. Note that $a_0(\xi, y)$ also satisfies estimates (4.152). Consider the following integral equation

$$w(t) = \mathcal{G}^{(j)}(t)\,\theta\delta_0(x) - \int_0^t \mathcal{G}^{(j)}(t-\tau)\,\mathcal{N}_0(w,w)(\tau)\,d\tau, \tag{4.155}$$

where

$$\mathcal{G}^{(j)}(t)\,\phi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-it\xi b^{(j)} - t\mu_j|\xi|^\delta} P^{(j)}(0)\,\widehat{\phi}(\xi)\right)$$

and

$$\mathcal{N}_0(w,w) = \overline{\mathcal{F}}_{\xi\to x} \sum_{k,l=1}^m \int_{\mathbf{R}^n} a_0^{kl}(\xi,y)\,\widehat{w}_k(t,\xi-y)\,\widehat{w}_l(t,y)\,dy.$$

Below in Section 4.5.3 we prove that there exists a unique self-similar solution to (4.155) in the form $w(t,x) = t^{-\frac{n}{\delta}} f\left(\left(x - b^{(j)}t\right) t^{-\frac{1}{\delta}}\right)$.

Theorem 4.38. *Suppose that the nonlinear operator $\mathcal{N}$ satisfies relationship (4.154) and estimates (4.152) with $\omega > 0$, $\tilde{\theta}+\sigma \in [0,\nu)$. Let the linear operator $\mathcal{L}$ satisfy conditions (4.150), (4.153) with $\delta = n+\alpha+\omega$, $\alpha \ge 0$. Let the initial data $\widetilde{u}$ be such that*

$$\|\widetilde{u}\|_{\mathbf{A}^{0,\infty}} + \|\widetilde{u}\|_{\mathbf{B}^{0,\infty}} + \|\widetilde{u}\|_{\mathbf{B}^{0,1}} \le \varepsilon,$$

where $\varepsilon > 0$ is sufficiently small. Also we suppose that

$$\|\widetilde{u} - \theta\delta_0\|_{\mathbf{A}^{-\gamma,\infty}} \leq \varepsilon,$$

where $\delta_0(x)$ is the Dirac delta-function, $\gamma \in \left(0, \frac{1}{\delta}(\min(1,\omega))^2\right)$. Then the solution $u(t,x)$ to the Cauchy problem (4.148) tends for large time to the superposition of the self-similar solutions $t^{-\frac{n}{\delta}} f^{(j)}\left(\left(x - b^{(j)}t\right)t^{-\frac{1}{\delta}}\right)$ of equations (4.155)

$$\left\| u(t,\cdot) - t^{-\frac{n}{\delta}} \sum_{j=1}^{m} f^{(j)}\left(\left(\cdot - b^{(j)}t\right)t^{-\frac{1}{\delta}}\right)\right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{n+\gamma}{\delta}}$$

for all $t \geq 1$.

Remark 4.39. The conditions of the theorems for the initial data $\widetilde{u}$ can also be expressed in terms of the standard weighted Sobolev spaces as follows

$$\|\widetilde{u}\|_{\mathbf{H}^{\beta,0}} + \|\widetilde{u}\|_{\mathbf{H}^{0,\beta+2\gamma}} \leq \varepsilon,$$

where $\beta > \frac{n}{2}$. However, the conditions on the initial data $\widetilde{u}$ are expressed more precisely in the norms $\mathbf{A}^{0,p}$ and $\mathbf{B}^{0,p}$.

As an example we apply Theorem 4.38 to the well-known Boussinesq system with viscosity

$$\begin{cases} \eta_t + (\eta\ v)_x - \mu\eta_{xx} + \beta^2 v_x + \vartheta^2 v_{xxx} = 0 \\ v_t + vv_x + \eta_x - \mu v_{xx} = 0, \end{cases} \tag{4.156}$$

and to the system describing surface water waves with allowance for viscosity and surface tension

$$\begin{cases} \eta_t + (\eta\ v)_x - \mu\eta_{xx} + \overline{\mathcal{F}}_{\xi\to x}\left(L_{12}(\xi)\,\widehat{v}(t,\xi)\right) = 0, \\ v_t + vv_x + \eta_x - \mu v_{xx} = 0. \end{cases} \tag{4.157}$$

We choose in system (4.148) the nonlinearity

$$a^{1,1} = 0,\ a^{1,2} = a^{2,1} = \begin{pmatrix} \frac{i\xi}{2} \\ 0 \end{pmatrix},\ a^{2,2} = \begin{pmatrix} 0 \\ \frac{i\xi}{2} \end{pmatrix},$$

and the linear operators

$$L_{11}(\xi) = L_{22}(\xi) = \mu|\xi|^2,\ L_{12}(\xi) = i\beta^2\xi\left(1 + \vartheta^2|\xi|^2\right),\ L_{21}(\xi) = i\xi$$

for system (4.156) and

$$L_{11}(\xi) = L_{22}(\xi) = \mu|\xi|^2,\ L_{12}(\xi) = i\beta^2\left(1 + \kappa|\xi|^2\right)\frac{\tanh\varrho\xi}{\varrho\xi},\ L_{21}(\xi) = i\xi$$

for the case of system (4.157) respectively. In these cases the eigenvalues $\lambda_j(\xi)$ are given by the formulas

$$\lambda_j(\xi) = \mu |\xi|^2 + i(-1)^j \beta\xi\sqrt{1+\vartheta^2 |\xi|^2}$$

and

$$\lambda_j(\xi) = \mu |\xi|^2 + i\beta\xi(-1)^j \sqrt{\left(1+\kappa |\xi|^2\right)\frac{\tanh \varrho\xi}{\varrho\xi}},$$

for $j = 1, 2$, respectively. So that we have $n = 1$, $\delta = 2$, $\nu = 2$, $\theta = 1$, $\omega = 1$, $\sigma = 0$, $\alpha = 0$, $A_0(\xi, y) = A(\xi, y)$ and $b^{(j)} = (-1)^j \beta$. Hence the conditions (4.150) - (4.152), (4.153) and (4.154) are fulfilled. Note that the solutions $w^{(j)}(t,x)$ to the integral equations (4.155) also satisfy the Burgers equation with transfer

$$\partial_t w^{(j)} + (-1)^j \frac{3}{4}\partial_x \left(w^{(j)}\right)^2 + (-1)^j \beta\partial_x w^{(j)} - \mu\partial_x^2 w^{(j)} = 0. \tag{4.158}$$

The Hopf-Cole substitution (see Hopf [1950]) yields the asymptotics of $w^{(j)}(t,x)$

$$w^{(j)}(t,x) = (\mu t)^{-\frac{1}{2}} A_j(\chi_j) + O\left(t^{-\frac{1}{2}-\gamma}\right), \tag{4.159}$$

where $A_j(z) = -\frac{4}{3}\mu(-1)^j \frac{\partial}{\partial z}\log H_j(z)$,

$$H_j(z) = e^{-\theta_j/2}\left(\cosh\frac{\theta_j}{2} - \sinh\frac{\theta_j}{2}\mathrm{Erf}\left(\frac{z}{2}\right)\right).$$

Here $\chi_j = \frac{x-(-1)^j\beta t}{\sqrt{\mu t}}$,

$$\mathrm{Erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-|x|^2}dx$$

is the error function, and

$$\theta_j \equiv \frac{1}{2b}\int_{\mathbf{R}} \widetilde{\eta}(x)\,dx + \frac{(-1)^j}{2}\int_{\mathbf{R}} \widetilde{v}(x)\,dx$$

is the total mass of the initial data. Then solutions (η, v) to the Cauchy problems (4.156) and (4.157) have the following asymptotic representation as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$

$$\eta(t,x) = \beta w_1(t,x) + \beta w_2(t,x) + O\left(t^{-\frac{1}{2}-\gamma}\right),$$
$$v(t,x) = w_2(t,x) - w_1(t,x) + O\left(t^{-\frac{1}{2}-\gamma}\right),$$

where $\gamma > 0$ and the functions $w^{(j)}(t,x)$ are the solutions of the Cauchy problem for the Burgers equation (4.158); their asymptotics is given by (4.159).

4.5.1 Preliminary Lemmas

The Green operator $\mathcal{G}$ is given by

$$\mathcal{G}(t)\phi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-L(\xi)t}\hat{\phi}(\xi)\right),\ e^{-tL(\xi)} = \sum_{j=1}^{m} e^{-t\lambda_j(\xi)}P^{(j)}(\xi).$$

In the next lemma we estimate the Green operator $\mathcal{G}(t)$ in the norms

$$\|\varphi(t)\|_{\mathbf{A}^{\rho,p}} = \||\cdot|^{\rho}\widehat{\varphi}(t,\cdot)\|_{\mathbf{L}^{p}_{\xi}(|\xi|\le 1)} \text{ and } \|\varphi(t)\|_{\mathbf{B}^{s,p}} = \||\cdot|^{s}\widehat{\varphi}(t,\cdot)\|_{\mathbf{L}^{p}_{\xi}(|\xi|\ge 1)},$$

where $s,\rho\in\mathbf{R}$, $1\le p\le\infty$. Using Lemmas
1.38 to 1.39 we get the following lemma.

Lemma 4.40. *Let the linear operator $\mathcal{L}$ satisfy the dissipation condition (4.150) with $\delta>0$. Then the following estimates are valid for all $t>0$, $1\le p\le q\le\infty$*

$$\|\mathcal{G}(t)\phi\|_{\mathbf{A}^{\rho,p}} \le C\langle t\rangle^{-\frac{\rho}{\delta}-\frac{n}{\delta}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\phi\|_{\mathbf{A}^{0,q}},$$

for $\rho\ge 0$,

$$\begin{aligned}&\left\|\int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau\right\|_{\mathbf{A}^{\rho,p}}\\ &\le C\langle t\rangle^{1-\lambda-\frac{\rho+\omega}{\delta}-\frac{n}{\delta p}}\sup_{p\le q\le\infty}\sup_{\tau>0}\left(\{\tau\}^{\kappa}\langle\tau\rangle^{\lambda+\frac{n}{\delta q}}\|\psi(\tau)\|_{\mathbf{A}^{-\omega,q}}\right),\end{aligned}$$

for $\kappa\in[0,1)$, $\lambda\in[0,1)$, $0\le\rho+\omega<\delta$;

$$\|\mathcal{G}(t)\phi\|_{\mathbf{B}^{s,p}} \le Ce^{-\frac{\mu}{2}t}\{t\}^{-\frac{s}{\nu}-\frac{n}{\nu}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\phi\|_{\mathbf{B}^{0,q}},$$

for $s\ge 0$, and

$$\begin{aligned}&\left\|\int_0^t \mathcal{G}(t-\tau)\psi(\tau)\,d\tau\right\|_{\mathbf{B}^{s,p}} \le C\{t\}^{1-\kappa-\frac{s+\theta}{\nu}-\frac{n}{\nu}\left(\frac{1}{p}-\frac{1}{q}\right)}\langle t\rangle^{-\widetilde{\lambda}}\\ &\times\left(\sup_{\tau>0}\left(\{\tau\}^{\kappa}\|\psi(\tau)\|_{\mathbf{B}^{-\theta,q}}\right)+\sup_{\tau>0}\left(\{\tau\}^{\kappa+\frac{n}{\nu}\left(\frac{1}{p}-\frac{1}{q}\right)}\langle\tau\rangle^{\widetilde{\lambda}}\|\psi(\tau)\|_{\mathbf{B}^{-\theta,p}}\right)\right),\end{aligned}$$

for $\kappa\in[0,1)$, $\widetilde{\lambda}\ge 0$, $s\ge 0$, $\theta\ge 0$ are such that $s+\theta<\nu$.

Now we estimate the nonlinearity

$$\mathcal{N}(\varphi,\phi) = \overline{\mathcal{F}}_{\xi\to x}\sum_{k,l=1}^{m}\int_{\mathbf{R}^n} a^{k,l}(t,\xi,y)\,\widehat{\varphi}_k(t,\xi-y)\,\widehat{\phi}_l(t,y)\,dy$$

in the norms $\|\cdot\|_{\mathbf{A}^{-\omega,p}}$ and $\|\cdot\|_{\mathbf{B}^{-\theta,p}}$.

Lemma 4.41. *Let the nonlinear operator $\mathcal{N}$ satisfy condition (4.152). Then the inequalities*

$$\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{A}^{-\omega,p}} \leq C\left(\|\varphi\|_{\mathbf{A}^{\alpha,1}}+\|\varphi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}}+\|\phi\|_{\mathbf{B}^{0,\infty}}\right)$$
$$+C\left(\|\phi\|_{\mathbf{A}^{\alpha,1}}+\|\phi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}}+\|\varphi\|_{\mathbf{B}^{0,\infty}}\right)$$

and

$$\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{B}^{-\theta,p}} \leq C\left(\|\varphi\|_{\mathbf{A}^{\alpha+\gamma,1}}+\|\varphi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}}+\|\phi\|_{\mathbf{B}^{0,p}}\right)$$
$$+C\left(\|\phi\|_{\mathbf{A}^{\alpha+\gamma,1}}+\|\phi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}}+\|\varphi\|_{\mathbf{B}^{0,p}}\right)$$

are valid for $1 \leq p \leq \infty$, where $\gamma > 0$, provided that the right-hand sides are bounded.

Proof. By virtue of condition (4.152) and by the Young inequality we obtain

$$\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{A}^{-\omega,p}}$$
$$\leq \sum_{k,l=1}^{m}\left\|\int_{\mathbf{R}^n}|\cdot|^{-\omega}\left|a^{k,l}(t,\cdot,y)\right|\left|\widehat{\varphi}_k(t,\cdot-y)\,\widehat{\phi}_l(t,y)\right|dy\right\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)}$$
$$\leq C\sum_{k,l=1}^{m}\left\|\int_{\mathbf{R}^n}\left(\langle\cdot-y\rangle^{\sigma}\{\cdot-y\}^{\alpha}+\langle y\rangle^{\sigma}\{y\}^{\alpha}\right)\right.$$
$$\left.\times\left|\widehat{\varphi}_k(t,\cdot-y)\,\widehat{\phi}_l(t,y)\right|dy\right\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)};$$

hence

$$\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{A}^{-\omega,p}}$$
$$\leq C\left\|\langle\cdot\rangle^{\sigma}\{\cdot\}^{\alpha}\widehat{\varphi}\right\|_{\mathbf{L}^1}\left(\left\|\widehat{\phi}\right\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)}+\left\|\widehat{\phi}\right\|_{\mathbf{L}^\infty_\xi(|\xi|>1)}\right)$$
$$+C\left\|\langle\cdot\rangle^{\sigma}\{\cdot\}^{\alpha}\widehat{\phi}\right\|_{\mathbf{L}^1}\left(\|\widehat{\varphi}\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)}+\|\widehat{\varphi}\|_{\mathbf{L}^\infty_\xi(|\xi|>1)}\right)$$
$$\leq C\left(\|\varphi\|_{\mathbf{A}^{\alpha,1}}+\|\varphi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}}+\|\phi\|_{\mathbf{B}^{0,\infty}}\right)$$
$$+C\left(\|\phi\|_{\mathbf{A}^{\alpha,1}}+\|\phi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}}+\|\varphi\|_{\mathbf{B}^{0,\infty}}\right),$$

and

$$\begin{aligned}
&\|\mathcal{N}(\varphi,\phi)\|_{\mathbf{B}^{-\theta,p}} \\
&\leq \sum_{k,l=1}^{m} \left\| \int_{\mathbf{R}^n} |\cdot|^{-\theta} \{\cdot\}^{\gamma} \left|a^{k,l}(t,\cdot,y)\right| \left|\widehat{\varphi}_k(t,\cdot-y)\,\widehat{\phi}_l(t,y)\right| dy \right\|_{\mathbf{L}^p_\xi(|\xi|\geq 1)} \\
&\leq C \left\| \int_{\mathbf{R}^n} \left(\langle \cdot - y\rangle^{\sigma} \{\cdot - y\}^{\alpha+\gamma} + \langle y\rangle^{\sigma}\{y\}^{\alpha+\gamma} \right) \right. \\
&\times \left. \left|\widehat{\varphi}_k(t,\cdot-y)\,\widehat{\phi}_l(t,y)\right| dy \right\|_{\mathbf{L}^p_\xi(|\xi|\geq 1)} \\
&\leq C \left\| \langle\cdot\rangle^{\sigma}\{\cdot\}^{\alpha+\gamma}\widehat{\varphi} \right\|_{\mathbf{L}^1} \left\|\widehat{\phi}\right\|_{\mathbf{L}^p} + C\|\widehat{\varphi}\|_{\mathbf{L}^p} \left\|\langle\cdot\rangle^{\sigma}\{\cdot\}^{\alpha+\gamma}\widehat{\phi}\right\|_{\mathbf{L}^1} \\
&\leq C\left(\|\varphi\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|\varphi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}} + \|\phi\|_{\mathbf{B}^{0,p}}\right) \\
&+ C\left(\|\phi\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|\phi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}} + \|\varphi\|_{\mathbf{B}^{0,p}}\right).
\end{aligned}$$

Thus the estimates of the lemma follow, and Lemma 4.41 is proved.

4.5.2 Proof of Theorem 4.36

Denote $\mathbf{X} = \{\phi \in \mathcal{S}' : \|\phi\|_{\mathbf{x}} < \infty\}$, with the norm

$$\begin{aligned}
\|\phi\|_{\mathbf{X}} = &\sup_{\rho\in[0,\alpha+\gamma]} \sup_{1\leq p\leq\infty} \sup_{t>0} \langle t\rangle^{\frac{\rho}{\delta}+\frac{n}{\delta p}} \|\phi(t)\|_{\mathbf{A}^{\rho,p}} \\
&+ \sup_{s\in[0,\sigma]} \sup_{1\leq p\leq\infty} \sup_{t>0} \{t\}^{\frac{s}{\nu}} \langle t\rangle^{\frac{n+\alpha+\gamma}{\delta}+\frac{n}{\delta p}} \|\phi(t)\|_{\mathbf{B}^{s,p}},
\end{aligned}$$

where $\nu \geq n+\sigma+\theta$, $\gamma \in \left(0, \frac{1}{\delta}(\min(1,\omega))^2\right)$. Here and below we take the critical value $\delta = n+\alpha+\omega$. Using the first and the third estimates of Lemma 4.40 with $q=\infty$ and $q=p$, respectively, we get

$$\langle t\rangle^{\frac{\rho}{\delta}+\frac{n}{\delta p}} \|\mathcal{G}(t)\widetilde{u}\|_{\mathbf{A}^{\rho,p}} \leq C\|\widetilde{u}\|_{\mathbf{A}^{0,\infty}}$$

and

$$\{t\}^{\frac{s}{\nu}} \langle t\rangle^{\frac{n+\alpha+\gamma}{\delta}+\frac{n}{\delta p}} \|\mathcal{G}(t)\widetilde{u}\|_{\mathbf{B}^{s,p}} \leq C\|\widetilde{u}\|_{\mathbf{B}^{0,p}},$$

therefore

$$\|\mathcal{G}(t)\widetilde{u}\|_{\mathbf{X}} \leq C\|\widetilde{u}\|_{\mathbf{A}^{0,\infty}} + C\|\widetilde{u}\|_{\mathbf{B}^{0,\infty}} + C\|\widetilde{u}\|_{\mathbf{B}^{0,1}}.$$

We denote also

$$\begin{aligned}
\|\psi\|_{\mathbf{Y}} = &\sup_{1\leq p\leq\infty} \sup_{t>0} \Big(\{t\}^{\kappa} \langle t\rangle^{\lambda+\frac{n}{\delta p}} \|\psi(t)\|_{\mathbf{A}^{-\omega,p}} \\
&+ \{t\}^{\kappa}\langle t\rangle^{\widetilde{\lambda}+\frac{n}{\delta p}} \|\psi(t)\|_{\mathbf{B}^{-\theta,p}} \Big),
\end{aligned}$$

where $\kappa = \frac{\sigma}{\nu}$, $\lambda = 1-\frac{\omega}{\delta}$, $\widetilde{\lambda} = \lambda + \frac{\gamma}{\delta}$. Then by virtue of the second and the fourth estimates of Lemma 4.40 with $q=p$ we have

$$\begin{aligned}&\langle t\rangle^{\frac{\rho}{\delta}+\frac{n}{\delta p}}\left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v,v)(\tau)\,d\tau\right\|_{\mathbf{A}^{\rho,p}}\\ &\leq C\sup_{p\leq q\leq\infty}\sup_{\tau>0}\left(\{\tau\}^{\kappa}\langle\tau\rangle^{\lambda+\frac{n}{\delta q}}\|\mathcal{N}(v,v)(\tau)\|_{\mathbf{A}^{-\omega,q}}\right)\leq C\|\mathcal{N}(v,v)\|_{\mathbf{Y}}\end{aligned}$$

and

$$\begin{aligned}&\{t\}^{\frac{s}{\nu}}\langle t\rangle^{\frac{n+\alpha+\gamma}{\delta}+\frac{n}{\delta p}}\left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v,v)(\tau)\,d\tau\right\|_{\mathbf{B}^{s,p}}\\ &\leq C\{t\}^{1-\kappa-\frac{\theta}{\nu}}\sup_{\tau>0}\left(\{\tau\}^{\kappa}\langle\tau\rangle^{\widetilde{\lambda}+\frac{n}{\delta p}}\|\mathcal{N}(v,v)(\tau)\|_{\mathbf{B}^{-\theta,p}}\right)\\ &\leq C\|\mathcal{N}(v,v)\|_{\mathbf{Y}},\end{aligned}$$

for all $t>0$, where $\rho\in[0,\alpha+\gamma]$, $s\in[0,\sigma]$, $1\leq p\leq\infty$, since $1-\kappa-\frac{\theta}{\nu}\geq 0$. Therefore,

$$\left\|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v,v)(\tau)\,d\tau\right\|_{\mathbf{X}}\leq C\|\mathcal{N}(v,v)\|_{\mathbf{Y}}.$$

Now by Lemma 4.41 we find

$$\begin{aligned}&\sup_{1\leq p\leq\infty}\sup_{t>0}\{t\}^{\kappa}\langle t\rangle^{\lambda+\frac{n}{\delta p}}\|\mathcal{N}(v,v)(t)\|_{\mathbf{A}^{-\omega,p}}\\ &\leq C\sup_{1\leq p\leq\infty}\sup_{t>0}\{t\}^{\kappa}\langle t\rangle^{\lambda}\left(\|v(t)\|_{\mathbf{A}^{\alpha,1}}+\|v(t)\|_{\mathbf{B}^{\sigma,1}}\right)\\ &\times\langle t\rangle^{\frac{n}{\delta p}}\left(\|v(t)\|_{\mathbf{A}^{0,p}}+\|v(t)\|_{\mathbf{B}^{0,\infty}}\right)\\ &\leq C\sup_{1\leq p\leq\infty}\sup_{t>0}\left(\langle t\rangle^{\frac{n+\alpha}{\delta}}\|v(t)\|_{\mathbf{A}^{\alpha,1}}+\{t\}^{\frac{\sigma}{\nu}}\langle t\rangle^{\frac{2n+\alpha+\gamma}{\delta}}\|v(t)\|_{\mathbf{B}^{\sigma,1}}\right)\\ &\times\left(\langle t\rangle^{\frac{n}{\delta p}}\|v(t)\|_{\mathbf{A}^{0,p}}+\langle t\rangle^{\frac{n+\alpha+\gamma}{\delta}}\|v(t)\|_{\mathbf{B}^{0,\infty}}\right)\leq C\|v\|_{\mathbf{X}}^2\end{aligned}$$

and

$$\begin{aligned}&\sup_{1\leq p\leq\infty}\sup_{t>0}\{t\}^{\kappa}\langle t\rangle^{\widetilde{\lambda}+\frac{n}{\delta p}}\|\mathcal{N}(v,v)(t)\|_{\mathbf{B}^{-\theta,p}}\\ &\leq C\sup_{1\leq p\leq\infty}\sup_{t>0}\{t\}^{\kappa}\langle t\rangle^{\widetilde{\lambda}}\left(\|v(t)\|_{\mathbf{A}^{\alpha+\gamma,1}}+\|v(t)\|_{\mathbf{B}^{\sigma,1}}\right)\\ &\times\langle t\rangle^{\frac{n}{\delta p}}\left(\|v(t)\|_{\mathbf{A}^{0,p}}+\|v(t)\|_{\mathbf{B}^{0,p}}\right)\\ &\leq C\sup_{1\leq p\leq\infty}\sup_{t>0}\left(\langle t\rangle^{\frac{n+\alpha+\gamma}{\delta}}\|v(t)\|_{\mathbf{A}^{\alpha+\gamma,1}}+\{t\}^{\frac{\sigma}{\nu}}\langle t\rangle^{\frac{2n+\alpha+\gamma}{\delta}}\|v(t)\|_{\mathbf{B}^{\sigma,1}}\right)\\ &\times\left(\langle t\rangle^{\frac{n}{\delta p}}\|v(t)\|_{\mathbf{A}^{0,p}}+\langle t\rangle^{\frac{n+\alpha+\gamma}{\delta}+\frac{n}{\delta p}}\|v(t)\|_{\mathbf{B}^{0,p}}\right)\leq C\|v\|_{\mathbf{X}}^2;\end{aligned}$$

hence we get

$$\|\mathcal{N}(v,v)\|_{\mathbf{Y}}\leq C\|v\|_{\mathbf{X}}^2.$$

$$\left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v)(\tau)\,d\tau \right\|_{\mathbf{X}} \le C\|v\|_{\mathbf{X}}^2 .$$

In the same manner we prove estimate

$$\left\| \int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(v_1)(\tau)-\mathcal{N}(v_2)(\tau))d\tau \right\|_{\mathbf{X}} \le C\|v_1-v_2\|_{\mathbf{X}}(\|v\|_{\mathbf{X}}+\|v\|_{\mathbf{X}}).$$

Therefore, as in the proof of Theorem 4.4, by applying the contraction mapping principle in $\mathbf{X}$ we prove the existence of a unique global solution $u(t,x)\in\mathbf{X}$ to the Cauchy problem (4.14). Since

$$\|u(t)\|_{\mathbf{L}^\infty} \le \|u(t)\|_{\mathbf{A}^{0,1}} + \|u(t)\|_{\mathbf{B}^{0,1}}$$

and $u(t,x)\in\mathbf{X}$ we have the estimate

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\langle t\rangle^{-\frac{n}{\delta}}$$

for all $t>0$. Thus the result of the theorem is true, and Theorem 4.36 is proved.

4.5.3 Self-similar solutions

In this subsection we construct special self-similar solutions which determine the large time asymptotic behavior. Consider the following integral equation

$$w(t) = \mathcal{G}^{(j)}(t)\,\theta\delta_0(x) - \int_0^t \mathcal{G}^{(j)}(t-\tau)\mathcal{N}_0(w,w)(\tau)\,d\tau, \tag{4.160}$$

where

$$\mathcal{G}^{(j)}(t)\,\phi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-it\xi b^{(j)}-t\mu_j|\xi|^\delta}P^{(j)}(0)\,\widehat{\phi}(\xi)\right)$$

and

$$\mathcal{N}_0(w,w) = \overline{\mathcal{F}}_{\xi\to x}\sum_{k,l=1}^m \int_{\mathbf{R}^n} a_0^{kl}(\xi,y)\,\widehat{w}_k(t,\xi-y)\,\widehat{w}_l(t,y)\,dy,$$

where a symbol $a_0(\xi,y)\in\mathbf{C}^1(\mathbf{R}^n\times\mathbf{R}^n)$ is homogeneous with respect to ξ and y of order $\omega+\alpha$, that is $a_0(t\xi,ty)=t^{\alpha+\omega}a_0(\xi,y)$ for all $\xi,y\in\mathbf{R}^n$, $t>0$. In addition we assume that $a_0(\xi,y)$ satisfies relation (4.154). As a consequence estimate (4.152) with $\theta=\omega$ and $\sigma=\alpha$ follows:

$$\left|a_0^{k,l}(\xi,y)\right| \le C|\xi|^\omega\left(|\xi-y|^\alpha+|y|^\alpha\right) \tag{4.161}$$

for all $\xi,y\in\mathbf{R}^n$, $k,l=1,...,m$. By homogeneity of $a_0(\xi,y)$ we also have estimate (4.161) for the derivatives

$$\left|((\xi\nabla_\xi)+(y\nabla_y))\,a_0^{k,l}(\xi,y)\right| \le C|\xi|^\omega\left(|\xi-y|^\alpha+|y|^\alpha\right) \tag{4.162}$$

for all $\xi, y \in \mathbf{R}^n$, $k, l = 1, ..., m$. Changing the variable of integration $\xi t^{\frac{1}{\delta}} = \eta$ we have

$$\mathcal{G}^{(j)}(t)\,\theta\delta_0(x) = P^{(j)}(0)\,\theta\,(2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{i\xi\left(x - tb^{(j)}\right) - t\mu_j|\xi|^\delta} d\xi$$
$$= t^{-\frac{n}{\delta}} G^{(j)}\left(\left(x - b^{(j)}t\right) t^{-\frac{1}{\delta}}\right) P^{(j)}(0)\,\theta,$$

here

$$G^{(j)}(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{i\eta x - \mu_j|\eta|^\delta} d\eta.$$

We look for the self-similar solution for (4.160) of the form

$$w(t, x) = t^{-\frac{n}{\delta}} f\left(\left(x - b^{(j)}t\right) t^{-\frac{1}{\delta}}\right).$$

Applying the Fourier transformation to the integral equation (4.160) we obtain

$$\widehat{w}(t, \xi) = e^{-it\xi b^{(j)} - t\mu_j|\xi|^\delta} P^{(j)}(0)\,\theta - \int_0^t d\tau e^{-i(t-\tau)\xi b^{(j)} - (t-\tau)\mu_j|\xi|^\delta} P^{(j)}(0)$$
$$\times \sum_{k,l=1}^m \int_{\mathbf{R}^n} a_0^{k,l}(\xi, y)\, \widehat{w}_k(\tau, \xi - y)\, \widehat{w}_l(\tau, y)\, dy. \tag{4.163}$$

The Fourier transform of $w(t, x) = t^{-\frac{n}{\delta}} f\left(\left(x - b^{(j)}t\right) t^{-\frac{1}{\delta}}\right)$ after a change $y = \left(x - b^{(j)}t\right) t^{-\frac{1}{\delta}}$ is equal to

$$\widehat{w}(t, \xi) = (2\pi)^{-\frac{n}{2}} t^{-\frac{n}{\delta}} \int_{\mathbf{R}^n} e^{-i\xi x} f\left(\left(x - b^{(j)}t\right) t^{-\frac{1}{\delta}}\right) dx$$
$$= (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-it\xi b^{(j)} - i\xi t^{\frac{1}{\delta}} y} f(y)\, dy = e^{-it\xi b^{(j)}} \widehat{f}\left(\xi t^{\frac{1}{\delta}}\right); \tag{4.164}$$

therefore, we get from (4.163)

$$\widehat{f}\left(\xi t^{\frac{1}{\delta}}\right) = e^{-t\mu_j|\xi|^\delta} P^{(j)}(0)\,\theta - \int_0^t d\tau e^{-(t-\tau)\mu_j|\xi|^\delta} P^{(j)}(0)$$
$$\times \sum_{k,l=1}^m \int_{\mathbf{R}^n} a_0^{k,l}(\xi, y)\, \widehat{f}_k\left((\xi - y)\,\tau^{\frac{1}{\delta}}\right) \widehat{f}_l\left(y\tau^{\frac{1}{\delta}}\right) dy.$$

Hence by changing $\eta = \xi t^{\frac{1}{\delta}}$, $\eta' = yt^{\frac{1}{\delta}}$, $\tau = tz$ and by taking into account that the symbols $a_0^{k,l}(\eta, \eta')$ are homogeneous of order $\omega + \alpha$, that is

$$a_0^{k,l}\left(\eta t^{-\frac{1}{\delta}}, \eta' t^{-\frac{1}{\delta}}\right) = t^{-\frac{\omega+\alpha}{\delta}} a_0^{k,l}(\eta, \eta'),$$

and that the critical $\delta = \omega + \alpha + n$, we find

$$\widehat{f}(\eta) = e^{-\mu_j|\eta|^\delta} P^{(j)}(0)\,\theta - \int_0^1 dze^{-(1-z)\mu_j|\eta|^\delta} P^{(j)}(0)$$
$$\times \sum_{k,l=1}^{m} \int_{\mathbf{R}^n} a_0^{k,l}(\eta,\eta')\, \widehat{f}_k\left((\eta-\eta')\, z^{\frac{1}{\delta}}\right) \widehat{f}_l\left(\eta' z^{\frac{1}{\delta}}\right) d\eta'. \qquad (4.165)$$

Denote by $\mathbf{Z} = \left\{\phi \in \mathbf{L}^1(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n)\right\}$ with the norm

$$\|\phi\|_{\mathbf{Z}} = \sup_{1\le p\le\infty} \left\|\langle\cdot\rangle^{2\delta}\phi\right\|_{\mathbf{L}^p} + \sup_{1\le p\le\infty} \left\|\langle\cdot\rangle^{2\delta}(\eta\nabla_\eta)\,\phi\right\|_{\mathbf{L}^p}.$$

The existence of a unique self-similar solution $\widehat{f} \in \mathbf{Z}$ follows by Lemma 4.40 and Lemma 4.3. Hence there exists a unique self-similar solution

$$w(t,x) = t^{-\frac{n}{\delta}} f\left(\left(x - b^{(j)}t\right) t^{-\frac{1}{\delta}}\right)$$

for equation (4.160) such that $\widehat{f} \in \mathbf{Z}$.

4.5.4 Proof of Theorem 4.38

Before proving Theorem 4.38 we prepare some preliminary estimates in the following two lemmas. First we estimate the difference $\mathcal{G}(t) - \mathcal{G}_0(t)$, where

$$\mathcal{G}_0(t)\,\phi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-tL_0(\xi)}\hat{\phi}(\xi)\right),\ e^{-tL_0(\xi)} = \sum_{j=1}^{m} e^{-it\xi b^{(j)} - t\mu_j|\xi|^\delta} P^{(j)}(0)$$

for all $\xi \in \mathbf{R}$, $t > 0$. From Lemma 1.39 we obtain the following lemma.

Lemma 4.42. *Let the linear operator $\mathcal{L}$ satisfy dissipation conditions (4.150), (4.151) and asymptotic representation (4.153). Then the estimates are valid for all $t > 0$, $1 \le p \le q \le \infty$*

$$\|(\mathcal{G}(t) - \mathcal{G}_0(t))\,\phi\|_{\mathbf{A}^{\rho,p}} \le C\,\langle t\rangle^{-\frac{\rho+\gamma}{\delta} - \frac{n}{\delta}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\phi\|_{\mathbf{A}^{0,q}},$$

where $\rho + \gamma \ge 0$, and

$$\left\|\int_0^t (\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau))\,\psi(\tau)\,d\tau\right\|_{\mathbf{A}^{\rho,p}}$$
$$\le C\,\langle t\rangle^{1-\lambda-\frac{\rho+\omega+\gamma}{\delta}-\frac{n}{\delta p}} \sup_{p\le q\le\infty}\sup_{\tau>0}\left(\{\tau\}^\kappa \langle\tau\rangle^{\lambda+\frac{n}{\delta q}} \|\psi(\tau)\|_{\mathbf{A}^{-\omega,q}}\right)$$

where $\kappa \in [0,1)$, $\lambda \in [0,1)$, $0 \le \rho + \omega + \gamma < \delta$.

Now we estimate the nonlinearities

$$\mathcal{N}^{(1)}(\varphi,\phi)=\overline{\mathcal{F}}_{\xi\to x}\sum_{k,l=1}^{m}\int_{|y|\le 2}a^{k,l}(t,\xi,y)\,\widehat{\varphi}_k(\xi-y)\,\widehat{\phi}_l(y)\,dy,$$

$$\mathcal{N}^{(2)}(\varphi,\phi)=\overline{\mathcal{F}}_{\xi\to x}\sum_{k,l=1}^{m}\int_{|y|\ge 2}a^{k,l}(t,\xi,y)\,\widehat{\varphi}_k(\xi-y)\,\widehat{\phi}_l(y)\,dy$$

in the norms $\mathbf{A}^{-\omega,p}$.

Lemma 4.43. *Let the symbols of the nonlinear operators $\mathcal{N}^{(1)}(\varphi,\phi)$ and $\mathcal{N}^{(2)}(\varphi,\phi)$ satisfy condition (4.152). Then the inequalities*

$$\left\|\mathcal{N}^{(1)}(\varphi,\phi)\right\|_{\mathbf{A}^{-\omega,p}}\le C\left(\|\varphi\|_{\mathbf{A}^{\alpha,1}}+\|\varphi\|_{\mathbf{B}^{0,\infty}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}}+\|\phi\|_{\mathbf{B}^{0,\infty}}\right)$$
$$+C\left(\|\phi\|_{\mathbf{A}^{\alpha,1}}+\|\phi\|_{\mathbf{B}^{0,\infty}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}}+\|\varphi\|_{\mathbf{B}^{0,\infty}}\right)$$

and

$$\left\|\mathcal{N}^{(2)}(\varphi,\phi)\right\|_{\mathbf{A}^{-\omega,p}}\le C\|\varphi\|_{\mathbf{B}^{\sigma,1}}\|\phi\|_{\mathbf{B}^{0,\infty}}+C\|\phi\|_{\mathbf{B}^{\sigma,1}}\|\varphi\|_{\mathbf{B}^{0,\infty}}$$

are valid for $1\le p\le\infty$, provided that the right-hand sides are bounded.

Proof. By virtue of condition (4.152) via the Young inequality we obtain as in the proof of Lemma 4.41

$$\left\|\mathcal{N}^{(1)}(\varphi,\phi)\right\|_{\mathbf{A}^{-\omega,p}}$$
$$\le C\sum_{k,l=1}^{m}\left\|\int_{|y|\le 2}\left(\{\cdot-y\}^{\alpha}+\{y\}^{\alpha}\right)\left|\widehat{\varphi}_k(\cdot-y)\,\widehat{\phi}_l(y)\right|dy\right\|_{\mathbf{L}^p_\xi(|\xi|\le 1)}$$
$$\le C\left(\||\cdot|^{\alpha}\widehat{\varphi}\|_{\mathbf{L}^1_\xi(|\xi|\le 1)}+\|\widehat{\varphi}\|_{\mathbf{L}^\infty_\xi(|\xi|\ge 1)}\right)$$
$$\times\left(\left\|\widehat{\phi}\right\|_{\mathbf{L}^p_\xi(|\xi|\le 1)}+\left\|\widehat{\phi}\right\|_{\mathbf{L}^\infty_\xi(|\xi|\ge 1)}\right)$$
$$+C\left(\left\||\cdot|^{\alpha}\widehat{\phi}\right\|_{\mathbf{L}^1_\xi(|\xi|\le 1)}+\left\|\widehat{\phi}\right\|_{\mathbf{L}^\infty_\xi(|\xi|\ge 1)}\right)$$
$$\times\left(\|\widehat{\varphi}\|_{\mathbf{L}^p_\xi(|\xi|\le 1)}+\|\widehat{\varphi}\|_{\mathbf{L}^\infty_\xi(|\xi|>1)}\right);$$

hence the first estimate of the lemma follows. Similarly,

$$\left\|\mathcal{N}^{(2)}(\varphi,\phi)\right\|_{\mathbf{A}^{-\omega,p}}$$
$$\le C\sum_{k,l=1}^{m}\left\|\int_{|y|\ge 2}\left(|\cdot-y|^{\sigma}+|y|^{\sigma}\right)\left|\widehat{\varphi}_k(t,\cdot-y)\,\widehat{\phi}_l(t,y)\right|dy\right\|_{\mathbf{L}^\infty_\xi(|\xi|\le 1)}$$
$$\le C\||\cdot|^{\sigma}\widehat{\varphi}\|_{\mathbf{L}^1_\xi(|\xi|\ge 1)}\left\|\widehat{\phi}\right\|_{\mathbf{L}^\infty_\xi(|\xi|\ge 1)}+C\left\||\cdot|^{\sigma}\widehat{\phi}\right\|_{\mathbf{L}^1_\xi(|\xi|\ge 1)}\|\widehat{\varphi}\|_{\mathbf{L}^\infty_\xi(|\xi|>1)}$$
$$\le C\|\varphi\|_{\mathbf{B}^{\sigma,1}}\|\phi\|_{\mathbf{B}^{0,\infty}}+C\|\phi\|_{\mathbf{B}^{\sigma,1}}\|\varphi\|_{\mathbf{B}^{0,\infty}}.$$

Thus, the second estimate of the lemma is true, and Lemma 4.43 is proved.

We now estimate the nonlinear term

$$\widetilde{\mathcal{N}}(\varphi,\phi)=\overline{\mathcal{F}}_{\xi\to x}\sum_{k,l=1}^{m}\int_{\mathbf{R}^n}B^{k,l}(t,\xi,y)\,\widehat{\varphi}_k(\xi-y)\,\widehat{\phi}_l(y)\,dy.$$

Denote $(\rho)_+=\max(0,\rho)$.

Lemma 4.44. *Let the symbols $B^{k,l}(t,\xi,y)$ of the nonlinear operator $\widetilde{\mathcal{N}}(\varphi,\phi)$ satisfy estimate*

$$\left|B^{k,l}(t,\xi,y)\right|\le Ct\langle\xi\rangle^{\theta}\{\xi\}^{\omega}\left(\langle\xi-y\rangle^{\sigma}\{\xi-y\}^{\alpha}+\langle y\rangle^{\sigma}\{y\}^{\alpha}\right)$$
$$\times\left(\langle t\xi\rangle^{-1}+\langle t(y-a\xi)\rangle^{-1}\right)\tag{4.166}$$

for all $\xi,y\in\mathbf{R}^n$, $t>0$. Then the inequality

$$\left\|\widetilde{\mathcal{N}}(\varphi,\phi)\right\|_{\mathbf{A}^{\rho-\omega,p}}$$
$$\le C\{t\}\langle t\rangle^{(1-\rho)_+}\left(\|\varphi\|_{\mathbf{A}^{\alpha+(\rho-1)_+,1}}+\|\varphi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}}+\|\phi\|_{\mathbf{B}^{0,\infty}}\right)$$
$$+C\{t\}\langle t\rangle^{(1-\rho)_+}\left(\|\phi\|_{\mathbf{A}^{\alpha+(\rho-1)_+,1}}+\|\phi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}}+\|\varphi\|_{\mathbf{B}^{0,\infty}}\right)$$
$$+C\{t\}\langle t\rangle^{\frac{1}{q}}\left(\|\varphi\|_{\mathbf{A}^{\rho+\alpha,q}}+\|\varphi\|_{\mathbf{B}^{\sigma,q}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}}+\|\phi\|_{\mathbf{B}^{0,\infty}}\right)$$
$$+C\{t\}\langle t\rangle^{\frac{1}{q}}\left(\|\phi\|_{\mathbf{A}^{\rho+\alpha,q}}+\|\phi\|_{\mathbf{B}^{\sigma,q}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}}+\|\varphi\|_{\mathbf{B}^{0,\infty}}\right)$$

is valid for all $t>0$, where $1\le p\le\infty$, $\rho\ge 0$, $1\le q<\infty$, provided that the right-hand side is bounded.

Proof. In the case of $t\in(0,1)$ as in Lemma 4.41 we obtain

$$\left\|\widetilde{\mathcal{N}}(\varphi,\phi)\right\|_{\mathbf{A}^{\rho-\omega,p}}\le Ct\left(\|\varphi\|_{\mathbf{A}^{\rho+\alpha,1}}+\|\varphi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}}+\|\phi\|_{\mathbf{B}^{0,\infty}}\right)$$
$$+C\left(\|\phi\|_{\mathbf{A}^{\rho+\alpha,1}}+\|\phi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}}+\|\varphi\|_{\mathbf{B}^{0,\infty}}\right).$$

By condition (4.166) and by the Young and Hölder inequalities we get

$$\left\|\widetilde{\mathcal{N}}(\varphi,\phi)\right\|_{\mathbf{A}^{\rho-\omega,p}}\le Ct\sum_{k,l=1}^{m}\left\||\cdot|^{\rho}\int_{\mathbf{R}^n}\left(\langle\cdot-y\rangle^{\sigma}\{\cdot-y\}^{\alpha}+\langle y\rangle^{\sigma}\{y\}^{\alpha}\right)\right.$$
$$\left.\times\left(\langle t\cdot\rangle^{-1}+\langle t(y-a\cdot)\rangle^{-1}\right)\left|\widehat{\varphi}_k(t,\cdot-y)\,\widehat{\phi}_l(t,y)\right|dy\right\|_{\mathbf{L}^p_{\xi}(|\xi|\le 1)}$$

for all $t\ge 1$. Hence

$$
\begin{aligned}
&\left\|\widetilde{\mathcal{N}}(\varphi,\phi)\right\|_{\mathbf{A}^{\rho-\omega,p}} \\
&\leq Ct^{(1-\rho)_+}\sum_{k,l=1}^{m}\left\|\int_{\mathbf{R}^n}\left(\langle\cdot-y\rangle^{\sigma}\{\cdot-y\}^{\alpha+(\rho-1)_+}+\langle y\rangle^{\sigma}\{y\}^{\alpha+(\rho-1)_+}\right)\right. \\
&\times\left.\left|\widehat{\varphi}_k(t,\cdot-y)\widehat{\phi}_l(t,y)\right|dy\right\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)} \\
&+Ct\sum_{k,l=1}^{m}\left\|\int_{\mathbf{R}^n}\frac{\langle\cdot-y\rangle^{\sigma}\{\cdot-y\}^{\rho+\alpha}+\langle y\rangle^{\sigma}\{y\}^{\rho+\alpha}}{\langle t(y-a\cdot)\rangle}\right. \\
&\times\left.\left|\widehat{\varphi}_k(t,\cdot-y)\widehat{\phi}_l(t,y)\right|dy\right\|_{\mathbf{L}^p_\xi(|\xi|\leq 1)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left\|\widetilde{\mathcal{N}}(\varphi,\phi)\right\|_{\mathbf{A}^{\rho-\omega,p}} \\
&\leq Ct^{(1-\rho)_+}\left(\|\varphi\|_{\mathbf{A}^{\alpha+(\rho-1)_+,1}}+\|\varphi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}}+\|\phi\|_{\mathbf{B}^{0,\infty}}\right) \\
&+Ct^{(1-\rho)_+}\left(\|\phi\|_{\mathbf{A}^{\alpha+(\rho-1)_+,1}}+\|\phi\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}}+\|\varphi\|_{\mathbf{B}^{0,\infty}}\right) \\
&+Ct^{\frac{1}{q}}\left(\|\varphi\|_{\mathbf{A}^{\rho+\alpha,q}}+\|\varphi\|_{\mathbf{B}^{\sigma,q}}\right)\left(\|\phi\|_{\mathbf{A}^{0,p}}+\|\phi\|_{\mathbf{B}^{0,\infty}}\right) \\
&+Ct^{\frac{1}{q}}\left(\|\phi\|_{\mathbf{A}^{\rho+\alpha,q}}+\|\phi\|_{\mathbf{B}^{\sigma,q}}\right)\left(\|\varphi\|_{\mathbf{A}^{0,p}}+\|\varphi\|_{\mathbf{B}^{0,\infty}}\right).
\end{aligned}
$$

Lemma 4.44 is proved.

Now we turn to the proof of Theorem 4.38. We consider the following integral equation

$$
w(t)=\mathcal{G}_0(t)\widetilde{u}-\int_0^t\mathcal{G}_0(t-\tau)\mathcal{N}_0(w,w)(\tau)\,d\tau. \tag{4.167}
$$

Note that Theorem 4.36 is also applicable to equation (4.167) if we take $\sigma=\alpha$, $\nu=\delta$ and $\theta=\omega$. Hence we obtain the estimate

$$
\|w\|_{\mathbf{X}_0}\leq C\varepsilon,
$$

where we introduce the norm

$$
\begin{aligned}
\|\phi\|_{\mathbf{X}_0}=&\sup_{\rho\in[0,\alpha+\gamma]}\sup_{1\leq p\leq\infty}\sup_{t>0}\langle t\rangle^{\frac{\rho}{\delta}+\frac{n}{\delta p}}\|\phi(t)\|_{\mathbf{A}^{\rho,p}} \\
&+\sup_{s\in[0,\alpha]}\sup_{1\leq p\leq\infty}\sup_{t>0}\{t\}^{\frac{s}{\delta}+\frac{n}{\delta p}}\langle t\rangle^{\frac{n+\alpha+\gamma}{\delta}+\frac{n}{\delta p}}\|\phi(t)\|_{\mathbf{B}^{s,p}}.
\end{aligned}
$$

For the solution $u(t,x)$ we obtained in Theorem 4.36 the estimate

$$
\|u\|_{\mathbf{X}}\leq C\varepsilon;
$$

therefore, we have in particular the estimates for all $t>0$

$$\|u(t)\|_{\mathbf{B}^{s,p}} \le C\varepsilon \{t\}^{-\frac{s}{\nu}} \langle t\rangle^{-\frac{n+\alpha+\gamma}{\delta}-\frac{n}{\delta p}},$$

where $s \in [0,\sigma]$, $1 \le p \le \infty$ and

$$\|w(t)\|_{\mathbf{B}^{s,p}} \le C\varepsilon \{t\}^{-\frac{s}{\delta}-\frac{n}{\delta p}} \langle t\rangle^{-\frac{n+\alpha+\gamma}{\delta}-\frac{n}{\delta p}},$$

where $s \in [0,\alpha]$, $1 \le p \le \infty$. These estimates are sufficient for obtaining the large time decay rate for the remainder $v(t) = u(t) - w(t)$ in $\mathbf{B}$ norms

$$\|v(t)\|_{\mathbf{B}^{0,1}} \le \|u(t)\|_{\mathbf{B}^{0,1}} + \|w(t)\|_{\mathbf{B}^{0,1}} \le C\varepsilon \left(1 + \{t\}^{-\frac{n}{\delta}}\right) \langle t\rangle^{-\frac{2n+\alpha+\gamma}{\delta}} \tag{4.168}$$

for all $t > 0$. Now we estimate the difference $v(t) = u(t) - w(t)$ in the norms $\mathbf{A}^{\rho,p}$ and show that

$$\|v(t)\|_{\mathbf{A}^{\rho,p}} \le C\varepsilon \langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{n}{\delta p}} \tag{4.169}$$

for all $t > 0$, where $\rho \in [0,\alpha]$, $1 \le p \le \infty$.

We represent the nonlinearities $\mathcal{N}(u) = \mathcal{N}^{(1)}(u,u) + \mathcal{N}^{(2)}(u,u)$ and $\mathcal{N}_0(w,w) = \mathcal{N}_0^{(1)}(w,w) + \mathcal{N}_0^{(2)}(w,w)$, where

$$\mathcal{N}^{(1)}(u,u) = \overline{\mathcal{F}}_{\xi\to x} \sum_{k,l=1}^{m} \int_{|y|\le 2} a^{k,l}(t,\xi,y)\, \widehat{u}_k(t,\xi-y)\, \widehat{u}_l(t,y)\, dy$$

$$\mathcal{N}^{(2)}(u,u) = \overline{\mathcal{F}}_{\xi\to x} \sum_{k,l=1}^{m} \int_{|y|\ge 2} a^{k,l}(t,\xi,y)\, \widehat{u}_k(t,\xi-y)\, \widehat{u}_l(t,y)\, dy.$$

The terms $\mathcal{N}_0^{(1)}(w,w)$, $\mathcal{N}_0^{(2)}(w,w)$ are defined similarly.

Then for the difference $v(t) = u(t) - w(t)$ we get

$$\begin{aligned} v(t) &= (\mathcal{G}(t) - \mathcal{G}_0(t))\,\widetilde{u} \\ &\quad - \int_0^t \left(\mathcal{G}(t-\tau)\mathcal{N}^{(2)}(u,u) - \mathcal{G}_0(t-\tau)\mathcal{N}_0^{(2)}(w,w)\right) d\tau \\ &\quad - \int_0^t (\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau))\,\mathcal{N}^{(1)}(u,u)(\tau)\, d\tau \\ &\quad - \int_0^t \mathcal{G}_0(t-\tau) \left(\mathcal{N}^{(1)}(u,u)(\tau) - \mathcal{N}_0^{(1)}(u,u)(\tau)\right) d\tau \\ &\quad - \int_0^t \mathcal{G}_0(t-\tau) \left(\mathcal{N}_0^{(1)}(u,u)(\tau) - \mathcal{N}_0^{(1)}(w,w)(\tau)\right) d\tau. \end{aligned} \tag{4.170}$$

By the first estimate of Lemma 4.42 we have

$$\|(\mathcal{G}(t) - \mathcal{G}_0(t))\,\widetilde{u}\|_{\mathbf{A}^{\rho,p}} \le C \langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{n}{\delta p}} \|\widetilde{u}\|_{\mathbf{A}^{0,\infty}}. \tag{4.171}$$

The second summand in (4.170) we estimate by Lemmas 4.40 and 4.43

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}(t-\tau)\,\mathcal{N}^{(2)}(u,u)\,d\tau \right\|_{\mathbf{A}^{\rho,p}} \\
&\le C\langle t\rangle^{1-\widetilde{\lambda}-\frac{\rho+\omega}{\delta}-\frac{n}{\delta p}} \sup_{p\le q\le\infty}\sup_{\tau>0}\left(\{\tau\}^{\kappa} \langle \tau\rangle^{\widetilde{\lambda}+\frac{n}{\delta q}} \left\| \mathcal{N}^{(2)}(u,u)\right\|_{\mathbf{A}^{-\omega,q}} \right) \\
&\le C\langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{n}{\delta p}} \sup_{\tau>0}\left(\{\tau\}^{\kappa} \langle \tau\rangle^{\widetilde{\lambda}+\frac{n}{\delta}} \|u\|_{\mathbf{B}^{\sigma,1}} \|u\|_{\mathbf{B}^{0,\infty}} \right) \\
&\le C\langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{n}{\delta p}} \|u\|_{\mathbf{X}}^2 \le C\varepsilon^2 \langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{n}{\delta p}}, \qquad (4.172)
\end{aligned}$$

where $\kappa = \frac{\sigma}{\nu}$, $\widetilde{\lambda} = \lambda + \frac{\gamma}{\delta} = \frac{n+\alpha+\gamma}{\delta}$, $\omega > \gamma > 0$. In the same manner we have

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}_0(t-\tau)\,\mathcal{N}_0^{(2)}(w,w)\,d\tau \right\|_{\mathbf{A}^{\rho,p}} \\
&\le C\langle t\rangle^{1-\widetilde{\lambda}-\frac{\rho+\omega}{\delta}-\frac{n}{\delta p}} \sup_{p\le q\le\infty}\sup_{\tau>0}\left(\{\tau\}^{\kappa_0} \langle \tau\rangle^{\widetilde{\lambda}+\frac{n}{\delta q}} \left\| \mathcal{N}_0^{(2)}(w,w)\right\|_{\mathbf{A}^{-\omega,q}} \right) \\
&\le C\langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{n}{\delta p}} \sup_{\tau>0}\left(\{\tau\}^{\kappa_0} \langle \tau\rangle^{\widetilde{\lambda}+\frac{n}{\delta}} \|w\|_{\mathbf{B}^{\alpha,1}} \|w\|_{\mathbf{B}^{0,\infty}} \right) \\
&\le C\langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{n}{\delta p}} \|w\|_{\mathbf{X}_0}^2 \le C\varepsilon^2 \langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{n}{\delta p}}, \qquad (4.173)
\end{aligned}$$

where $\kappa_0 = \frac{\alpha+n}{\delta}$. By Lemma 4.42 we obtain with $\lambda = 1 - \frac{\omega}{\delta}$

$$\begin{aligned}
&\left\| \int_0^t \left(\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau)\right)\mathcal{N}^{(1)}(u,u)(\tau)\,d\tau \right\|_{\mathbf{A}^{\rho,p}} \\
&\le C\langle t\rangle^{1-\lambda-\frac{\rho+\omega+\gamma}{\delta}-\frac{n}{\delta p}} \sup_{p\le q\le\infty}\sup_{\tau>0}\left(\langle \tau\rangle^{\lambda+\frac{n}{\delta q}} \left\| \mathcal{N}^{(1)}(u,u)(\tau)\right\|_{\mathbf{A}^{-\omega,q}} \right) \\
&\le C\varepsilon^2 \langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{n}{\delta p}}, \qquad (4.174)
\end{aligned}$$

since via Lemma 4.43 we have

$$\begin{aligned}
&\langle \tau\rangle^{\lambda+\frac{n}{\delta p}} \left\| \mathcal{N}^{(1)}(u,u)(\tau)\right\|_{\mathbf{A}^{-\omega,p}} \le C\langle \tau\rangle^{\frac{n+\alpha}{\delta}} \left(\|u(\tau)\|_{\mathbf{A}^{\alpha,1}} + \|u(\tau)\|_{\mathbf{B}^{0,\infty}} \right) \\
&\times \langle \tau\rangle^{\frac{n}{\delta p}} \left(\|u(\tau)\|_{\mathbf{A}^{0,p}} + \|u(\tau)\|_{\mathbf{B}^{0,\infty}} \right) \le C\|u\|_{\mathbf{X}}^2 \le C\varepsilon^2.
\end{aligned}$$

As above the application of Lemma 4.40 yields

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}_0(t-\tau)\left(\mathcal{N}^{(1)}(u,u)(\tau) - \mathcal{N}_0^{(1)}(u,u)(\tau)\right) d\tau \right\|_{\mathbf{A}^{\rho,p}} \\
&\le C\langle t\rangle^{1-\widetilde{\lambda}-\frac{\rho+\omega}{\delta}-\frac{n}{\delta p}} \\
&\times \sup_{p\le q\le\infty}\sup_{\tau>0}\left(\langle \tau\rangle^{\widetilde{\lambda}+\frac{n}{\delta q}} \left\| \mathcal{N}^{(1)}(u,u)(\tau) - \mathcal{N}_0^{(1)}(u,u)(\tau)\right\|_{\mathbf{A}^{-\omega,q}} \right) \\
&\le C\varepsilon^2 \langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{n}{\delta p}}, \qquad (4.175)
\end{aligned}$$

since by Lemma 4.41 and Lemma 4.43 in view of condition (4.154) we have

$$
\begin{aligned}
&\langle t\rangle^{\widetilde{\lambda}+\frac{n}{\delta p}} \left\| \mathcal{N}^{(1)}(u,u)(t) - \mathcal{N}_0^{(1)}(u,u)(t) \right\|_{\mathbf{A}^{-\omega,p}} \\
&\leq C \langle t\rangle^{\frac{\alpha+\gamma+n}{\delta}+\frac{n}{\delta p}} \left(\|u(t)\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|u(t)\|_{\mathbf{B}^{0,\infty}} \right) \left(\|u(t)\|_{\mathbf{A}^{0,p}} + \|u(t)\|_{\mathbf{B}^{0,\infty}} \right) \\
&+ C \langle t\rangle^{\frac{\alpha+n}{\delta}+\frac{n}{\delta p}} \left(\|u(t)\|_{\mathbf{A}^{\alpha,1}} + \|u(t)\|_{\mathbf{B}^{0,\infty}} \right) \left(\|u(t)\|_{\mathbf{A}^{0,p}} + \|u(t)\|_{\mathbf{B}^{0,\infty}} \right) \\
&\leq C \|u\|_{\mathbf{X}}^2 .
\end{aligned}
$$

Finally, by Lemma 4.40 we find for $\rho \in [0,\alpha]$, $1 \leq p \leq \infty$

$$
\begin{aligned}
&\langle t\rangle^{\frac{\rho+\gamma}{\delta}+\frac{n}{\delta p}} \left\| \int_0^t \mathcal{G}_0(t-\tau) \left(\mathcal{N}_0^{(1)}(u,u)(\tau) - \mathcal{N}_0^{(1)}(w,w)(\tau) \right) d\tau \right\|_{\mathbf{A}^{\rho,p}} \\
&\leq C \sup_{p\leq q\leq\infty} \sup_{\tau>0} \left(\langle \tau\rangle^{\widetilde{\lambda}+\frac{n}{\delta q}} \left\| \left(\mathcal{N}_0^{(1)}(u,u)(\tau) - \mathcal{N}_0^{(1)}(w,w)(\tau) \right) \right\|_{\mathbf{A}^{-\omega,q}} \right) \\
&\leq C\varepsilon^2 + C\varepsilon \sup_{\rho\in[0,\alpha]} \sup_{1\leq q\leq\infty} \langle \tau\rangle^{\frac{\rho+\gamma}{\delta}+\frac{n}{\delta p}} \|v(\tau)\|_{\mathbf{A}^{\rho,q}} ,
\end{aligned}
\tag{4.176}
$$

since by using Lemma 4.43 we get

$$
\begin{aligned}
&\langle \tau\rangle^{\widetilde{\lambda}+\frac{1}{\delta q}} \left\| \left(\mathcal{N}_0^{(1)}(u,u)(\tau) - \mathcal{N}_0^{(1)}(w,w)(\tau) \right) \right\|_{\mathbf{A}^{-\omega,q}} \\
&\leq C \langle \tau\rangle^{\frac{n+\alpha+\gamma}{\delta}} \left(\|v(\tau)\|_{\mathbf{A}^{\alpha,1}} + \|u\|_{\mathbf{B}^{0,\infty}} + \|w\|_{\mathbf{B}^{0,\infty}} \right) \\
&\times \langle \tau\rangle^{\frac{1}{\delta q}} \left(\|u\|_{\mathbf{A}^{0,q}} + \|w\|_{\mathbf{A}^{0,q}} + \|u\|_{\mathbf{B}^{0,\infty}} + \|w\|_{\mathbf{B}^{0,\infty}} \right) \\
&+ C \langle \tau\rangle^{\frac{\alpha+n}{\delta}} \left(\|u\|_{\mathbf{A}^{\alpha,1}} + \|w\|_{\mathbf{A}^{\alpha,1}} + \|u\|_{\mathbf{B}^{0,\infty}} + \|w\|_{\mathbf{B}^{0,\infty}} \right) \\
&\times \langle \tau\rangle^{\frac{\gamma}{\delta}+\frac{n}{\delta q}} \left(\|v(\tau)\|_{\mathbf{A}^{0,q}} + \|u\|_{\mathbf{B}^{0,\infty}} + \|w\|_{\mathbf{B}^{0,\infty}} \right) \\
&\leq \left(\|u\|_{\mathbf{X}} + \|w\|_{\mathbf{X}_0} \right)^2 \\
&+ C \sup_{\rho\in[0,\alpha]} \sup_{1\leq p\leq q} \langle \tau\rangle^{\frac{\rho+\gamma}{\delta}+\frac{n}{\delta p}} \|v(\tau)\|_{\mathbf{A}^{\rho,p}} \left(\|u\|_{\mathbf{X}} + \|w\|_{\mathbf{X}_0} \right) \\
&\leq C\varepsilon^2 + C\varepsilon \sup_{\rho\in[0,\alpha]} \sup_{1\leq p\leq q} \langle \tau\rangle^{\frac{\rho+\gamma}{\delta}+\frac{n}{\delta p}} \|v(\tau)\|_{\mathbf{A}^{\rho,p}} .
\end{aligned}
$$

Therefore by collecting inequalities (4.171) - (4.176) with representation (4.170) we get

$$
\begin{aligned}
&\sup_{\rho\in[0,\alpha]} \sup_{1\leq p\leq\infty} \sup_{t>0} \langle t\rangle^{\frac{\rho+\gamma}{\delta}+\frac{n}{\delta p}} \|v(t)\|_{\mathbf{A}^{\rho,p}} \\
&\leq C\varepsilon^2 + C\varepsilon \sup_{\rho\in[0,\alpha]} \sup_{1\leq p\leq\infty} \sup_{t>0} \langle t\rangle^{\frac{\rho+\gamma}{\delta}+\frac{n}{\delta p}} \|v(t)\|_{\mathbf{A}^{\rho,p}} ;
\end{aligned}
$$

hence estimate (4.169) follows.

Now we construct the main term of the large time asymptotics as a superposition of the self-similar solutions

$$
h(t,x) = \sum_{j=1}^{m} h^{(j)}(t,x) = t^{-\frac{n}{\delta}} \sum_{j=1}^{m} f^{(j)} \left(\left(x - b^{(j)} t \right) t^{-\frac{1}{\delta}} \right) .
$$

The functions $h^{(j)}$ were defined in Section 4.5.3 by equations

$$h^{(j)}(t) = \mathcal{G}^{(j)}(t)\,\theta\delta_0(x) - \int_0^t \mathcal{G}^{(j)}(t-\tau)\,\mathcal{N}_0\left(h^{(j)}, h^{(j)}\right)(\tau)\,d\tau$$

for $j = 1, ..., m$, where

$$\mathcal{G}^{(j)}(t)\,\phi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-it\xi b^{(j)} - t\mu_j|\xi|^\delta} P^{(j)}(0)\,\hat{\phi}(\xi)\right),$$

so that

$$\mathcal{G}_0(t) = \sum_{j=1}^m \mathcal{G}^{(j)}(t).$$

Then for the difference $h(t) - w(t)$ we get from (4.167)

$$\begin{aligned} h(t) - w(t) &= \mathcal{G}_0(t)\,\mathring{u} \\ &- \int_0^t \mathcal{G}_0(t-\tau)\left(\mathcal{N}_0(h,h)(\tau) - \mathcal{N}_0(w,w)(\tau)\right)d\tau \\ &- \sum_{j=1}^m \int_0^t \mathcal{G}^{(j)}(t-\tau)\left(\mathcal{N}_0\left(h^{(j)}, h^{(j)}\right)(\tau) - \mathcal{N}_0(h,h)(\tau)\right)d\tau, \end{aligned} \qquad (4.177)$$

where $\mathring{u}(x) = \widetilde{u}(x) - \theta\delta_0(x)$. By the condition of Theorem 4.38 we have estimate

$$\|\mathring{u}\|_{\mathbf{A}^{-\gamma,\infty}} + \|\mathring{u}\|_{\mathbf{B}^{0,\infty}} \le \varepsilon$$

which means that the initial data $\widetilde{u}(x)$ have the following asymptotic representation $\widehat{\widetilde{u}}(\xi) = \theta + O(|\xi|^\gamma)$ as $\xi \to 0$. By the first and third estimates of Lemma 4.40 we find

$$\|\mathcal{G}_0(t)\,\mathring{u}\|_{\mathbf{X}_\gamma} \le C\left(\|\mathring{u}\|_{\mathbf{A}^{-\gamma,\infty}} + \|\mathring{u}\|_{\mathbf{B}^{0,\infty}}\right), \qquad (4.178)$$

where we denote

$$\begin{aligned} \|\phi\|_{\mathbf{X}_\gamma} = &\sup_{\rho\in[0,\alpha]}\sup_{1\le p\le\infty}\sup_{t>0}\langle t\rangle^{\frac{\rho+\gamma}{\delta}+\frac{n}{\delta p}}\|\phi(t)\|_{\mathbf{A}^{\rho,p}} \\ &+ \sup_{s\in[0,\alpha]}\sup_{1\le p\le\infty}\sup_{t>0}\{t\}^{\frac{s}{\delta}+\frac{n}{\delta p}}\langle t\rangle^{\frac{n+\alpha+\gamma}{\delta}+\frac{n}{\delta p}}\|\phi(t)\|_{\mathbf{B}^{s,p}}. \end{aligned}$$

For the second summand in equation (4.177) we estimate by virtue of Lemma 4.40 with $q = \infty$

$$\begin{aligned} &\left\|\int_0^t \mathcal{G}_0(t-\tau)\left(\mathcal{N}_0(h,h)(\tau) - \mathcal{N}_0(w,w)(\tau)\right)d\tau\right\|_{\mathbf{A}^{\rho,p}} \\ \le &C\langle t\rangle^{1-\widetilde{\lambda}-\frac{\rho+\omega}{\delta}-\frac{n}{\delta p}} \\ \times &\sup_{p\le q\le\infty}\sup_{\tau>0}\left(\{\tau\}^{\kappa_0}\langle\tau\rangle^{\widetilde{\lambda}+\frac{n}{\delta q}}\|(\mathcal{N}_0(h,h)(\tau) - \mathcal{N}_0(w,w)(\tau))\|_{\mathbf{A}^{-\omega,q}}\right) \end{aligned}$$

and

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}_0 (t-\tau) \left(\mathcal{N}_0 (h,h)(\tau) - \mathcal{N}_0 (w,w)(\tau) \right) d\tau \right\|_{\mathbf{B}^{s,p}} \\
\le & C \{t\}^{1-\kappa_0 - \frac{s+\omega}{\delta} - \frac{n}{\delta p}} \langle t \rangle^{-\widetilde{\lambda} - \frac{n}{\delta p}} \\
& \times \left(\sup_{\tau>0} \left(\{\tau\}^{\kappa_0} \left\| \left(\mathcal{N}_0 (h,h)(\tau) - \mathcal{N}_0 (w,w)(\tau) \right) \right\|_{\mathbf{B}^{-\omega,\infty}} \right) \right. \\
& \left. + \sup_{\tau>0} \left(\{\tau\}^{\kappa_0 + \frac{n}{\delta p}} \langle \tau \rangle^{\widetilde{\lambda} + \frac{n}{\delta p}} \left\| \left(\mathcal{N}_0 (h,h)(\tau) - \mathcal{N}_0 (w,w)(\tau) \right) \right\|_{\mathbf{B}^{-\omega,p}} \right) \right),
\end{aligned}$$

for all $t > 0$, $s, \rho \in [0, \alpha]$, $1 \le p \le \infty$, where $\kappa_0 = 1 - \frac{\omega}{\delta}$, $\widetilde{\lambda} = \lambda + \frac{\gamma}{\delta}$, $\lambda = 1 - \frac{\omega}{\delta}$. Since

$$\begin{aligned}
& \mathcal{N}_0 (h,h) - \mathcal{N}_0 (w,w) \\
& = \frac{1}{2} \mathcal{N}_0 (h-w, h+w) + \frac{1}{2} \mathcal{N}_0 (h+w, h-w),
\end{aligned}$$

by applying Lemma 4.41 we have

$$\begin{aligned}
& \{\tau\}^{\kappa_0} \langle \tau \rangle^{\widetilde{\lambda} + \frac{n}{\delta q}} \left\| \left(\mathcal{N}_0 (h,h)(\tau) - \mathcal{N}_0 (w,w)(\tau) \right) \right\|_{\mathbf{A}^{-\omega,q}} \\
& \le C \{\tau\}^{\frac{n+\alpha}{\delta}} \langle \tau \rangle^{\frac{n+\alpha+\gamma}{\delta}} \left(\|h-w\|_{\mathbf{A}^{\alpha,1}} + \|h-w\|_{\mathbf{B}^{\alpha,1}} \right) \\
& \times \langle \tau \rangle^{\frac{n}{\delta q}} \left(\|h+w\|_{\mathbf{A}^{0,q}} + \|h+w\|_{\mathbf{B}^{0,\infty}} \right) \\
& + C \{\tau\}^{\frac{n+\alpha}{\delta}} \langle \tau \rangle^{\frac{n+\alpha}{\delta}} \left(\|h+w\|_{\mathbf{A}^{\alpha,1}} + \|h+w\|_{\mathbf{B}^{\alpha,1}} \right) \\
& \times \langle \tau \rangle^{\frac{\gamma}{\delta} + \frac{n}{\delta q}} \left(\|h-w\|_{\mathbf{A}^{0,q}} + \|h-w\|_{\mathbf{B}^{0,\infty}} \right) \\
& \le C \|h-w\|_{\mathbf{X}_\gamma} \left(\|h\|_{\mathbf{X}_0} + \|w\|_{\mathbf{X}_0} \right) + C \left(\|h\|_{\mathbf{X}_0} + \|w\|_{\mathbf{X}_0} \right)^2
\end{aligned}$$

and

$$\begin{aligned}
& \{\tau\}^{\kappa_0 + \frac{n}{\delta q}} \langle \tau \rangle^{\widetilde{\lambda} + \frac{n}{\delta q}} \left\| \left(\mathcal{N}_0 (h,h)(\tau) - \mathcal{N}_0 (w,w)(\tau) \right) \right\|_{\mathbf{B}^{-\omega,q}} \\
& \le C \{\tau\}^{\frac{n+\alpha}{\delta}} \langle \tau \rangle^{\frac{n+\alpha+\gamma}{\delta}} \left(\|h\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|w\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|h\|_{\mathbf{B}^{\alpha,1}} + \|w\|_{\mathbf{B}^{\alpha,1}} \right) \\
& \times \tau^{\frac{n}{\delta q}} \left(\|h\|_{\mathbf{A}^{0,q}} + \|w\|_{\mathbf{A}^{0,q}} + \|h\|_{\mathbf{B}^{0,q}} + \|w\|_{\mathbf{B}^{0,q}} \right) \\
& \le C \left(\|h\|_{\mathbf{X}_0} + \|w\|_{\mathbf{X}_0} \right)^2.
\end{aligned}$$

Hence

$$\begin{aligned}
& \left\| \int_0^t \mathcal{G}_0 (t-\tau) \left(\mathcal{N}_0 (h,h)(\tau) - \mathcal{N}_0 (w,w)(\tau) \right) d\tau \right\|_{\mathbf{X}_\gamma} \\
& \le C\varepsilon^2 + C\varepsilon \|h-w\|_{\mathbf{X}_\gamma}.
\end{aligned} \tag{4.179}$$

For the last summand in (4.177) we have

$$
\begin{aligned}
&- \int_0^t \mathcal{G}^{(j)}(t-\tau)\left(\mathcal{N}_0\left(h^{(j)}, h^{(j)}\right)(\tau) - \mathcal{N}_0(h,h)(\tau)\right) d\tau \\
&= \sum_{k,l=1,\ l\neq j}^{m} \int_0^t \mathcal{G}^{(j)}(t-\tau)\mathcal{N}_0\left(h^{(k)}, h^{(l)}\right)(\tau)\, d\tau \\
&+ \sum_{k=1,\ k\neq j}^{m} \int_0^t \mathcal{G}^{(j)}(t-\tau)\mathcal{N}_0\left(h^{(j)}, h^{(j)}\right)(\tau)\, d\tau.
\end{aligned}
$$

The application of the Fourier transformation yields

$$
\begin{aligned}
&\mathcal{F}_{x\to\xi} \int_0^t \mathcal{G}^{(j)}(t-\tau)\mathcal{N}_0\left(h^{(k)}, h^{(l)}\right)(\tau)\, d\tau \\
&= \int_0^t d\tau e^{-i(t-\tau)\xi b^{(j)} - (t-\tau)\mu_j|\xi|^{\delta}} P^{(j)}(0) \\
&\times \sum_{k',l'=1}^{m} \int_{\mathbf{R}^n} A_0^{k',l'}(\xi,y)\, \widehat{f_{k'}^{(k)}}\left(\tau^{\frac{1}{\delta}}(\xi-y)\right) \widehat{f_{l'}^{(l)}}\left(\tau^{\frac{1}{\delta}}y\right) e^{-i\tau(\xi-y)b^{(k)} - i\tau y b^{(l)}}\, dy,
\end{aligned}
$$

since by (4.164) we have

$$
\mathcal{F}_{x\to\xi} h^{(k)} = e^{-it\xi b^{(k)}} \widehat{f^{(k)}}\left(t^{\frac{1}{\delta}}\xi\right).
$$

To be able to estimate the time decay for $t \geq 1$, we integrate by parts via the identity

$$
e^{\tau S_{j,k,l}} = \left(1 + \tau S_{j,k,l}\right)^{-1} \frac{d}{d\tau}\left(\tau e^{\tau S_{j,k,l}}\right),
$$

where

$$
S_{j,k,l} = i\left(b^{(j)} - b^{(k)}\right)\xi + i\left(b^{(k)} - b^{(l)}\right)y + \mu_j |\xi|^{\delta},
$$

with $\left(b^{(j)} - b^{(k)}\right)^2 + \left(b^{(k)} - b^{(l)}\right)^2 \neq 0$, to obtain

$$
\begin{aligned}
&\mathcal{F}_{x\to\xi} \int_0^t \mathcal{G}^{(j)}(t-\tau)\mathcal{N}_0\left(h^{(k)}, h^{(l)}\right)(\tau)\, d\tau \\
&= e^{-it\xi b^{(j)} - t\mu_j|\xi|^{\delta}} P^{(j)}(0) \\
&\times \sum_{k',l'=1}^{m} \int_{\mathbf{R}^n} e^{tS_{j,k,l}} \frac{tA_0^{k',l'}(\xi,y)}{1 + tS_{j,k,l}} \widehat{f_{k'}^{(k)}}\left(t^{\frac{1}{\delta}}(\xi-y)\right) \widehat{f_{l'}^{(l)}}\left(t^{\frac{1}{\delta}}y\right) dy \\
&- e^{-it\xi b^{(j)} - t\mu_j|\xi|^{\delta}} P^{(j)}(0) \int_0^t d\tau \\
&\times \sum_{k',l'=1}^{m} \int_{\mathbf{R}^n} \tau e^{\tau S_{j,k,l}} A_0^{k',l'}(\xi,y) \\
&\times \frac{d}{d\tau}\left(\left(1 + \tau S_{j,k,l}\right)^{-1} \widehat{f_{k'}^{(k)}}\left(\tau^{\frac{1}{\delta}}(\xi-y)\right) \widehat{f_{l'}^{(l)}}\left(\tau^{\frac{1}{\delta}}y\right)\right) dy.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_0^t \mathcal{G}^{(j)}(t-\tau)\mathcal{N}_0\left(h^{(k)},h^{(l)}\right)(\tau)\,d\tau \\
&= \overline{\mathcal{F}}_{x\to\xi}P^{(j)}(0)\sum_{k',l'=1}^{m}\int_{\mathbf{R}^n}\frac{tA_0^{k',l'}(\xi,y)}{1+tS_{j,k,l}}e^{it(\xi-y)b^{(k)}+ityb^{(l)}} \\
&\times \widehat{f_{k'}^{(k)}}\left(t^{\frac{1}{\delta}}(\xi-y)\right)\widehat{f_{l'}^{(l)}}\left(t^{\frac{1}{\delta}}y\right)dy \\
&+\overline{\mathcal{F}}_{x\to\xi}\sum_{k',l'=1}^{m}\int_0^t d\tau e^{-i(t-\tau)\xi b^{(j)}-(t-\tau)\mu_j|\xi|^{\delta}}P^{(j)}(0) \\
&\times\int_{\mathbf{R}^n}\frac{\tau A_0^{k',l'}(\xi,y)\,S_{j,k,l}(\xi,y)}{(1+\tau S_{j,k,l}(\xi,y))^2}e^{it(\xi-y)b^{(k)}+ityb^{(l)}} \\
&\times \widehat{f_{k'}^{(k)}}\left(\tau^{\frac{1}{\delta}}(\xi-y)\right)\widehat{f_{l'}^{(l)}}\left(\tau^{\frac{1}{\delta}}y\right)dy \\
&+\overline{\mathcal{F}}_{x\to\xi}\sum_{k',l'=1}^{m}\int_0^t d\tau e^{-i(t-\tau)\xi b^{(j)}-(t-\tau)\mu_j|\xi|^{\delta}}P^{(j)}(0) \\
&\times\int_{\mathbf{R}^n}\frac{\tau A_0^{k',l'}(\xi,y)}{1+\tau S_{j,k,l}(\xi,y)}e^{it(\xi-y)b^{(k)}+ityb^{(l)}} \\
&\times\partial_\tau\left(\widehat{f_{k'}^{(k)}}\left(\tau^{\frac{1}{\delta}}(\xi-y)\right)\widehat{f_{l'}^{(l)}}\left(\tau^{\frac{1}{\delta}}y\right)\right)dy.
\end{aligned}
$$

We introduce the operator $\mathcal{D}_k=\partial_\tau+b^{(k)}\nabla_x$, then by (4.164) we get

$$
\begin{aligned}
\partial_\tau\widehat{f^{(k)}}\left(\tau^{\frac{1}{\delta}}\xi\right) &= \partial_\tau\left(e^{i\tau\xi b^{(k)}}\mathcal{F}_{x\to\xi}h^{(k)}\right) \\
&= e^{i\tau\xi b^{(k)}}\left(i\xi b^{(k)}+\partial_\tau\right)\mathcal{F}_{x\to\xi}h^{(k)} = e^{i\tau\xi b^{(k)}}\mathcal{F}_{x\to\xi}\mathcal{D}_k h^{(k)}.
\end{aligned}
$$

Thus denoting the nonlinearities $\mathcal{N}_{j,k,l}$

$$
\begin{aligned}
\mathcal{N}_{j,k,l}(\varphi,\phi) &= \overline{\mathcal{F}}_{\xi\to x}\sum_{k',l'=1}^{m}\int_{\mathbf{R}^n}B_{j,k,l}^{k',l'}(t,\xi,y)\,\widehat{\varphi}_{k'}(t,\xi-y)\,\widehat{\phi}_{l'}(t,y)\,dy, \\
\widetilde{\mathcal{N}}_{j,k,l}(\varphi,\phi) &= \overline{\mathcal{F}}_{\xi\to x}\sum_{k',l'=1}^{m}\int_{\mathbf{R}^n}\widetilde{B}_{j,k,l}^{k',l'}(t,\xi,y)\,\widehat{\varphi}_{k'}(t,\xi-y)\,\widehat{\phi}_{l'}(t,y)\,dy
\end{aligned}
$$

with symbols

$$
B_{j,k,l}^{k',l'}(t,\xi,y)=\frac{tA_0^{k',l'}(\xi,y)}{1+tS_{j,k,l}(\xi,y)},\ \widetilde{B}_{j,k,l}^{k',l'}(t,\xi,y)=\frac{tA_0^{k',l'}(\xi,y)\,S_{j,k,l}(\xi,y)}{(1+tS_{j,k,l}(\xi,y))^2},
$$

we get

$$\int_0^t \mathcal{G}^{(j)}(t-\tau)\mathcal{N}_0\left(h^{(k)},h^{(l)}\right)(\tau)\,d\tau = \mathcal{G}^{(j)}(0)\widetilde{\mathcal{N}}_{j,k,l}\left(h^{(k)},h^{(l)}\right)$$
$$+\int_0^t \mathcal{G}^{(j)}(t-\tau)\widetilde{\mathcal{N}}_{j,k,l}\left(h^{(k)},h^{(l)}\right)(\tau)\,d\tau$$
$$+\int_0^t \mathcal{G}^{(j)}(t-\tau)\mathcal{N}_{j,k,l}\left(\mathcal{D}_k h^{(k)},h^{(l)}\right)(\tau)\,d\tau$$
$$+\int_0^t \mathcal{G}^{(j)}(t-\tau)\mathcal{N}_{j,k,l}\left(h^{(k)},\mathcal{D}_l h^{(l)}\right)(\tau)\,d\tau.$$

Applying the second estimate of Lemma 4.40 with $\lambda_1 = 1 - \frac{\omega-\beta-\gamma}{\delta}$, $\beta+\gamma < \min(1,\omega)$, $\kappa_1 \in (0,1)$ we obtain

$$\left\|\int_0^t \mathcal{G}^{(j)}(t-\tau)\mathcal{N}_0\left(h^{(k)},h^{(l)}\right)(\tau)\,d\tau\right\|_{\mathbf{A}^{\rho,p}} \le \left\|\mathcal{N}_{j,k,l}\left(h^{(k)},h^{(l)}\right)\right\|_{\mathbf{A}^{\rho,p}}$$
$$+\langle t\rangle^{1-\lambda_1-\frac{\rho+\omega-\beta}{\delta}-\frac{n}{\delta p}} \sup_{p\le q\le\infty}\sup_{\tau>0}\{\tau\}^{\kappa_1}\langle\tau\rangle^{\lambda_1+\frac{n}{\delta q}}$$
$$\times\left(\left\|\widetilde{\mathcal{N}}_{j,k,l}\left(h^{(k)},h^{(l)}\right)(\tau)\right\|_{\mathbf{A}^{\beta-\omega,q}} + \left\|\mathcal{N}_{j,k,l}\left(\mathcal{D}_k h^{(k)},h^{(l)}\right)(\tau)\right\|_{\mathbf{A}^{\beta-\omega,q}}\right.$$
$$\left.+\left\|\mathcal{N}_{j,k,l}\left(h^{(k)},\ \mathcal{D}_l h^{(l)}\right)(\tau)\right\|_{\mathbf{A}^{\beta-\omega,q}}\right)$$

for all $t>0$, $\rho\in[0,\alpha]$, $1\le p\le\infty$. Note that by (4.164) we have

$$\tau\mathcal{F}_{x\to\xi}\mathcal{D}_k h^{(k)}(\tau) = e^{-i\tau\xi b^{(k)}}\tau\partial_\tau\widehat{f^{(k)}}\left(\tau^{\frac{1}{\delta}}\xi\right) = \delta e^{-i\tau\xi b^{(k)}}(\xi\nabla_\xi)\widehat{f^{(k)}}\left(\xi\tau^{\frac{1}{\delta}}\right);$$

hence, by virtue of the result of Section 4.5.3 the estimates of the norms

$$\langle\tau\rangle^{\frac{\rho}{\delta}+\frac{n}{\delta p}}\left(\left\|h^{(k)}(\tau)\right\|_{\mathbf{A}^{\rho,p}} + \tau\left\|\mathcal{D}_k h^{(k)}(\tau)\right\|_{\mathbf{A}^{\rho,p}}\right)$$
$$+\{\tau\}^{\frac{s}{\delta}+\frac{n}{\delta p}}\langle\tau\rangle^{\frac{n+\alpha+\gamma}{\delta}+\frac{n}{\delta p}}\left(\left\|h^{(k)}(\tau)\right\|_{\mathbf{B}^{s,p}} + \tau\left\|\mathcal{D}_k h^{(k)}(\tau)\right\|_{\mathbf{B}^{s,p}}\right)$$
$$\le C\left\|\widehat{f^{(k)}}\right\|_{\mathbf{Z}} \le C\varepsilon$$

are valid for all $\tau>0$, $\rho\in[0,2\delta]$, $1\le p\le\infty$. Since the symbols $B_{j,k,l}^{k',l'}(t,\xi,y)$, $\widetilde{B}_{j,k,l}^{k',l'}(t,\xi,y)$ satisfy conditions (4.166), by Lemma 4.44 we get

$$\left\|\mathcal{N}_{j,k,l}\left(h^{(k)},h^{(l)}\right)(t)\right\|_{\mathbf{A}^{\rho,p}}$$
$$\le Ct^{(1-\rho-\omega)_+}\left(\|h\|_{\mathbf{A}^{\alpha+(\rho+\omega-1)_+,1}} + \|h\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|h\|_{\mathbf{A}^{0,p}} + \|h\|_{\mathbf{B}^{0,\infty}}\right)$$
$$+Ct^{\frac{1}{q}}\left(\|h\|_{\mathbf{A}^{\rho+\omega+\alpha,q}} + \|h\|_{\mathbf{B}^{\sigma,q}}\right)\left(\|h\|_{\mathbf{A}^{0,p}} + \|h\|_{\mathbf{B}^{0,\infty}}\right)$$
$$\le C\varepsilon^2 t^{(1-\rho-\omega)_+-\frac{\alpha+(\rho+\omega-1)_+}{\delta}-\frac{n}{\delta}-\frac{n}{\delta p}} + C\varepsilon^2 t^{\frac{1}{q}-\frac{\rho+\omega+\alpha}{\delta}-\frac{n}{\delta q}-\frac{n}{\delta p}}$$
$$\le C\varepsilon^2 t^{-\frac{\rho+\gamma}{\delta}-\frac{n}{\delta p}}$$

for $t \geq 1$, since

$$(1-\rho-\omega)_{+} - \frac{\alpha+(\rho+\omega-1)_{+}}{\delta} - \frac{n}{\delta} < -\frac{\rho+\gamma}{\delta}$$

and $\frac{1}{q} - \frac{\rho+\omega+\alpha}{\delta} - \frac{n}{\delta q} < -\frac{\rho+\gamma}{\delta}$ for all $\rho \geq 0$, if we choose

$$0 < \gamma \leq (\delta-1)\min(1,\omega), \ \gamma \leq (\omega+\alpha)\left(1-\frac{1}{q}\right).$$

Thus we assume that $0 < \gamma < \frac{1}{\delta}(\min(1,\omega))^2$. Also by Lemma 4.44 with

$$\lambda_1 = 1 - \frac{\omega-\beta-\gamma}{\delta}, \beta+\gamma < \min(1,\omega),$$
$$\kappa_1 = \frac{n+\alpha}{\delta} + \beta, \beta \in \left(0, \frac{\omega}{\delta}\right),$$

we have

$$\begin{aligned}
&\{\tau\}^{\kappa_1} \langle\tau\rangle^{\lambda_1+\frac{n}{\delta p}} \left\| \widetilde{\mathcal{N}}_{j,k,l}\left(h^{(k)}, h^{(l)}\right)(\tau) \right\|_{\mathbf{A}^{\beta-\omega,p}} \\
&\leq C\langle\tau\rangle^{-\beta} \{\tau\}^{\kappa_1} \langle\tau\rangle^{\lambda_1+\frac{n}{\delta p}} \left(\left\| h^{(k)} \right\|_{\mathbf{A}^{\alpha,1}} + \left\| h^{(k)} \right\|_{\mathbf{B}^{\alpha,1}} \right) \\
&\times \left(\left\| h^{(l)} \right\|_{\mathbf{A}^{0,p}} + \left\| h^{(l)} \right\|_{\mathbf{B}^{0,\infty}} \right) \\
&+ C\langle\tau\rangle^{\frac{1}{q}-1} \{\tau\}^{\kappa_1} \langle\tau\rangle^{\lambda_1+\frac{n}{\delta p}} \left(\left\| h^{(k)} \right\|_{\mathbf{A}^{\beta+\alpha,q}} + \left\| h^{(k)} \right\|_{\mathbf{B}^{\alpha,q}} \right) \\
&\times \left(\left\| h^{(l)} \right\|_{\mathbf{A}^{0,p}} + \left\| h^{(l)} \right\|_{\mathbf{B}^{0,\infty}} \right) \\
&\leq C\varepsilon^2 \langle\tau\rangle^{-\beta+\lambda_1-\frac{n+\alpha}{\delta}} + C\varepsilon^2 \langle\tau\rangle^{\frac{1}{q}-1+\lambda_1-\frac{\beta+\alpha}{\delta}-\frac{n}{\delta q}} \leq C\varepsilon^2,
\end{aligned}$$

since $1-\beta-\frac{\omega-\beta-\gamma}{\delta}-\frac{\alpha+n}{\delta} \leq 0$ and $\frac{1}{q}-\frac{\omega-\beta-\gamma}{\delta}-\frac{\beta+\alpha}{\delta}-\frac{n}{\delta q} \leq 0$, if we choose $0 < \gamma \leq \beta(\delta-1)$ and $\gamma \leq (\omega+\alpha)\left(1-\frac{1}{q}\right)$. In addition by Lemma 4.44 we get with $\kappa_1 = \frac{n+\alpha}{\delta}+\beta < 1$, $\beta \in \left(0, \frac{\omega}{\delta}\right)$

$$\begin{aligned}
&\{\tau\}^{\kappa_1} \langle\tau\rangle^{\lambda_1+\frac{n}{\delta p}} \left\| \mathcal{N}_{j,k,l}\left(\mathcal{D}_k h^{(k)}, h^{(l)}\right)(\tau) + \mathcal{N}_{j,k,l}\left(h^{(k)}, \mathcal{D}_l h^{(l)}\right)(\tau) \right\|_{\mathbf{A}^{\beta-\omega,p}} \\
&\leq C\tau\langle\tau\rangle^{-\beta} \{\tau\}^{\kappa_1} \langle\tau\rangle^{1-\frac{\omega-\beta-\gamma}{\delta}} \left(\left\| \mathcal{D}_k h^{(k)} \right\|_{\mathbf{A}^{\alpha,1}} + \left\| \mathcal{D}_k h^{(k)} \right\|_{\mathbf{B}^{\sigma,1}} \right) \\
&\times \langle\tau\rangle^{\frac{n}{\delta p}} \left(\left\| h^{(l)} \right\|_{\mathbf{A}^{0,p}} + \left\| h^{(l)} \right\|_{\mathbf{B}^{0,\infty}} \right) \\
&+ C\tau\langle\tau\rangle^{\frac{1}{q}} \{\tau\}^{\kappa_1} \langle\tau\rangle^{-\frac{\omega-\beta-\gamma}{\delta}} \left(\left\| \mathcal{D}_k h^{(k)} \right\|_{\mathbf{A}^{\alpha+\beta,q}} + \left\| \mathcal{D}_k h^{(k)} \right\|_{\mathbf{B}^{\alpha,q}} \right) \\
&\times \langle\tau\rangle^{\frac{n}{\delta p}} \left(\left\| h^{(l)} \right\|_{\mathbf{A}^{0,p}} + \left\| h^{(l)} \right\|_{\mathbf{B}^{0,\infty}} \right) \\
&\leq C\varepsilon^2 \langle\tau\rangle^{-\beta+\lambda_1-\frac{n+\alpha}{\delta}} + C\varepsilon^2 \langle\tau\rangle^{\frac{1}{q}-1+\lambda_1-\frac{\beta+\alpha}{\delta}-\frac{n}{\delta q}} \leq C\varepsilon^2,
\end{aligned}$$

for all $\tau > 0$, since

$$1 - \beta - \frac{\omega - \beta - \gamma}{\delta} - \frac{\alpha + n}{\delta} \le 0 \text{ and}$$
$$\frac{1}{q} - \frac{\omega - \beta - \gamma}{\delta} - \frac{\beta + \alpha}{\delta} - \frac{n}{\delta q} \le 0,$$

if we choose $0 < \gamma \le \beta(\delta - 1)$ and $\gamma \le (\omega + \alpha)\left(1 - \frac{1}{q}\right)$. All these conditions for γ are fulfilled if we take $0 < \gamma < \frac{1}{\delta}(\min(1, \omega))^2$ and $\beta < \frac{\omega}{\delta}$. Thus we get

$$\left\| \int_0^t \mathcal{G}^{(j)}(t - \tau) \mathcal{N}_0\left(h^{(k)}, h^{(l)}\right)(\tau)\, d\tau \right\|_{\mathbf{A}^{\rho,p}} \le C\varepsilon^2 \langle t \rangle^{-\frac{\rho+\gamma}{\delta} - \frac{n}{\delta p}}.$$

As above by virtue of Lemma 4.40 with $q = \infty$, $\kappa_2 = \widetilde{\lambda} = 1 - \frac{\omega - \gamma}{\delta}$ we obtain

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}^{(j)}(t - \tau) \left(\mathcal{N}_0\left(h^{(j)}, h^{(j)}\right)(\tau) - \mathcal{N}_0(h, h)(\tau)\right) d\tau \right\|_{\mathbf{B}^{s,p}} \\
&\le C \{t\}^{1 - \kappa_2 - \frac{s+\omega}{\delta} - \frac{n}{\delta p}} \langle t \rangle^{-\widetilde{\lambda} - \frac{n}{\delta p}} \sup_{p \le q \le \infty} \sup_{\tau > 0} \{\tau\}^{\kappa_2 + \frac{n}{\delta q}} \langle \tau \rangle^{\widetilde{\lambda} + \frac{n}{\delta q}} \\
&\times \left\| \mathcal{N}_0\left(h^{(j)}, h^{(j)}\right)(\tau) - \mathcal{N}_0(h, h)(\tau) \right\|_{\mathbf{B}^{-\omega,q}}
\end{aligned}$$

for all $t > 0$, $s \in [0, \alpha]$, $1 \le p \le \infty$.

In view of Lemma 4.41

$$\begin{aligned}
&\{\tau\}^{\kappa_2 + \frac{n}{\delta q}} \langle \tau \rangle^{\widetilde{\lambda} + \frac{n}{\delta q}} \left\| \mathcal{N}_0\left(h^{(j)}, h^{(j)}\right)(\tau) - \mathcal{N}_0(h, h)(\tau) \right\|_{\mathbf{B}^{-\omega,q}} \\
&\le C\tau^{\frac{n+\alpha+\gamma}{\delta} + \frac{n}{\delta q}} \left(\left\| h^{(j)} \right\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|h\|_{\mathbf{A}^{\alpha+\gamma,1}} + \left\| h^{(j)} \right\|_{\mathbf{B}^{\alpha,1}} + \|h\|_{\mathbf{B}^{\alpha,1}} \right) \\
&\times \left(\left\| h^{(j)} \right\|_{\mathbf{A}^{0,q}} + \|h\|_{\mathbf{A}^{0,q}} + \left\| h^{(j)} \right\|_{\mathbf{B}^{0,q}} + \|h\|_{\mathbf{B}^{0,q}} \right) \\
&< C \left\| h^{(j)} \right\|_{\mathbf{X}_0}^2 + C \|h\|_{\mathbf{X}_0}^2 \le C\varepsilon^2
\end{aligned}$$

for all $\tau > 0$, $1 \le q \le \infty$. Thus by virtue of the inequalities

$$1 - \kappa_2 - \frac{s + \omega}{\delta} - \frac{n}{\delta p} \le 0$$

and $-\widetilde{\lambda} - \frac{n}{\delta p} \le 0$ we get

$$\left\| \int_0^t \mathcal{G}^{(j)}(t - \tau) \left(\mathcal{N}_0\left(h^{(j)}, h^{(j)}\right)(\tau) - \mathcal{N}_0(h, h)(\tau)\right) d\tau \right\|_{\mathbf{X}_\gamma} \le C\varepsilon^2. \quad (4.180)$$

Collecting now estimates (4.178), (4.179) and (4.180) we obtain from (4.177)

$$\|h - w\|_{\mathbf{X}_\gamma} \le C\varepsilon. \quad (4.181)$$

Via (4.168), (4.169) and (4.181) we have the estimate

$$\|u(t) - h(t)\|_{\mathbf{L}^\infty} \le \|u(t) - h(t)\|_{\mathbf{A}^{0,1}} + \|u(t) - h(t)\|_{\mathbf{B}^{0,1}} \le C\varepsilon t^{-\frac{n+\gamma}{\delta}}$$

for all $t > 0$. Theorem 4.38 is then proved.

4.6 Comments

Section 4.1.

We relate some known results on the large time asymptotic behavior of solutions to nonlinear convection-diffusion type equations in the critical case. By the Hopf-Cole Hopf [1950] transformation the large time asymptotics of solutions to the Burgers equation (4.16) was computed

$$u(t,x) = \frac{1}{\sqrt{t}} f_\theta \left(\frac{x}{\sqrt{t}} \right) + O\left(t^{-1}\right).$$

Here $\theta = \int_{\mathbf{R}} u_0(x)\,dx$ is the total mass of the initial data $u_0(x)$,

$$f_\theta(x) = -2\partial_x \log \left(\cosh \frac{\theta}{4} - \sinh\left(\frac{\theta}{4}\right) \operatorname{Erf}\left(\frac{x}{2}\right) \right)$$

is the self-similar solution, and $\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$ is the error function.

In paper Dix [1991] it was proved that for small initial data $u_0(x)$ such that $u_0 \in \mathbf{L}^{1,a}(\mathbf{R})$ with $a \in (0,1)$, the solutions of the Cauchy problem for the Benjamin-Ono-Burgers equation (4.19) are close for large time values to the corresponding self-similar solution $t^{-\frac{1}{2}} \widetilde{w}_\theta \left((\cdot)\, t^{-\frac{1}{2}} \right)$ of (4.19) defined by the total mass of the initial data $\int_{\mathbf{R}} \widetilde{w}_\theta(x)\,dx = \theta = \int_{\mathbf{R}} u_0(x)\,dx$, more precisely

$$\left\| u(t) - t^{-\frac{1}{2}} \widetilde{w}_\theta \left((\cdot)\, t^{-\frac{1}{2}} \right) \right\|_{\mathbf{L}^\infty} = O\left(t^{-\frac{1+a}{2}} \right)$$

as $t \to \infty$.

The convection-diffusion equation

$$u_t + (\lambda \cdot \nabla)\left(|u|^\sigma u\right) - \Delta u = 0, \tag{4.182}$$

where λ is a constant vector in $\mathbf{R}^n$, was considered in papers Escobedo et al. [1994], Escobedo et al. [1993a], Escobedo and Zuazua [1991], Escobedo and Zuazua [1997], Liu and Pierre [1984], Zuazua [1995], Zuazua [1994] in the critical case of $\sigma = \frac{1}{n}$, when the initial data $u_0 \in \mathbf{L}^1(\mathbf{R}^n)$. It was proved that the large time asymptotic behavior of solutions with not small initial data from $\mathbf{L}^1(\mathbf{R}^n)$ is given by a self-similar solution $t^{-\frac{n}{2}} w_\theta \left((\cdot)\, t^{-\frac{1}{2}} \right)$ defined uniquely by the total mass of the initial data:

$$\left\| u(t) - t^{-\frac{n}{2}} w_\theta \left((\cdot)\, t^{-\frac{1}{2}} \right) \right\|_{\mathbf{L}^\infty} = o\left(t^{-\frac{n}{2}} \right).$$

Similar results were obtained in Biler et al. [2000] for the supercritical case of $\sigma > \frac{\varrho - 1}{n}$ and in Biler et al. [1998] for the critical case of $\sigma = \frac{\varrho-1}{n}$ for solutions to (4.182) with $(-\Delta)^{\frac{\varrho}{2}}$, $1 < \varrho < 2$, instead of the Laplacian $-\Delta$.

Section 4.2.

Large time behavior of solutions to the convection-diffusion equation with fractional Laplacian

$$u_t + (\lambda \cdot \nabla)\left(|u|^\sigma u\right) + (-\Delta)^{\frac{\alpha}{2}} u = 0,\ x \in \mathbf{R}^n,\ t > 0, \tag{4.183}$$

where $1 < \alpha < 2$ and λ is a constant vector in $\mathbf{R}^n$, was considered in the critical case of $\sigma = \frac{\alpha - 1}{n}$ in papers Biler et al. [2001b], Biler et al. [2001a]. It was proved that for any not small initial data $u_0 \in \mathbf{L}^1(\mathbf{R}^n)$ with nonzero total mass $\int_{\mathbf{R}^n} u_0(x)\,dx = \theta$ the large time asymptotics of solutions is given by the self-similar solution $t^{-\frac{n}{\alpha}} \widetilde{v_\theta}\left((\cdot)\,t^{-\frac{1}{\alpha}}\right)$ for equation (4.183)

$$\left\| u(t) - t^{-\frac{n}{\alpha}} \widetilde{v_\theta}\left((\cdot)\,t^{-\frac{1}{\alpha}}\right)\right\|_{\mathbf{L}^\infty} = o\left(t^{-\frac{n}{\alpha}}\right)$$

as $t \to \infty$. For some other results for critical convective nonlinear dissipative equations we refer to papers Amick et al. [1989], Biler [1984], Biler et al. [2000], Bona et al. [1999], Carpio [1996], Dix [1991], Dix [1997], Galaktionov et al. [1985], Galaktionov and Vázquez [1991], Gmira and Véron [1984], Hayashi et al. [2000], Hayashi and Naumkin [2003], Il′in and Oleĭnik [1960], Karch [1999b], Kavian [1987], Naumkin and Shishmarev [1994a], Zuazua [1994].

Section 4.3.

For the global existence and smoothing effects of solutions to the Cauchy problem for the Korteweg-de Vries-Burgers equation we refer to Naumkin and Shishmarev [1994b], Saut [1979]. Thus we know that there exists a unique solution $u(t,x) \in \mathbf{C}^\infty((0,\infty);\mathbf{H}^\infty(\mathbf{R}))$ to the Cauchy problem for the Korteweg-de Vries-Burgers equation (4.36) with initial data $u_0 \in \mathbf{H}^s(\mathbf{R})$, $s > -\frac{1}{2}$. In this section we are interested in the large time asymptotics for the case of the initial data having an arbitrary size.

We now refer to some known results about the large time asymptotic behavior of solutions to (4.36). In paper Naumkin [1993] it was proved that for small initial data $u_0 \in \mathbf{L}^1(\mathbf{R}) \cap \mathbf{H}^7(\mathbf{R})$ such that $xu_0(x) \in \mathbf{L}^1(\mathbf{R})$, the solutions of Korteweg-de Vries-Burgers equation (4.36) have the asymptotics

$$u(t) = t^{-\frac{1}{2}} f_\theta\left((\cdot)\,t^{-\frac{1}{2}}\right) + O\left(t^{-\frac{1}{2}-\gamma}\right)$$

as $t \to \infty$, where $\gamma \in \left(0, \frac{1}{2}\right)$ and f_θ is the self-similar solution for the Burgers equation, defined by the total mass θ of the initial data. The conditions on the initial data were generalized in paper Karch [1999b], where it was proved that the solution of (4.36) with small initial data $u_0 \in \mathbf{L}^1(\mathbf{R}) \cap \mathbf{L}^2(\mathbf{R})$ have asymptotics

$$u(t) = t^{-\frac{1}{2}} f_\theta\left((\cdot)\,t^{-\frac{1}{2}}\right) + o\left(t^{-\frac{1}{2}}\right) \tag{4.184}$$

as $t \to \infty$. As regards to the question of how to remove the restrictions on the size of the initial data we note that in paper Karch [1999b] a conditional result was proved: if the initial data $u_0 \in \mathbf{L}^1(\mathbf{R}) \cap \mathbf{L}^2(\mathbf{R})$ and the optimal time decay estimate for $\mathbf{L}^2(\mathbf{R})$ norm of the solution is fulfilled

$$\|u(t)\|_{\mathbf{L}^2} = O\left(t^{-\frac{1}{4}}\right),$$

then the asymptotics (4.184) is also true for the case of large initial data. The smallness condition on the initial data was removed in paper Hayashi and Naumkin [2006b], where Theorem 4.18 was proved.

In paper Naumkin and Shishmarev [1994a], it was proved that for small initial data $u_0 \in \mathbf{L}^{1,1}(\mathbf{R}) \cap \mathbf{H}^5(\mathbf{R})$ such that $u_0'' \in \mathbf{L}^1(\mathbf{R})$ the solutions of (4.36) have the following two terms of the large time asymptotics

$$u(t) = t^{-\frac{1}{2}} f_\theta\left((\cdot)\,t^{-\frac{1}{2}}\right) + \frac{\log t}{t} \widetilde{f_\theta}\left((\cdot)\,t^{-\frac{1}{2}}\right) + O\left(\frac{\sqrt{\log t}}{t}\right)$$

as $t \to \infty$ uniformly with respect to $x \in R$, where

$$\widetilde{f_\theta}(x) = -\frac{\left(f_\theta(x) - \frac{x}{2}\right) e^{-x^2/4}}{2\sqrt{\pi} H(x)} \int_{\mathbf{R}} H(y) f_\theta^3(y)\, dy$$

with

$$H(x) = \cosh\frac{\theta}{4} - \sinh\frac{\theta}{4} \operatorname{Erf}\left(\frac{x}{2}\right).$$

In paper Kaikina and Ruiz-Paredes [2005] this result was extended for the case when the initial data have an arbitrary size.

Section 4.4.

Optimal time decay estimate

$$\|u(t)\|_{\mathbf{L}^2} = O\left(t^{-\frac{1}{4}}\right)$$

for solutions to the Cauchy problem for the Benjamin-Bona-Mahony- Burgers equation (4.18) was obtained in paper Amick et al. [1989] in the case of not small initial data $u_0 \in \mathbf{L}^1(\mathbf{R}) \cap \mathbf{H}^{2,0}(\mathbf{R})$. In paper Naumkin [1993] it was proved that for small initial data $u_0 \in \mathbf{L}^{1,1}(\mathbf{R}) \cap \mathbf{H}^{7,0}(\mathbf{R})$, the solutions of (4.18) have asymptotics (4.101). In paper Prado and Zuazua [2002] it was obtained the asymptotic expansion of solutions to the generalized n-dimensional Benjamin-Bona-Mahony-Burgers equations and the Korteweg-de Vries-Burgers equations.

Some other results for dissipative equations with critical nonlinearities were obtained in papers Amick et al. [1989], Bona et al. [1999], Carpio [1996], Escobedo et al. [1993a], Il′in and Oleĭnik [1960], Mei [1999], Mei and Schmeiser [2001], Zuazua [1993].

Section 4.5.

Large time asymptotics for the Boussinesq system of equations was obtained first in paper Naumkin and Shishmarev [1995]. Theorems 4.36 and 4.38 were proved in paper Kaĭkina et al. [2004b]. Similar results are valid for the Boussinesq equation with damping

$$u_{tt} - \vartheta \Delta u + \varrho \Delta^2 u - \Delta u + \Delta\left(u^2\right) = 0,$$

where $x \in \mathbf{R}^2, t > 0$, in the case $\varrho > \frac{\vartheta^2}{4}$ (see Varlamov [1996]). The material of this section was taken from paper Kaĭkina et al. [2004b].

5

Subcritical Nonconvective Equations

In this chapter we study the large time asymptotic behavior of solutions to the Cauchy problem for various nonlinear dissipative equations in the subcritical case, when the nonlinear term essentially determines the asymptotic properties of solutions. This case also can be called the case of strong nonlinearity. The character of the large time asymptotic behavior for nonconvective type equations in the subcritical case is determined by the special self-similar solutions. By taking into account some additional properties such as the maximum principle and positivity of solutions, or by applying the weighted energy type estimates, we will remove the requirement of smallness of the initial data.

5.1 General approach

In this section we give a general approach for obtaining global existence and large time asymptotic representation of solutions to the Cauchy problem 1.7)

$$\begin{cases} u_t + \mathcal{N}(u) + \mathcal{L}u = 0, \ x \in \mathbf{R}^n, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^n, \end{cases} \tag{5.1}$$

in the case of nonconvective subcritical nonlinearity $\mathcal{N}(u)$.

We fix a metric space $\mathbf{Z}$ of functions defined on $\mathbf{R}^n$ and a complete metric space $\mathbf{X}$ of functions defined on $[0,\infty) \times \mathbf{R}^n$. We denote as above by $G_0 \in \mathbf{X}$ the asymptotic kernel for the Green operator $\mathcal{G}$ in spaces $\mathbf{X}$, $\mathbf{Z}$ and by f the corresponding functional (see Definition 2.1.)

Definition 5.1. *We call the nonlinearity $\mathcal{N}$ in equation (5.1) subcritical nonconvective if the linear continuous functional f is such that the estimate is true*

$$\frac{\sigma}{\theta} \operatorname{Re} f(\mathcal{N}(\theta G_0(t))) \geq \frac{\eta}{2} \theta^{\sigma} t^{\mu-1} \tag{5.2}$$

for all $t > 0$, $\theta > 0$ with some positive constants η, σ and $\mu \in (0,1)$.

Now we prove the global existence of small solutions to the Cauchy problem (5.1) with a subcritical nonlinearity of nonconvective type.

Theorem 5.2. *Assume that the linear operator* $\mathcal{L}$ *is such that* $f(\mathcal{L}(u)) = 0$ *for any* $u \in \mathbf{X}$. *Let the nonlinearity* $\mathcal{N}(u)$ *in equation (5.1) be subcritical nonconvective. Assume that*

$$e^{z}\mathcal{N}(ue^{-z}) = e^{-\sigma \operatorname{Re} z}\mathcal{N}(u) \tag{5.3}$$

for any $z \in \mathbf{C}$ *and* $u \in \mathbf{X}$, *where* $\sigma > 0$. *Suppose that the estimates are valid*

$$\begin{aligned}&\nu(t)\left|f\left(\mathcal{N}(v(t)) - \mathcal{N}(w(t))\right)\right| \\ &\leq C\{t\}^{-\alpha}\langle t\rangle^{\mu-1}\|\nu(t)(v(t)-w(t))\|_{\mathbf{X}}\left(\|v\|_{\mathbf{X}}^{\sigma} + \|w\|_{\mathbf{X}}^{\sigma}\right)\end{aligned} \tag{5.4}$$

for all $t > 0$ *and for any* $v, w \in \mathbf{X}$, *where* $\nu(t) > 0$, $\alpha < 1$, $\sigma > 0$, $\mu \in (0,1)$, *and*

$$\begin{aligned}&\left\|\int_0^t \left|\mathcal{G}(t-\tau)\left(\mathcal{K}(v(\tau)) - \mathcal{K}(w(\tau))\right)\right|\tau^{-\mu}d\tau\right\|_{\mathbf{X}} \\ &\leq C\|v-w\|_{\mathbf{X}}\left(\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}\right)^{\sigma}\left(1 + \frac{\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}}{\theta}\right)\end{aligned} \tag{5.5}$$

for any $v, w \in \mathbf{X}$ *such that* $f(v) = \theta = f(w)$, or $w \equiv 0$, *where* $\sigma > 0$ *and* $\mathcal{K}(v) = \mathcal{N}(v) - \frac{v}{\theta}f(\mathcal{N}(v))$. *Let the initial data* $u_0 \in \mathbf{Z}$ *have a small norm* $\|u_0\|_{\mathbf{Z}} \leq \varepsilon$ *and the mean value* $\theta \equiv |f(u_0)| \geq C\varepsilon > 0$ *with some* $C > 0$. *Suppose also that* $\mu \in (0,1)$ *in (5.2) and (5.4) is sufficiently small. Then there exists a unique global solution* $u \in \mathbf{X}$ *to the Cauchy problem (5.1) satisfying the time decay estimate*

$$\left\|t^{\frac{\mu}{\sigma}}u(t)\right\|_{\mathbf{X}} \leq C\varepsilon.$$

Proof. As in the proof of Theorem 3.2 we change the dependent variable $u(t,x) = v(t,x)\,e^{-\varphi(t)+i\psi(t)}$ in equation (5.1), and arrive to the Cauchy problem (3.7). Then we obtain the integral equation (3.8)

$$v(t) = \mathcal{G}(t)\,v_0 - \int_0^t \mathcal{G}(t-\tau)\,\mathcal{K}(v(\tau))\,\frac{d\tau}{h_v(\tau)}, \tag{5.6}$$

where the nonlinearity

$$\mathcal{K}(v(\tau)) = \mathcal{N}(v(\tau)) - \frac{v(\tau)}{\theta}f(\mathcal{N}(v(\tau)))$$

and the functional

$$h_v(t) = 1 + \frac{\sigma}{\theta}\int_0^t \operatorname{Re} f(\mathcal{N}(v(\tau)))\,d\tau.$$

Also we note that the function $\psi(t)$ satisfies

$$\psi'(t) = -\frac{1}{\theta h_v(t)} \operatorname{Im} f(\mathcal{N}(v(t))), \ \psi(0) = \arg f(u_0). \tag{5.7}$$

We now prove the existence of the solution $v(t,x)$ for integral equation (5.6) by the contraction mapping principle. We define the transformation $\mathcal{M}(w)$ by the formula

$$\mathcal{M}(w) = \mathcal{G}(t)v_0 - \int_0^t \mathcal{G}(t-\tau)\mathcal{K}(w(\tau))\frac{d\tau}{h_w(\tau)}, \tag{5.8}$$

for any $w \in \mathbf{B}$, where

$$\mathbf{B} = \Big\{ w \in \mathbf{X}: \ f(w) = \theta, \ \|w\|_{\mathbf{X}} \le C\varepsilon, \\ \|w - \mathcal{G}(t)v_0\|_{\mathbf{X}} \le C\mu\varepsilon, \ \sup_{t>0} t^{-\mu}h_w(t) \ge \frac{\eta}{3\mu}\theta^{\sigma} \Big\},$$

where

$$h_w(t) = 1 + \frac{\sigma}{\theta}\int_0^t \operatorname{Re} f(\mathcal{N}(w(\tau)))\,d\tau.$$

First we check that the mapping $\mathcal{M}$ transforms the set $\mathbf{B}$ into itself. Since $f(w) = \theta$ we have

$$f(\mathcal{K}(w)) = f(\mathcal{N}(w)) - \frac{1}{\theta}f(\mathcal{N}(w))f(w) = 0;$$

hence from the integral equation (5.8) we see that

$$f(\mathcal{M}(w)) = f(\mathcal{G}(t)v_0) = f(v_0) = \theta.$$

By the definition of the asymptotic kernel (see (2.3)) we have

$$\|\langle t\rangle^{\gamma}(\mathcal{G}(t)v_0 - \theta G_0(t)\|_{\mathbf{X}} \le C\|v_0\|_{\mathbf{Z}} \le C\varepsilon,$$

and since $w \in \mathbf{B}$ by condition (5.5) of the theorem we get

$$\begin{aligned} &\left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{K}(w(\tau))\frac{d\tau}{h_w(\tau)} \right\|_{\mathbf{X}} \\ &\le \frac{3\mu}{\eta}\theta^{-\sigma}\left\| \int_0^t \tau^{-\mu}|\mathcal{G}(t-\tau)\mathcal{K}(w(\tau))|\,d\tau \right\|_{\mathbf{X}} \\ &\le \frac{C\mu}{\theta^{\sigma}}\|w\|_{\mathbf{X}}^{\sigma+1}\left(1 + \frac{\|w\|_{\mathbf{X}}}{\theta}\right) \le C\mu\varepsilon. \end{aligned}$$

Hence we see that

$$\|\mathcal{M}(w) - \mathcal{G}(t)v_0\|_{\mathbf{X}} \le \left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{K}(w(\tau))\frac{d\tau}{h_w(\tau)} \right\|_{\mathbf{X}} \le C\mu\varepsilon$$

and

$$\|\mathcal{M}(w)\|_{\mathbf{X}} \leq \|\mathcal{G}(t) v_0\|_{\mathbf{X}} + \|\mathcal{M}(w) - \mathcal{G}(t) v_0\|_{\mathbf{X}} \leq C\varepsilon + C\mu\varepsilon \leq C\varepsilon.$$

It remains to prove the estimate

$$h_{\mathcal{M}(w)}(t) = 1 + \frac{\sigma}{\theta}\int_0^t \operatorname{Re} f(\mathcal{N}(\mathcal{M}(w)))\, d\tau \geq \frac{\eta}{3\mu}\theta^\sigma t^\mu$$

for all $t > 0$. We have by condition (5.2)

$$\begin{aligned} &\frac{\sigma}{\theta}\int_0^t \operatorname{Re} f(\mathcal{N}(\mathcal{M}(w)))\, d\tau = \frac{\sigma}{\theta}\int_0^t \operatorname{Re} f(\mathcal{N}(\theta G_0(\tau)))\, d\tau \\ &+ \frac{\sigma}{\theta}\int_0^t \operatorname{Re} f(\mathcal{N}(\mathcal{M}(w)) - \mathcal{N}(\theta G_0(\tau)))\, d\tau \geq \frac{\eta}{2\mu}\theta^\sigma t^\mu + R(t), \end{aligned} \tag{5.9}$$

where

$$R(t) = \frac{\sigma}{\theta}\int_0^t \operatorname{Re} f(\mathcal{N}(\mathcal{M}(w)) - \mathcal{N}(\theta G_0(\tau)))\, d\tau.$$

By estimate (5.4) we have

$$\begin{aligned} |R(t)| &\leq \frac{C}{\theta}\|\mathcal{M}(w) - \mathcal{G}(t) v_0\|_{\mathbf{X}} \left(\|\mathcal{M}(w)\|_{\mathbf{X}}^\sigma + \|\mathcal{G}(t) v_0\|_{\mathbf{X}}^\sigma\right) \\ &\times \int_0^t \{\tau\}^{-\alpha} \langle\tau\rangle^{\mu-1}\, d\tau \\ &+ \frac{C}{\theta}\|\langle t\rangle^\gamma (\mathcal{G}(t) v_0 - \theta G_0(t))\|_{\mathbf{X}} \left(\|\mathcal{G}(t) v_0\|_{\mathbf{X}}^\sigma + \|\theta G_0\|_{\mathbf{X}}^\sigma\right) \\ &\times \int_0^t \{\tau\}^{-\alpha} \langle\tau\rangle^{\mu-\gamma-1}\, d\tau \leq C\varepsilon^\sigma \langle\tau\rangle^\mu. \end{aligned}$$

Therefore by virtue of (5.9) we find that

$$h_{\mathcal{M}(w)}(t) \geq 1 + \frac{\eta}{2\mu}\theta^\sigma t^\mu - C\varepsilon^\sigma \langle\tau\rangle^\mu \geq \frac{\eta}{3\mu}\theta^\sigma t^\mu$$

for all $t > 0$, since $\mu \in (0, 1)$ is small. Thus we see that $\mathcal{M}$ transforms the closed set $\mathbf{B}$ of a complete metric space $\mathbf{X}$ into itself. Now by virtue of (5.5) let us estimate the difference

$$\begin{aligned} &\|\mathcal{M}(v) - \mathcal{M}(w)\|_{\mathbf{X}} \\ &= \left\|\int_0^t \mathcal{G}(t-\tau)\left(\mathcal{K}(v(\tau))\frac{1}{h_v(\tau)} - \mathcal{K}(w(\tau))\frac{1}{h_w(\tau)}\right) d\tau\right\|_{\mathbf{X}} \\ &\leq \frac{C\mu}{\theta^\sigma}\left\|\int_0^t \tau^{-\mu}\left|\mathcal{G}(t-\tau)(\mathcal{K}(v(\tau)) - \mathcal{K}(w(\tau)))\right| d\tau\right\|_{\mathbf{X}} \\ &+ \frac{C\mu^2}{\theta^{2\sigma}}\left\|\int_0^t \tau^{-2\mu}\left|\mathcal{G}(t-\tau)\mathcal{K}(w(\tau))\right| \left|h_v(\tau) - h_w(\tau)\right| d\tau\right\|_{\mathbf{X}} \\ &\leq C\frac{\mu}{\theta^\sigma}\|v - w\|_{\mathbf{X}}\left(\varepsilon^\sigma + \theta^{-1}\left\|\int_0^t \tau^{-\mu}\left|\mathcal{G}(t-\tau)\mathcal{K}(w(\tau))\right| d\tau\right\|_{\mathbf{X}}\right) \\ &\leq C\mu\|v - w\|_{\mathbf{X}} \leq \frac{1}{2}\|v - w\|_{\mathbf{X}}, \end{aligned}$$

where in view of condition (5.4) we used the estimate

$$|h_v(t) - h_w(t)| \le \frac{C}{\theta}\left|\int_0^t f(\mathcal{N}(v(\tau)) - \mathcal{N}(w(\tau)))\, d\tau\right|$$
$$\le \frac{C}{\theta}\|v-w\|_{\mathbf{X}}\int_0^t \{\tau\}^{-\alpha}\langle\tau\rangle^{\mu-1}d\tau \le \frac{C\varepsilon^{\sigma}t^{\mu}}{\mu\theta}\|v-w\|_{\mathbf{X}}$$

for all $v, w \in \mathbf{B}$. Therefore $\mathcal{M}$ is a contraction mapping in the closed set $\mathbf{B}$ of a complete metric space $\mathbf{X}$. Hence there exists a unique global solution $v \in \mathbf{B}$ to integral equation (5.6) such that

$$\|v\|_{\mathbf{X}} \le C\varepsilon,\ \ \|v - \mathcal{G}(t)v_0\|_{\mathbf{X}} \le C\mu\varepsilon,\ \ h_v(t) \ge \frac{\eta}{3\mu}\theta^{\sigma}t^{\mu}.$$

Using the relation $u(t,x) = v(t,x)e^{i\psi(t)}h_v^{-\frac{1}{\sigma}}(t)$ we obtain the existence of the solution to the integral equation (5.6) satisfying the following time decay estimates

$$\left\|t^{\frac{\mu}{\sigma}}u\right\|_{\mathbf{X}} \le C\varepsilon.$$

This completes the proof of Theorem 5.2.

We now obtain the large time asymptotic representation of solutions to the Cauchy problem (5.1) in the case of subcritical nonlinearity $\mathcal{N}(u)$ of nonconvective type.

Definition 5.3. *We call the operator*

$$\mathcal{G}_0(t)\phi = t^{-\frac{n}{\delta}}\int_{\mathbf{R}^n}\widetilde{G}\left((x-y)t^{-\frac{1}{\delta}}\right)\phi(y)\,dy$$

with some $\delta > 0$ *a self-similar asymptotic operator for the Green operator* $\mathcal{G}(t)$ *in spaces* $\mathbf{X}$, $\mathbf{Z}$ *if the estimates are true*

$$\|\mathcal{G}_0(t)\phi\|_{\mathbf{X}} + \|\langle t\rangle^{\gamma}(\mathcal{G}(t) - \mathcal{G}_0(t))\phi\|_{\mathbf{X}} \le C\|\phi\|_{\mathbf{Z}} \tag{5.10}$$

for any $\phi \in \mathbf{Z}$, *where* $\gamma > 0$. *Also we assume that the asymptotic kernel (see Definition 2.1) has a self-similar form* $G_0(t,x) = t^{-\alpha}\widetilde{G_0}\left(xt^{-\frac{1}{\delta}}\right)$ *and*

$$f\left(\phi\left((\cdot)t^{-\frac{1}{\delta}}\right)\right) = t^{\alpha}f(\phi(\cdot)) \tag{5.11}$$

for all $t > 0$, $\phi \in \mathbf{Z}$, *with some* $\alpha > 0$.

We now fix a metric space $\mathbf{Q}$ of functions defined on $\mathbf{R}^n$ and a complete metric space $\widetilde{\mathbf{X}} \subset \mathbf{X}$ of functions defined on $[0,\infty)\times\mathbf{R}^n$, such that the norm of $\mathbf{Q}$ is induced by the norm of $\widetilde{\mathbf{X}}$ by

$$\|\phi\|_{\mathbf{Q}} = \left\|t^{-\alpha}\phi\left((\cdot)t^{-\frac{1}{\delta}}\right)\right\|_{\widetilde{\mathbf{X}}}$$

and

$$\|\phi\|_{\mathbf{X}} \le C \|\phi\|_{\widetilde{\mathbf{X}}}.$$

First we prove the existence of particular solutions of equation (5.1) having a self-similar form.

Theorem 5.4. *Let conditions (5.3), (5.4) and (5.11) be fulfilled. Also we assume that the estimate is true*

$$\left\| \int_0^t \left| \mathcal{G}_0 (t-\tau) \left(\mathcal{K}(v(\tau)) - \mathcal{K}(w(\tau)) \right) \right| \tau^{-\mu} d\tau \right\|_{\widetilde{\mathbf{X}}} \\ \le C \|v-w\|_{\widetilde{\mathbf{X}}} \left(\|v\|_{\widetilde{\mathbf{X}}} + \|w\|_{\widetilde{\mathbf{X}}} \right)^{\sigma} \left(1 + \frac{\|v\|_{\widetilde{\mathbf{X}}} + \|w\|_{\widetilde{\mathbf{X}}}}{\theta} \right) \tag{5.12}$$

for any $v, w \in \widetilde{\mathbf{X}}$ *such that* $f(v) = 1 = f(w)$ or $w \equiv 0$, *where* $\sigma > 0$ *and* $\mathcal{K}(v) = \mathcal{N}(v) - \frac{v}{f(v)} f(\mathcal{N}(v))$. *Then there exist a unique solution* $V \in \mathbf{Q}$ *to the integral equation*

$$V = \widetilde{G_0} - \frac{\mu}{\sigma \operatorname{Re} f(\mathcal{N}(V))} \int_0^1 z^{-\mu} \mathcal{G}_0 (1-z) \mathcal{K}\left(z^{-\alpha} V\left((\cdot) z^{-\frac{1}{\delta}} \right) \right) dz \tag{5.13}$$

with $\mathcal{K}(V) = \mathcal{N}(V) - \frac{V}{f(V)} f(\mathcal{N}(V))$.

Proof. We define the transformation $\mathcal{R}(V)$ by

$$\mathcal{R}(V) = \widetilde{G_0} - \frac{\mu}{\sigma \operatorname{Re} f(\mathcal{N}(V))} \int_0^1 z^{-\mu} \mathcal{G}_0 (1-z) \mathcal{K}\left(z^{-\alpha} V\left((\cdot) z^{-\frac{1}{\delta}} \right) \right) dz,$$

for any $V \in \mathbf{A}$, where

$$\mathbf{A} = \Big\{ V \in \mathbf{Q} : \ f(V) = 1, \ \|V\|_{\mathbf{Q}} \le C, \\ \left\| V - \widetilde{G_0} \right\|_{\mathbf{Q}} \le C\mu, \ \operatorname{Re} f(\mathcal{N}(V)) \ge \frac{\eta}{3\sigma} \Big\}.$$

Since $f(V) = 1$ by the definition of $\mathcal{K}(V)$, we see that $f(\mathcal{K}(V)) = 0$; hence $f(\mathcal{R}(V)) = f\left(\widetilde{G_0}\right) + f(\mathcal{K}(V)) = 1$.

Note that by conditions (5.3) and (5.11) we have

$$f(\mathcal{N}(V)) = f\left(t^{-\alpha} \mathcal{N}\left(V\left((\cdot) t^{-\frac{1}{\delta}} \right) \right) \right) = t^{\alpha\sigma} f\left(\mathcal{N}\left(t^{-\alpha} V\left((\cdot) t^{-\frac{1}{\delta}} \right) \right) \right)$$

and condition (5.2) with $\mu = 1 - \alpha\sigma$ implies

$$\operatorname{Re} f\left(\mathcal{N}\left(\widetilde{G_0} \right) \right) \ge \frac{\eta}{2\sigma} > 0.$$

Applying the property of self-similarity we have, by taking $\xi = xt^{-\frac{1}{\delta}}$ and by changing the variables of integration $\tau = zt$, $y = y' t^{\frac{1}{\delta}}$,

$$\int_0^t \tau^{-\mu}\mathcal{G}_0(t-\tau)(x)\mathcal{K}\left(\tau^{-\alpha}V\left((\cdot)\tau^{-\frac{1}{\delta}}\right)\right)d\tau$$
$$=\int_0^t \tau^{-\mu}(t-\tau)^{-\frac{n}{\delta}}\int_{\mathbf{R}^n}\widetilde{G_0}\left((x-y)(t-\tau)^{-\frac{1}{\delta}}\right)\mathcal{K}\left(\tau^{-\alpha}V\left(y\tau^{-\frac{1}{\delta}}\right)\right)dyd\tau$$
$$=t^{1-\mu-\sigma\alpha-\alpha}\int_0^1 dzz^{-\mu}(1-z)^{-\frac{n}{\delta}}$$
$$\times\int_{\mathbf{R}^n}\widetilde{G_0}\left((\xi-y')(1-z)^{-\frac{1}{\delta}}\right)\mathcal{K}\left(z^{-\alpha}V\left(y'z^{-\frac{1}{\delta}}\right)\right)dy'$$
$$=t^{-\alpha}\int_0^1 z^{-\mu}\mathcal{G}_0(1-z)\left(xt^{-\frac{1}{\delta}}\right)\mathcal{K}\left(z^{-\alpha}V\left((\cdot)z^{-\frac{1}{\delta}}\right)\right)dz,$$

since $\mu=1-\alpha\sigma$. Hence using the relation of the norms

$$\left\|t^{-\alpha}\phi\left(xt^{-\frac{1}{\delta}}\right)\right\|_{\widetilde{\mathbf{X}}}=\|\phi(\xi)\|_{\mathbf{Q}},$$

by estimate (5.12) we get

$$\left\|\mathcal{R}(V)-\widetilde{G_0}\right\|_{\mathbf{Q}}$$
$$=\frac{\mu}{\sigma|f(\mathcal{N}(V))|}\left\|\int_0^1 z^{-\mu}\mathcal{G}_0(1-z)\mathcal{K}\left(z^{-\alpha}V\left((\cdot)z^{-\frac{1}{\delta}}\right)\right)dz\right\|_{\mathbf{Q}}$$
$$=C\mu\left\|\int_0^t \tau^{-\mu}\mathcal{G}_0(t-\tau)\mathcal{K}\left(\tau^{-\alpha}V\left((\cdot)\tau^{-\frac{1}{\delta}}\right)\right)d\tau\right\|_{\widetilde{\mathbf{X}}}$$
$$\le C\mu\left\|t^{-\alpha}V\left(xt^{-\frac{1}{\delta}}\right)\right\|_{\widetilde{\mathbf{X}}}^{\sigma+1}=C\mu\|V\|_{\mathbf{Q}}^{\sigma+1}\le C\mu$$

and, in particular,

$$\|\mathcal{R}(V)\|_{\mathbf{Q}}\le\left\|\widetilde{G_0}\right\|_{\mathbf{Q}}+\left\|\mathcal{R}(V)-\widetilde{G_0}\right\|_{\mathbf{Q}}\le C+C\mu\le C.$$

Applying conditions (5.2), (5.4) and (5.11) we obtain

$$\operatorname{Re}f(\mathcal{N}(V))=t^{1-\mu}\operatorname{Re}f\left(\mathcal{N}\left(t^{-\alpha}\widetilde{G_0}\left((\cdot)t^{-\frac{1}{\delta}}\right)\right)\right)$$
$$+t^{1-\mu}\operatorname{Re}f\left(\mathcal{N}\left(t^{-\alpha}V\left((\cdot)t^{-\frac{1}{\delta}}\right)\right)-\mathcal{N}\left(t^{-\alpha}\widetilde{G_0}\left((\cdot)t^{-\frac{1}{\delta}}\right)\right)\right)$$
$$\ge\frac{\eta}{2\sigma}-\left\|V-\widetilde{G_0}\right\|_{\mathbf{Q}}\left(\|V\|_{\mathbf{Q}}^{\sigma}+\left\|\widetilde{G_0}\right\|_{\mathbf{Q}}^{\sigma}\right)\ge\frac{\eta}{2\sigma}-C\mu\ge\frac{\eta}{3\sigma}.$$

Thus we see that $\mathcal{R}(V)\in\mathbf{A}$.

Now we prove the estimate for the difference denoting $v(t)=\frac{1}{t^{\alpha}}V\left((\cdot)t^{-\frac{1}{\delta}}\right)$ and $w(t)=\frac{1}{t^{\alpha}}W\left((\cdot)t^{-\frac{1}{\delta}}\right)$

$$
\begin{aligned}
&\|\mathcal{R}(V)-\mathcal{R}(W)\|_{\mathbf{Q}} \\
&=\frac{\mu}{\sigma}\left\|\int_0^1 \mathcal{G}_0(1-z)\left(\frac{\mathcal{K}\left(\frac{1}{z^\alpha}V\left((\cdot)z^{-\frac{1}{\delta}}\right)\right)}{\operatorname{Re} f(\mathcal{N}(V))}-\frac{\mathcal{K}\left(\frac{1}{z^\alpha}W\left((\cdot)z^{-\frac{1}{\delta}}\right)\right)}{\operatorname{Re} f(\mathcal{N}(W))}\right)\frac{dz}{z^\mu}\right\|_{\mathbf{Q}} \\
&=\frac{\mu}{\sigma}\left\|\int_0^t \tau^{-\mu}\left|\mathcal{G}_0(t-\tau)\left(\mathcal{K}(v(\tau))-\mathcal{K}(w(\tau))\right)\right|d\tau\right\|_{\widetilde{\mathbf{X}}} \\
&+\frac{\mu}{\sigma}\left\|\int_0^t \tau^{-\mu}\left|\mathcal{G}_0(t-\tau)\mathcal{K}(w(\tau))\right|\left|f(\mathcal{N}(v))-f(\mathcal{N}(w))\right|d\tau\right\|_{\widetilde{\mathbf{X}}} \\
&\le C\mu\|v-w\|_{\widetilde{\mathbf{X}}}\left(1+\left\|\int_0^t \tau^{-\mu}\left|\mathcal{G}_0(t-\tau)\mathcal{K}(w(\tau))\right|d\tau\right\|_{\widetilde{\mathbf{X}}}\right) \\
&\le C\mu\|V-W\|_{\mathbf{Q}}\le\frac{1}{2}\|V-W\|_{\mathbf{Q}}.
\end{aligned}
$$

Here in view of (5.4) and (5.11) we used the estimate

$$|f(\mathcal{N}(V))-f(\mathcal{N}(W))|\le C\|v-w\|_{\mathbf{Q}}.$$

Therefore $\mathcal{R}(V)$ is a contraction mapping in the set $\mathbf{A}$. Hence there exists a unique solution $V\in\mathbf{A}$ to the integral equation (5.10), and Theorem 5.4 is proved.

In the next theorem we find the large time asymptotics for the solutions of the Cauchy problem (5.1) with a subcritical nonconvective nonlinearity.

Theorem 5.5. *Assume that the linear operator $\mathcal{L}$ is such that $f(\mathcal{L}(u))=0$ for any $u\in\mathbf{X}$. Let the nonlinearity $\mathcal{N}(u)$ in equation (5.1) be subcritical nonconvective. Assume that conditions (5.3) and (5.4) are fulfilled and the following estimates are true*

$$
\begin{aligned}
&\left\|\langle t\rangle^\gamma\int_0^t\left|\mathcal{G}(t-\tau)\left(\mathcal{K}(v(\tau))-\mathcal{K}(w(\tau))\right)\right|\tau^{-\mu}d\tau\right\|_{\mathbf{X}} \\
&\le C\left\|\langle t\rangle^\gamma(v(t)-w(t))\right\|_{\mathbf{X}}\left(\|v\|_{\mathbf{X}}+\|w\|_{\mathbf{X}}\right)^\sigma\left(1+\frac{\|v\|_{\mathbf{X}}+\|w\|_{\mathbf{X}}}{\theta}\right), \quad (5.14)
\end{aligned}
$$

$$\left\|\langle t\rangle^\gamma\int_0^t\left|\mathcal{G}(t-\tau)\mathcal{K}(v(\tau))\right|\tau^{-\mu-\gamma}d\tau\right\|_{\mathbf{X}}\le C\|v\|_{\mathbf{X}}^{\sigma+1} \quad (5.15)$$

and

$$\left\|\langle t\rangle^\gamma\int_0^t\left(\mathcal{G}(t-\tau)-\mathcal{G}_0(t-\tau)\right)\mathcal{K}(v(\tau))\tau^{-\mu}d\tau\right\|_{\mathbf{X}}\le C\|v\|_{\mathbf{X}}^{1+\sigma} \quad (5.16)$$

for any $v,w\in\mathbf{X}$ such that $f(v)=\theta=f(w)$, or $w\equiv 0$, *where $\sigma>0$ and $\mathcal{K}(v)=\mathcal{N}(v)-\frac{v}{\theta}f(\mathcal{N}(v))$; here $\mathcal{G}_0$ is the asymptotic self-similar operator for the Green operator $\mathcal{G}$ in spaces $\mathbf{X}$, $\mathbf{Z}$. Let the initial data $u_0\in\mathbf{Z}$ have* *a small norm* $\|u_0\|_{\mathbf{Z}} \leq \varepsilon$ *and the mean value* $\theta \equiv |f(u_0)| \geq C\varepsilon > 0$ *with some* $C > 0$. *Suppose also that* $\mu \in (0,1)$ *is sufficiently small. Then there exist numbers* A, ω *and a function* $V \in \mathbf{Q}$ *(as a unique solution of the integral equation (5.13)), such that for the solution* $u \in \mathbf{X}$ *of the Cauchy problem (5.1) the asymptotics is valid*

$$\left\| \langle t \rangle^{\gamma + \frac{\mu}{\sigma}} \left(u - A t^{-\frac{1}{\sigma}} e^{i\omega \log t} V\left((\cdot)\, t^{-\frac{1}{\delta}} \right) \right) \right\|_{\mathbf{X}} \leq C \tag{5.17}$$

with some $\gamma > 0$.

Proof. First let us prove the estimate

$$\| \langle t \rangle^{\gamma} (v(t) - w(t)) \|_{\mathbf{X}} \leq C, \tag{5.18}$$

with some $\gamma \in (0, \mu)$, where the function v satisfies the integral equation (5.6), $w(t) = t^{-\alpha} \theta V\left((\cdot)\, t^{-\frac{1}{\delta}} \right)$, and V satisfies the integral equation (5.13) so that we have

$$w(t) = \theta G_0(t) - \int_0^t \mathcal{G}_0(t-\tau)\, \mathcal{K}(w(\tau)) \frac{d\tau}{\widetilde{h_w}(\tau)}, \tag{5.19}$$

with $G_0(t) = t^{-\alpha} \widetilde{G}_0\left((\cdot)\, t^{-\frac{1}{\delta}} \right)$ and $\widetilde{h_w}(t) = \frac{\sigma}{\mu\theta} t \operatorname{Re} f(\mathcal{N}(w(t)))$. Note that in view of the relation of the norms we have estimates

$$\|w\|_{\mathbf{X}} \leq C \|w\|_{\widetilde{\mathbf{X}}} = C\theta \|V\|_{\mathbf{Q}} \leq C\theta.$$

Then by (5.6) and (5.19) we get

$$\begin{aligned} \| \langle t \rangle^{\gamma} (v(t) - w(t)) \|_{\mathbf{X}} &\leq \| \langle t \rangle^{\gamma} (\mathcal{G}(t) v_0 - \theta G_0(t)) \|_{\mathbf{X}} \\ &+ \left\| \langle t \rangle^{\gamma} \int_0^t \left(\mathcal{G}_0(t-\tau)\, \mathcal{K}(w(\tau)) \frac{1}{\widetilde{h_w}(\tau)} \right.\right. \\ &\left.\left. - \mathcal{G}(t-\tau)\, \mathcal{K}(v(\tau)) \frac{1}{h_v(\tau)} \right) d\tau \right\|_{\mathbf{X}}. \end{aligned} \tag{5.20}$$

By the definition of the asymptotic kernel (see Definition 2.1) we have the estimate of the first summand in the right-hand side of (5.20)

$$\| \langle t \rangle^{\gamma} (\mathcal{G}(t) v_0 - \theta G_0(t)) \|_{\mathbf{X}} \leq C \|v_0\|_{\mathbf{Z}} \leq C\varepsilon.$$

Then we rewrite (5.20) in the form

$$\begin{aligned}
&\left\|\langle t\rangle^{\gamma}\left(v\left(t\right)-w\left(t\right)\right)\right\|_{\mathbf{X}}\leq C\varepsilon\\
&+\left\|\langle t\rangle^{\gamma}\int_{0}^{t}\left(\mathcal{G}\left(t-\tau\right)-\mathcal{G}_{0}\left(t-\tau\right)\right)\mathcal{K}\left(w\left(\tau\right)\right)\frac{d\tau}{\widetilde{h_{w}}\left(\tau\right)}\right\|_{\mathbf{X}}\\
&+\left\|\langle t\rangle^{\gamma}\int_{0}^{t}\mathcal{G}\left(t-\tau\right)\left(\mathcal{K}\left(v\left(\tau\right)\right)-\mathcal{K}\left(w\left(\tau\right)\right)\right)\frac{d\tau}{\widetilde{h_{w}}\left(\tau\right)}\right\|_{\mathbf{X}}\\
&+\left\|\langle t\rangle^{\gamma}\int_{0}^{t}\left(\frac{1}{h_{v}\left(\tau\right)}-\frac{1}{\widetilde{h_{w}}\left(\tau\right)}\right)\mathcal{G}\left(t-\tau\right)\mathcal{K}\left(v\left(\tau\right)\right)d\tau\right\|_{\mathbf{X}}\\
&\equiv I_{1}+I_{2}+I_{3}. \qquad (5.21)
\end{aligned}$$

By (5.16) we get

$$\begin{aligned}
I_{1}&=\left\|\langle t\rangle^{\gamma}\int_{0}^{t}\left(\mathcal{G}\left(t-\tau\right)-\mathcal{G}_{0}\left(t-\tau\right)\right)\mathcal{K}\left(w\left(\tau\right)\right)\frac{d\tau}{\widetilde{h_{w}}\left(\tau\right)}\right\|_{\mathbf{X}}\\
&\leq\frac{C\mu}{\theta^{\sigma}}\left\|\langle t\rangle^{\gamma}\int_{0}^{t}\left(\mathcal{G}\left(t-\tau\right)-\mathcal{G}_{0}\left(t-\tau\right)\right)\mathcal{K}\left(v\left(\tau\right)\right)\tau^{-\mu}d\tau\right\|_{\mathbf{X}}\\
&\leq\frac{C\mu}{\theta^{\sigma}}\left\|v\right\|_{\mathbf{X}}^{\sigma+1}\leq C\mu\varepsilon.
\end{aligned}$$

In view of (5.11) we also find

$$\begin{aligned}
I_{2}&=\left\|\langle t\rangle^{\gamma}\int_{0}^{t}\mathcal{G}\left(t-\tau\right)\left(\mathcal{K}\left(v\left(\tau\right)\right)-\mathcal{K}\left(w\left(\tau\right)\right)\right)\frac{d\tau}{\widetilde{h_{w}}\left(\tau\right)}\right\|_{\mathbf{X}}\\
&\leq\frac{C\mu}{\theta^{\sigma}}\left\|\langle t\rangle^{\gamma}\int_{0}^{t}\left|\mathcal{G}\left(t-\tau\right)\left(\mathcal{K}\left(v\left(\tau\right)\right)-\mathcal{K}\left(w\left(\tau\right)\right)\right)\right|\tau^{-\mu}d\tau\right\|_{\mathbf{X}}\\
&\leq\frac{C\mu}{\theta^{\sigma}}\left\|\langle t\rangle^{\gamma}\left(v\left(t\right)-w\left(t\right)\right)\right\|_{\mathbf{X}}\left(\left\|v\right\|_{\mathbf{X}}+\left\|w\right\|_{\mathbf{X}}\right)^{\sigma}\left(1+\frac{\left\|v\right\|_{\mathbf{X}}+\left\|w\right\|_{\mathbf{X}}}{\theta}\right)\\
&\leq C\mu\left\|\langle t\rangle^{\gamma}\left(v\left(t\right)-w\left(t\right)\right)\right\|_{\mathbf{X}}.
\end{aligned}$$

Now we estimate the difference $h_{v}\left(t\right)-\widetilde{h_{w}}\left(t\right)$. Since

$$\begin{aligned}
\widetilde{h_{w}}\left(t\right)&=\frac{\sigma}{\mu\theta}t\operatorname{Re}f\left(\mathcal{N}\left(w\left(t\right)\right)\right)=\frac{\sigma}{\mu\theta}t^{\mu}\operatorname{Re}f\left(\mathcal{N}\left(V\right)\right)\\
&=\frac{\sigma}{\theta}\int_{0}^{t}t^{\mu-1}\operatorname{Re}f\left(\mathcal{N}\left(V\right)\right)d\tau=\frac{\sigma}{\theta}\int_{0}^{t}\operatorname{Re}f\left(\mathcal{N}\left(w\left(\tau\right)\right)\right)d\tau,
\end{aligned}$$

then in view of estimate (5.4) we obtain

$$\left|h_v(t)-\widetilde{h_w}(t)\right| \le 1+\frac{\sigma}{\theta}\int_0^t |f(\mathcal{N}(v(\tau))-\mathcal{N}(w(\tau)))|\,d\tau$$
$$\le 1+\frac{C}{\theta}\varepsilon^{\sigma}\left\|\langle t\rangle^{\gamma}(v(t)-w(t))\right\|_{\mathbf{X}}\int_0^t \{\tau\}^{-\alpha}\langle\tau\rangle^{\mu-\gamma-1}\,d\tau$$
$$\le 1+\frac{C}{\theta(\mu-\gamma)}\varepsilon^{\sigma}t^{\mu-\gamma}\left\|\langle t\rangle^{\gamma}(v(t)-w(t))\right\|_{\mathbf{X}} \tag{5.22}$$

because $\gamma<\mu$. Therefore by (5.15) we find

$$I_3=\left\|\langle t\rangle^{\gamma}\int_0^t\left(\frac{1}{h_v(\tau)}-\frac{1}{\widetilde{h_w}(\tau)}\right)\mathcal{G}(t-\tau)\mathcal{K}(v(\tau))\,d\tau\right\|_{\mathbf{X}}$$
$$\le C\mu^2\theta^{-2\sigma}\left\|\langle t\rangle^{\gamma}\int_0^t\left|h_v(\tau)-\widetilde{h_w}(\tau)\right|\left|\mathcal{G}(t-\tau)\mathcal{K}(v(\tau))\right|\tau^{-2\mu}d\tau\right\|_{\mathbf{X}}$$
$$\le C\mu^2\theta^{-2\sigma}\left\|\langle t\rangle^{\gamma}\int_0^t\left|\mathcal{G}(t-\tau)\mathcal{K}(v(\tau))\right|\tau^{-2\mu}d\tau\right\|_{\mathbf{X}}$$
$$+\frac{C\mu}{\theta^{1+\sigma}}\left\|\langle t\rangle^{\gamma}(v(t)-w(t))\right\|_{\mathbf{X}}\left\|\langle t\rangle^{\gamma}\int_0^t\left|\mathcal{G}(t-\tau)\mathcal{K}(v(\tau))\right|\tau^{-\mu-\gamma}d\tau\right\|_{\mathbf{X}}$$
$$\le C\mu^2\theta^{1-\sigma}+C\mu\left\|\langle t\rangle^{\gamma}(v(t)-w(t))\right\|_{\mathbf{X}}.$$

Collecting these estimates we get from (5.21)

$$\left\|\langle t\rangle^{\gamma}(v(t)-w(t))\right\|_{\mathbf{X}}\le C\varepsilon+C\mu^2\theta^{1-\sigma}+C\mu\left\|\langle t\rangle^{\gamma}(v(t)-w(t))\right\|_{\mathbf{X}}.$$

Hence considering the fact that μ is small enough, we arrive at the estimate (5.18).

Via (5.7) we get

$$\psi(t)=\arg f(u_0)-\frac{1}{\theta}\int_0^t\frac{1}{h_v(\tau)}\operatorname{Im} f(\mathcal{N}(v(\tau)))\,d\tau. \tag{5.23}$$

By using estimate (5.22) we have the asymptotic estimate

$$\left|h_v(t)-\beta\theta^{\sigma}\langle t\rangle^{\mu}\right|\le C\langle t\rangle^{\mu-\gamma} \tag{5.24}$$

for all $t>0$, where $\beta=\frac{\sigma}{\mu}\operatorname{Re} f(\mathcal{N}(V))>\frac{\eta}{2\mu}>0$. In addition by (5.4) we get

$$\left|\operatorname{Im} f(\mathcal{N}(v(\tau)))-\chi\theta^{\sigma+1}\langle t\rangle^{\mu-1}\right|\le C\varepsilon^{\sigma+1}\langle t\rangle^{\mu-\gamma}$$

for all $t>0$, where $\chi=\operatorname{Im} f(\mathcal{N}(V))$. Substitution of these estimates into (5.23) yields

$$
\begin{aligned}
\psi(t) &= -\frac{\chi}{\beta}\int_0^t \langle\tau\rangle^{-1}\,d\tau + \arg f(u_0) \\
&- \frac{1}{\theta}\int_0^{\infty}\left(\frac{1}{h_v(\tau)}\operatorname{Im} f(\mathcal{N}(v(\tau))) - \frac{\chi}{\beta}\langle\tau\rangle^{-1}\right)d\tau \\
&+ \frac{1}{\theta}\int_t^{\infty}\left(\frac{1}{h_v(\tau)}\operatorname{Im} f(\mathcal{N}(v(\tau))) - \frac{\chi}{\beta}\langle\tau\rangle^{-1}\right)d\tau \\
&= \omega\log t + \Psi + O\left(\int_t^{\infty}\langle\tau\rangle^{-1-\gamma}\,d\tau\right) \\
&= \omega\log t + \Psi + O\left(t^{-\gamma}\right),
\end{aligned} \tag{5.25}
$$

where $\omega = -\frac{\chi}{\beta}$ and

$$
\Psi \equiv \arg f(u_0) - \frac{1}{\theta}\int_0^{\infty}\left(\frac{1}{h_v(\tau)}\operatorname{Im} f(\mathcal{N}(v(\tau))) - \frac{\chi}{\beta}\langle\tau\rangle^{-1}\right)d\tau.
$$

Therefore via the formula $u(t,x) = e^{i\psi(t)}h_v^{-\frac{1}{\sigma}}(t)\,v(t,x)$ and asymptotic estimates (5.18), (5.24) and (5.25) we obtain the asymptotics (5.17) with a constant $A = \beta^{-\frac{1}{\sigma}}e^{i\Psi}$. Theorem 5.5 is proved.

Example 5.6. **Large time asymptotics for the nonlinear heat equations in the subcritical case**

Consider the Cauchy problem for the nonlinear heat equation

$$
\begin{cases} u_t - \Delta u = \lambda|u|^{\sigma}u, \ x\in\mathbf{R}^n, \ t>0, \\ u(0,x) = u_0(x), \ x\in\mathbf{R}^n, \end{cases} \tag{5.26}
$$

in the subcritical case $\sigma < \frac{2}{n}$, where $\lambda < 0$.

We define $f(\phi) = \int_{\mathbf{R}^n}\phi(x)\,dx$ and $\mu = 1 - \frac{n}{2}\sigma > 0$. Denote

$$
\theta = |f(u_0)| = \left|\int_{\mathbf{R}^n} u_0(x)\,dx\right|.
$$

Theorem 5.7. *Let $\lambda < 0$ and $\sigma < \frac{2}{n}$ such that $\mu = 1 - \frac{n}{2}\sigma > 0$ is sufficiently small. Assume that the initial data $u_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n)\cap\mathbf{L}^p(\mathbf{R}^n)$ with $a\in(0,1]$, $p>1$, have a small norm $\|u_0\|_{\mathbf{L}^{1,a}} + \|u_0\|_{\mathbf{L}^p} \le \varepsilon$, and the mean value $\theta \ge C\varepsilon > 0$ with some $C > 0$. Then the Cauchy problem (5.26) has a unique global solution $u(t,x)\in\mathbf{C}\left([0,\infty);\mathbf{L}^{1,a}(\mathbf{R}^n)\cap\mathbf{L}^p(\mathbf{R}^n)\right)$ $\cap\,\mathbf{C}((0,\infty);\mathbf{L}^{\infty}(\mathbf{R}^n))$ satisfying the asymptotics*

$$
u(t,x) = At^{-\frac{1}{\sigma}}V\left(\frac{x}{\sqrt{t}}\right) + O\left(t^{-\frac{1}{\sigma}-\gamma}\right),
$$

for $t\to\infty$ uniformly with respect to $x\in\mathbf{R}^n$, where $A\in\mathbf{R}$ and $V(\xi)$ is the solution of the integral equation

$$V(\xi)=\frac{1}{(4\pi)^{\frac{n}{2}}}e^{-\frac{|\xi|^2}{4}}-\frac{\mu}{\beta(4\pi)^{\frac{n}{2}}}\int_0^1\frac{dz}{z(1-z)^{\frac{n}{2}}}\int_{\mathbf{R}^n}e^{-\frac{|\xi-y\sqrt{z}|^2}{4(1-z)}}K(y)\,dy,$$

where $\beta=\sigma\int_{\mathbf{R}^n}|V(y)|^{\sigma}V(y)\,dy,$ *and*

$$K(y)=|V(y)|^{\sigma}V(y)-V(y)\int_{\mathbf{R}^n}|V(\xi)|^{\sigma}V(\xi)\,d\xi.$$

Proof. To apply Theorems 5.2 - 5.5 we choose the space $\mathbf{Z}=\mathbf{L}^{1,a}(\mathbf{R}^n)\cap\mathbf{L}^p(\mathbf{R}^n)$ with $a\in(0,1]$, $p>1$ and the space

$$\mathbf{X}=\{\phi\in\mathbf{C}([0,\infty);\mathbf{Z})\cap\mathbf{C}((0,\infty);\mathbf{L}^{\infty}(\mathbf{R}^n)):\ \ \|\phi\|_{\mathbf{X}}<\infty\},$$

where the norm

$$\|\phi\|_{\mathbf{X}}=\sup_{t>0}\Big(\langle t\rangle^{-\frac{a}{2}}\|\phi(t)\|_{\mathbf{L}^{1,a}}+\langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{p}\right)}\|\phi(t)\|_{\mathbf{L}^p}+\{t\}^{\frac{n}{2p}}\langle t\rangle^{\frac{n}{2}}\|\phi(t)\|_{\mathbf{L}^{\infty}}\Big).$$

Also we consider the norm for nonlinearity

$$\|\phi\|_{\mathbf{Y}}=\sup_{t>0}\{t\}^{\frac{n}{2p}\sigma}\langle t\rangle^{\frac{n}{2}\sigma}\Big(\langle t\rangle^{-\frac{a}{2}}\|\phi(t)\|_{\mathbf{L}^{1,a}}+\langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{p}\right)}\|\phi(t)\|_{\mathbf{L}^p}+\{t\}^{\frac{n}{2p}}\langle t\rangle^{\frac{n}{2}}\|\phi(t)\|_{\mathbf{L}^{\infty}}\Big).$$

We also define the space

$$\widetilde{\mathbf{X}}=\left\{\phi\in\mathbf{C}\left([0,\infty);\mathbf{L}^{1,a}(\mathbf{R}^n)\right)\cap\mathbf{C}((0,\infty);\mathbf{L}^{\infty}(\mathbf{R}^n)):\ \ \|\phi\|_{\widetilde{\mathbf{X}}}<\infty\right\},$$

where the norm

$$\|\phi\|_{\widetilde{\mathbf{X}}}=\sup_{t>0}\left(t^{-\frac{a}{2}}\||\cdot|^a\phi(t)\|_{\mathbf{L}^1}+t^{\frac{n}{2}}\|\phi(t)\|_{\mathbf{L}^{\infty}}\right).$$

Also we consider the norm for the nonlinearity $\|\phi\|_{\widetilde{\mathbf{Y}}}=\left\|t^{\frac{n}{2}\sigma}\phi(t)\right\|_{\widetilde{\mathbf{X}}}$.

The Green operator $\mathcal{G}(t)$ of the linear heat equation has the form

$$\mathcal{G}(t)\phi=\int_{\mathbf{R}^n}G(t,x-y)\phi(y)\,dy,$$

where the heat kernel $G(t,x)$ is

$$G(t,x)=(4\pi t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}}.$$

We can take now the heat kernel $\widetilde{G}(x)=(4\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4}}$ as the self-similar asymptotic kernel (see Definition 5.3) and $\widetilde{G_0}(x)=(4\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4}}$. Note that condition (5.11) is true if we take $\delta=2$ and $\alpha=\frac{n}{2}$.

By a direct computation we see that

$$\begin{aligned}&\operatorname{Re} f\left(\mathcal{N}\left(\theta G_0\left(\tau\right)\right)\right) d\tau = \left|\lambda\right| \theta^{\sigma+1} \int_{\mathbf{R}^n} G^{\sigma+1}\left(t,x\right) dx \\ &= \left|\lambda\right| \theta^{\sigma+1} \left(4\pi\right)^{-\frac{n}{2}\left(\sigma+1\right)} t^{-\frac{n}{2}\left(\sigma+1\right)} \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4t}\left(\sigma+1\right)} dx \\ &= \left|\lambda\right| \theta^{\sigma+1} \left(4\pi\right)^{-\frac{n}{2}\sigma} \left(1+\sigma\right)^{-\frac{n}{2}} t^{-\frac{n}{2}\sigma};\end{aligned}$$

hence condition (5.2) with $\eta = 2\sigma\left|\lambda\right|\left(4\pi\right)^{-\frac{n}{2}\sigma}\left(1+\sigma\right)^{-\frac{n}{2}}$ and $\mu = 1 - \frac{n}{2}\sigma$ is fulfilled. For any u, $z \in \mathbf{R}$ we get

$$e^z \mathcal{N}\left(ue^{-z}\right) = -\lambda e^z \left|ue^{-z}\right|^{\sigma} ue^{-z} = e^{-\sigma z}\mathcal{N}\left(u\right),$$

therefore condition (5.3) is valid.

By Lemma 1.28 and Lemma 1.30 with $\delta = \nu = 2$ applied to the case of the Green operator for the heat equation we obtain

$$\left\|\mathcal{G}\left(t\right)\phi\right\|_{\mathbf{L}^q} \leq Ct^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)} \left\|\phi\right\|_{\mathbf{L}^r} \tag{5.27}$$

and

$$\left\|\left|\cdot\right|^b\left(\mathcal{G}\left(t\right)\phi - \vartheta G_0\left(t\right)\right)\right\|_{\mathbf{L}^q} \leq Ct^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-\frac{a-b}{2}} \left\|\left|\cdot\right|^a \phi\right\|_{\mathbf{L}^1} \tag{5.28}$$

for all $t > 0$, where $1 \leq r \leq q \leq \infty$, $\beta \geq 0$, $0 \leq b \leq a$, $\vartheta = \int_{\mathbf{R}^n} \phi\left(x\right) dx$, and the asymptotic kernel is

$$G_0\left(t,x\right) = \left(4\pi t\right)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$

Now we prove the following auxiliary result.

Lemma 5.8. *Let the function $\phi\left(t,x\right)$ have a zero mean value $\int_{\mathbf{R}^n} \phi\left(t,x\right) dx = 0$. Then the following inequalities are valid*

$$\left\|\left\langle t\right\rangle^{\nu} \int_0^t \mathcal{G}\left(t-\tau\right)\phi\left(\tau\right)\tau^{-\mu}\left\langle\tau\right\rangle^{-\nu} d\tau\right\|_{\mathbf{X}} \leq C\left\|\phi\right\|_{\mathbf{Y}}$$

and

$$\left\|\int_0^t \mathcal{G}\left(t-\tau\right)\phi\left(\tau\right)\tau^{-\mu} d\tau\right\|_{\widetilde{\mathbf{X}}} \leq C\left\|\phi\right\|_{\widetilde{\mathbf{Y}}}$$

for all $t > 0$, $0 \leq \nu < \frac{a}{2}$, provided that the right-hand sides are finite.

Proof. Since $\vartheta = \int_{\mathbf{R}^n} \phi\left(t,x\right) dx = 0$, by estimates (5.28) and (1.2) we get

$$\begin{aligned}&\left\|\int_0^t \mathcal{G}\left(t-\tau\right)\phi\left(\tau\right)\tau^{-\mu}\left\langle\tau\right\rangle^{-\nu} d\tau\right\|_{\mathbf{L}^{1,a}} \leq \int_0^t \left\|\phi\left(\tau\right)\right\|_{\mathbf{L}^{1,a}} \tau^{-\mu}\left\langle\tau\right\rangle^{-\nu} d\tau \\ &\leq \int_0^t \left\{\tau\right\}^{-\frac{n}{2p}\sigma} \tau^{-\mu}\left\langle\tau\right\rangle^{\frac{a}{2}-\frac{n}{2}\sigma-\nu} d\tau \sup_{t>0} \left\{t\right\}^{\frac{n}{2p}\sigma}\left\langle t\right\rangle^{\frac{n}{2}\sigma-\frac{a}{2}} \left\|\phi\left(t\right)\right\|_{\mathbf{L}^{1,a}} \\ &\leq C\left\langle t\right\rangle^{\frac{a}{2}-\nu}\left\|\phi\right\|_{\mathbf{Y}}\end{aligned}$$

for all $t > 0$, because $\frac{n}{2p}\sigma + \mu = 1 - \frac{n}{2}\sigma\left(1 - \frac{1}{p}\right) < 1$. In the same manner by virtue of (5.27) we have

$$\left\| \int_0^t \mathcal{G}(t-\tau)\phi(\tau)\tau^{-\mu}\langle\tau\rangle^{-\nu}d\tau \right\|_{\mathbf{L}^p} \le \int_0^t \|\phi(\tau)\|_{\mathbf{L}^p}\tau^{-\mu}d\tau$$

$$\le \int_0^t \tau^{-\frac{n}{2p}\sigma-\mu}d\tau \sup_{0<t\le 1}\{t\}^{\frac{n}{2p}\sigma}\|\phi(t)\|_{\mathbf{L}^p} \le C\|\phi\|_{\mathbf{Y}} \tag{5.29}$$

for all $0 < t \le 1$, and

$$\left\| \int_0^t \mathcal{G}(t-\tau)\phi(\tau)\tau^{-\mu}\langle\tau\rangle^{-\nu}d\tau \right\|_{\mathbf{L}^p} \le \int_{\frac{t}{2}}^t \|\phi(\tau)\|_{\mathbf{L}^p}\langle\tau\rangle^{-\mu-\nu}d\tau$$

$$+ \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{a}{2}}\|\phi(\tau)\|_{\mathbf{L}^{1,a}}\tau^{-\mu}\langle\tau\rangle^{-\nu}d\tau$$

$$\le \int_{\frac{t}{2}}^t \tau^{-1-\nu-\frac{n}{2}\left(1-\frac{1}{p}\right)}d\tau \sup_{t>0}\{t\}^{\frac{n}{2p}\sigma}\langle t\rangle^{\frac{n}{2}\sigma+\frac{n}{2}\left(1-\frac{1}{p}\right)}\|\phi(t)\|_{\mathbf{L}^p}$$

$$+ t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{a}{2}}\int_0^{\frac{t}{2}}\{\tau\}^{-\frac{n}{2p}\sigma-\mu}\langle\tau\rangle^{\frac{a}{2}-1-\nu}d\tau \sup_{t>0}\{t\}^{\frac{n}{2p}\sigma}\langle t\rangle^{\frac{n}{2}\sigma-\frac{a}{2}}\|\phi(\tau)\|_{\mathbf{L}^{1,a}}$$

$$\le C\langle t\rangle^{-\nu-\frac{n}{2}\left(1-\frac{1}{p}\right)}\|\phi\|_{\mathbf{Y}}$$

for all $t > 1$. Also by estimate (5.27) we find

$$\left\| \int_0^t \mathcal{G}(t-\tau)\phi(\tau)\tau^{-\mu}\langle\tau\rangle^{-\nu}d\tau \right\|_{\mathbf{L}^\infty}$$

$$\le \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2p}}\|\phi(\tau)\|_{\mathbf{L}^p}\tau^{-\mu}d\tau + \int_{\frac{t}{2}}^t \|\phi(\tau)\|_{\mathbf{L}^\infty}\tau^{-\mu}d\tau$$

$$\le C\|\phi\|_{\mathbf{Y}}\left(\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{n}{2p}}\tau^{-\frac{n}{2p}\sigma-\mu}d\tau + \int_{\frac{t}{2}}^t \tau^{-\frac{n}{2p}(\sigma+1)-\mu}d\tau\right)$$

$$\le Ct^{-\frac{n}{2p}}\|\phi\|_{\mathbf{Y}} \tag{5.30}$$

for all $0 < t \le 1$, and

$$\left\| \int_0^t \mathcal{G}(t-\tau)\phi(\tau)\tau^{-\mu}\langle\tau\rangle^{-\nu}d\tau \right\|_{\mathbf{L}^\infty} \le \int_{\frac{t}{2}}^t \|\phi(\tau)\|_{\mathbf{L}^\infty}\langle\tau\rangle^{-\mu-\nu}d\tau$$

$$+ \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}-\frac{a}{2}}\|\phi(\tau)\|_{\mathbf{L}^{1,a}}\tau^{-\mu}\langle\tau\rangle^{-\nu}d\tau$$

$$\le \int_{\frac{t}{2}}^t \tau^{-1-\nu-\frac{n}{2}}d\tau \sup_{t>0}\{t\}^{\frac{n}{2p}(\sigma+1)}\langle t\rangle^{\frac{n}{2}\sigma+\frac{n}{2}}\|\phi(t)\|_{\mathbf{L}^\infty}$$

$$+ t^{-\frac{n}{2}-\frac{a}{2}}\int_0^{\frac{t}{2}}\{\tau\}^{-\frac{n}{2p}\sigma-\mu}\langle\tau\rangle^{\frac{a}{2}-1-\nu}d\tau \sup_{t>0}\{t\}^{\frac{n}{2p}\sigma}\langle t\rangle^{\frac{n}{2}\sigma-\frac{a}{2}}\|\phi(\tau)\|_{\mathbf{L}^{1,a}}$$

$$\le C\langle t\rangle^{-\nu-\frac{n}{2}}\|\phi\|_{\mathbf{Y}}$$

for all $t > 1$.

Likewise we obtain estimates in the norms $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{Y}}$

$$\left\| |\cdot|^a \int_0^t \mathcal{G}(t-\tau)\phi(\tau)\tau^{-\mu}d\tau \right\|_{\mathbf{L}^1} \leq \int_0^t \| |\cdot|^a \phi(\tau)\|_{\mathbf{L}^1} \tau^{-\mu}d\tau$$
$$\leq \int_0^t \tau^{\frac{a}{2}-\frac{n}{2}\sigma-\mu}d\tau \sup_{t>0} t^{\frac{n}{2}\sigma-\frac{a}{2}} \| |\cdot|^a \phi(t)\|_{\mathbf{L}^1} \leq Ct^{\frac{a}{2}} \|\phi\|_{\widetilde{\mathbf{Y}}}$$

for all $t > 0$, since $\frac{n}{2}\sigma + \mu - \frac{a}{2} = 1 - \frac{a}{2} < 1$. As well by estimates (5.28) and (5.27) with $r = \infty$, we find

$$\left\| \int_0^t \mathcal{G}(t-\tau)\phi(\tau)\tau^{-\mu}d\tau \right\|_{\mathbf{L}^\infty}$$
$$\leq \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}-\frac{a}{2}} \| |\cdot|^a \phi(\tau)\|_{\mathbf{L}^1} \tau^{-\mu}d\tau + \int_{\frac{t}{2}}^t \|\phi(\tau)\|_{\mathbf{L}^\infty} \tau^{-\mu}d\tau$$
$$\leq C\|\phi\|_{\widetilde{\mathbf{Y}}} \left(\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}-\frac{a}{2}} \tau^{-\frac{n}{2}\sigma-\mu+\frac{a}{2}}d\tau + \int_{\frac{t}{2}}^t \tau^{-\frac{n}{2}(\sigma+1)-\mu}d\tau \right)$$
$$\leq Ct^{-\frac{n}{2}} \|\phi\|_{\widetilde{\mathbf{Y}}}$$

for all $t > 0$, since by changing the variable of integration $\tau = tz$ we get

$$\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}-\frac{a}{2}} \tau^{\frac{a}{2}-1}d\tau = t^{-\frac{n}{2}} \int_0^{\frac{1}{2}} (1-z)^{-\frac{n}{2}-\frac{a}{2}} z^{\frac{a}{2}-1}dz \leq Ct^{-\frac{n}{2}}.$$

Hence, the result of the lemma follows, and Lemma 5.8 is proved.

Since by interpolation inequality (1.4)

$$\begin{aligned} &\nu(t)\|\mathcal{N}(v(t)) - \mathcal{N}(w(t))\|_{\mathbf{L}^1} \\ &\leq C\nu(t)(\|v(t)\|_{\mathbf{L}^\infty}^\sigma + \|w(t)\|_{\mathbf{L}^\infty}^\sigma)\|v(t) - w(t)\|_{\mathbf{L}^1} \\ &\leq C\{t\}^{-\frac{n}{2p}\sigma}\langle t\rangle^{-\frac{n}{2}\sigma}\|\nu(t)(v-w)\|_{\mathbf{X}}(\|v\|_{\mathbf{X}}^\sigma + \|w\|_{\mathbf{X}}^\sigma); \end{aligned} \tag{5.31}$$

condition (5.4) with $\alpha = \frac{n}{2p}\sigma < 1$ is true. By estimates (1.24) we have

$$\begin{aligned} &\|\mathcal{K}(v) - \mathcal{K}(w)\|_{\mathbf{Y}} \leq \|\mathcal{N}(v) - \mathcal{N}(w)\|_{\mathbf{Y}} \\ &+ \frac{1}{\theta}(\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}) \sup_{t>0}\{t\}^{\frac{1}{p}}\langle t\rangle\|\mathcal{N}(v(t)) - \mathcal{N}(w(t))\|_{\mathbf{L}^1} \\ &+ \frac{1}{\theta}\|v-w\|_{\mathbf{X}} \sup_{t>0}\{t\}^{\frac{1}{p}}\langle t\rangle(\|\mathcal{N}(v(t))\|_{\mathbf{L}^1} + \|\mathcal{N}(w(t))\|_{\mathbf{L}^1}) \\ &\leq C\|v-w\|_{\mathbf{X}}(\|v\|_{\mathbf{X}}^\sigma + \|w\|_{\mathbf{X}}^\sigma)\left(1 + \frac{1}{\theta}\|v\|_{\mathbf{X}} + \frac{1}{\theta}\|w\|_{\mathbf{X}}\right). \end{aligned} \tag{5.32}$$

Therefore by Lemma 5.8 we see that conditions (5.5), (5.14), (5.15) and (5.16) are fulfilled. Condition (5.10) follows from (5.27). The rest condition (5.12). is a consequence of the second estimate of Lemma 5.8 and the estimates

$$\begin{aligned}&\|\mathcal{N}(w(\tau))-\mathcal{N}(v(\tau))\|_{\mathbf{L}^{1,a}} \le C\|w-v\|_{\mathbf{L}^{1,a}}\left(\|w\|_{\mathbf{L}^\infty}^\sigma+\|v\|_{\mathbf{L}^\infty}^\sigma\right)\\&\le C\tau^{\frac{a}{2}-\frac{n\sigma}{2}}\|w-v\|_{\widetilde{\mathbf{X}}}\left(\|w\|_{\widetilde{\mathbf{X}}}^\sigma+\|v\|_{\widetilde{\mathbf{X}}}^\sigma\right)\end{aligned}$$

and

$$\begin{aligned}&\|\mathcal{N}(w(\tau))-\mathcal{N}(v(\tau))\|_{\mathbf{L}^\infty}\le C\|w-v\|_{\mathbf{L}^\infty}\left(\|w\|_{\mathbf{L}^\infty}+\|v\|_{\mathbf{L}^\infty}\right)^\sigma\\&\le C\tau^{-\frac{n}{2}(\sigma+1)}\;\|w-v\|_{\widetilde{\mathbf{X}}}\left(\|w\|_{\widetilde{\mathbf{X}}}^\sigma+\|v\|_{\widetilde{\mathbf{X}}}^\sigma\right).\end{aligned}$$

Thus

$$\|\mathcal{N}(w)-\mathcal{N}(v)\|_{\widetilde{\mathbf{Y}}}\le C\|w-v\|_{\widetilde{\mathbf{X}}}\left(\|w\|_{\widetilde{\mathbf{X}}}^\sigma+\|v\|_{\widetilde{\mathbf{X}}}^\sigma\right),\tag{5.33}$$

and then

$$\begin{aligned}&\|\mathcal{K}(v)-\mathcal{K}(w)\|_{\widetilde{\mathbf{Y}}}\le\|\mathcal{N}(v)-\mathcal{N}(w)\|_{\widetilde{\mathbf{Y}}}\\&+\frac{1}{\theta}\left(\|v\|_{\widetilde{\mathbf{X}}}+\|w\|_{\widetilde{\mathbf{X}}}\right)\sup_{t>0}t\,\|\mathcal{N}(v(t))-\mathcal{N}(w(t))\|_{\mathbf{L}^1}\\&+\frac{1}{\theta}\|v-w\|_{\widetilde{\mathbf{X}}}\sup_{t>0}t\left(\|\mathcal{N}(v(t))\|_{\mathbf{L}^1}+\|\mathcal{N}(w(t))\|_{\mathbf{L}^1}\right)\\&\le C\|v-w\|_{\widetilde{\mathbf{X}}}\left(\|v\|_{\widetilde{\mathbf{X}}}^\sigma+\|w\|_{\widetilde{\mathbf{X}}}^\sigma\right)\left(1+\frac{1}{\theta}\|v\|_{\widetilde{\mathbf{X}}}+\frac{1}{\theta}\|w\|_{\widetilde{\mathbf{X}}}\right).\end{aligned}\tag{5.34}$$

Now (5.12) is a consequence of the second estimate of Lemma 5.8. Applying Theorems 5.2 - 5.5 we get the result of Theorem 5.7 which is then proved.

Example 5.9. **The case of odd solutions to the nonlinear heat equation**

Now let us consider problem (5.26) with the initial data $u_0(x)$, which are odd functions in $\mathbf{R}^n$: $u_0(x_1,...,-x_j,...,x_n)=-u_0(x_1,...,x_j,...,x_n)$, for every $j=1,2,...,n$. In this case the solutions $u(t,x)$ also will be odd functions with respect to $x\in\mathbf{R}^n$.

Denote

$$\theta=\left|\int_{\mathbf{R}^n}u_0(x)\prod_{j=1}^n x_j dx\right|.$$

Theorem 5.10. *Let $\lambda<0$ and $\sigma<\frac{1}{n}$ such that $\mu=1-n\sigma>0$ is sufficiently small. Assume that the initial data u_0 are odd functions in $\mathbf{R}^n$, and $u_0\in\mathbf{L}^{1,a}(\mathbf{R}^n)\cap\mathbf{L}^p(\mathbf{R}^n)$, with $a\in(n,n+1]$, and $p>1$. Also suppose that u_0 have a sufficiently small norm $\|u_0\|_{\mathbf{L}^{1,a}}+\|u_0\|_{\mathbf{L}^p}$, and the mean value $\theta\ge C\varepsilon>0$ with some $C>0$. Then the Cauchy problem (5.26) has a unique global solution $u(t,x)\in\mathbf{C}\left([0,\infty);\mathbf{L}^{1,a}(\mathbf{R}^n)\cap\mathbf{L}^p(\mathbf{R}^n)\right)\cap\mathbf{C}\left((0,\infty);\mathbf{L}^\infty(\mathbf{R}^n)\right)$ satisfying the asymptotics*

$$u(t,x)=At^{-\frac{1}{\sigma}}V\left(\frac{x}{\sqrt{t}}\right)+O\left(t^{-\frac{1}{\sigma}-\gamma}\right),$$

for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$, where $A \in \mathbf{R}$ and $V(\xi)$ is the solution of the integral equation

$$V(\xi) = \widetilde{G_0}(\xi) - \frac{\mu}{\beta} \int_0^1 \frac{dz}{z(1-z)^{\frac{n}{2}}} \int_{\mathbf{R}^n} \widetilde{G}\left(\frac{\xi - y\sqrt{z}}{\sqrt{1-z}}\right) K(y)\, dy,$$

where $\beta = \sigma \int_{\mathbf{R}^n} |V(y)|^{\sigma} V(y)\, dy$,

$$\widetilde{G_0}(x) = \frac{1}{4^n \pi^{\frac{n}{2}}} e^{-\frac{|x|^2}{4}} \prod_{j=1}^{n} x_j,\ \widetilde{G}(x) = \frac{1}{2^n \pi^{\frac{n}{2}}} e^{-\frac{|x|^2}{4}}$$

and

$$K(y) = |V(y)|^{\sigma} V(y) - V(y) \int_{\mathbf{R}^n} |V(\xi)|^{\sigma} V(\xi)\, d\xi.$$

Proof. To apply Theorems 5.2 - 5.5 we choose the space $\mathbf{Z}$

$$\mathbf{Z} = \left\{\phi \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^p(\mathbf{R}^n) :\ \phi \text{ is odd function in } \mathbf{R}^n\right\},$$

where now $a \in (n, n+1]$, and $p > 1$ and the space

$$\mathbf{X} = \{\phi \in \mathbf{C}([0,\infty); \mathbf{Z}) \cap \mathbf{C}((0,\infty); \mathbf{L}^{\infty}(\mathbf{R}^n)) : \\ \phi \text{ is odd function in } \mathbf{R}^n \text{ and } \|\phi\|_{\mathbf{X}} < \infty\},$$

where the norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t \geq 0} \left(\langle t\rangle^{-\frac{a-n}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} + \langle t\rangle^{n - \frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^p}\right) \\ + \sup_{t>0} \{t\}^{\frac{n}{2p}} \langle t\rangle^{n} \|\phi(t)\|_{\mathbf{L}^{\infty}}.$$

Also we define the norm to estimate the nonlinearity

$$\|\phi\|_{\mathbf{Y}} = \sup_{t>0} \{t\}^{\frac{n}{2p}\sigma} \langle t\rangle^{n\sigma} \left(\langle t\rangle^{-\frac{a-n}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} \right. \\ \left. + \langle t\rangle^{n - \frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^p} + \{t\}^{\frac{n}{2p}} \langle t\rangle^{n} \|\phi(t)\|_{\mathbf{L}^{\infty}}\right).$$

We also define the space

$$\widetilde{\mathbf{X}} = \left\{\phi \in \mathbf{C}\left([0,\infty); \mathbf{L}^{1,a}(\mathbf{R}^n)\right) \cap \mathbf{C}((0,\infty); \mathbf{L}^{\infty}(\mathbf{R}^n)) : \right. \\ \left. \phi \text{ is odd function in } \mathbf{R}^n \text{ and } \|\phi\|_{\widetilde{\mathbf{X}}} < \infty\right\},$$

where the norm

$$\|\phi\|_{\widetilde{\mathbf{X}}} = \sup_{t>0} \left(t^{-\frac{a-n}{2}} \||\cdot|^{a} \phi(t)\|_{\mathbf{L}^1} + t^{n} \|\phi(t)\|_{\mathbf{L}^{\infty}}\right).$$

Also we consider the norm for the nonlinearity $\|\phi\|_{\widetilde{\mathbf{Y}}} = \left\|t^{\frac{n}{2}\sigma} \phi(t)\right\|_{\widetilde{\mathbf{X}}}$.

By a direct computation we see that

$$\operatorname{Re} f\left(\mathcal{N}\left(\theta G_0\left(t\right)\right)\right) d\tau = \left|\lambda\right| \theta^{\sigma+1} \int_{\mathbf{R}^n} \left|G_0\left(t,x\right)\right|^{\sigma} G_0\left(t,x\right) \prod_{j=1}^{n} x_j dx$$

$$= \left|\lambda\right| \theta^{\sigma+1} \left(16\pi\right)^{-\frac{n}{2}\left(\sigma+1\right)} t^{-\frac{3n}{2}\left(\sigma+1\right)} \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4t}\left(\sigma+1\right)} \prod_{j=1}^{n} \left|x_j\right|^{\sigma+2} dx$$

$$= \eta \theta^{\sigma+1} t^{-n\sigma};$$

hence, condition (5.2) with

$$\eta = \left|\lambda\right| \left(4\pi\right)^{-\frac{n}{2}\left(\sigma+1\right)} \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4t}\left(\sigma+1\right)} \prod_{j=1}^{n} \left|x_j\right|^{\sigma+2} dx$$

and $\mu = 1 - n\sigma$ is fulfilled.

By Lemma 1.28 with $\delta = \nu = 2$ we obtain for any odd function ϕ

$$\left\| \left|\cdot\right|^{b} \left(\mathcal{G}\left(t\right)\phi - \vartheta G_0\left(t\right)\right) \right\|_{\mathbf{L}^q} \leq C t^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-\frac{a-b}{2}} \left\| \left|\cdot\right|^{a} \phi \right\|_{\mathbf{L}^1} \tag{5.35}$$

for all $t > 0$, where $\vartheta = \int_{\mathbf{R}^n} \phi\left(x\right) \prod_{j=1}^{n} x_j dx$, $1 \leq r \leq q \leq \infty$, $\beta \geq 0$, $0 \leq b \leq a$. Now we prove the following auxiliary result.

Lemma 5.11. *Let the odd function $\phi\left(t,x\right)$ have a zero mean value*

$$\int_{\mathbf{R}^n} \phi\left(t,x\right) \prod_{j=1}^{n} x_j dx = 0.$$

Then the following inequalities are valid

$$\left\| \langle t\rangle^{\nu} \int_0^t \mathcal{G}\left(t-\tau\right)\phi\left(\tau\right)\tau^{-\mu}\langle\tau\rangle^{-\nu} d\tau \right\|_{\mathbf{X}} \leq C \left\|\phi\right\|_{\mathbf{Y}}$$

and

$$\left\| \int_0^t \mathcal{G}\left(t-\tau\right)\phi\left(\tau\right)\tau^{-\mu} d\tau \right\|_{\widetilde{\mathbf{X}}} \leq C \left\|\phi\right\|_{\widetilde{\mathbf{Y}}}$$

for all $t > 0$, $0 \leq \nu < \frac{a}{2}$, provided that the right-hand sides are finite.

Proof. By estimates (5.35) and (1.2) we get

$$\left\| \int_0^t \mathcal{G}\left(t-\tau\right)\phi\left(\tau\right)\tau^{-\mu}\langle\tau\rangle^{-\nu} d\tau \right\|_{\mathbf{L}^{1,a}} \leq \int_0^t \left\|\phi\left(\tau\right)\right\|_{\mathbf{L}^{1,a}} \tau^{-\mu}\langle\tau\rangle^{-\nu} d\tau$$

$$\leq \int_0^t \left\{\tau\right\}^{-\frac{n}{2p}\sigma} \tau^{-\mu} \langle\tau\rangle^{\frac{a-n}{2}-n\sigma-\nu} d\tau \sup_{t>0} \left\{t\right\}^{\frac{n}{2p}\sigma} \langle t\rangle^{n\sigma-\frac{a-n}{2}} \left\|\phi\left(t\right)\right\|_{\mathbf{L}^{1,a}}$$

$$\leq C \langle t\rangle^{\frac{a-n}{2}-\nu} \left\|\phi\right\|_{\mathbf{Y}}$$

for all $t>0$, since $\vartheta=0$ and $\frac{n}{2p}\sigma+\mu=1-n\sigma\left(1-\frac{1}{2p}\right)<1$. Similarly by virtue of (5.35), we have (5.29) for all $0<t\leq 1$, and

$$\left\|\int_0^t \mathcal{G}(t-\tau)\phi(\tau)\tau^{-\mu}\langle\tau\rangle^{-\nu}d\tau\right\|_{\mathbf{L}^p} \leq \int_{\frac{t}{2}}^t \|\phi(\tau)\|_{\mathbf{L}^p}\langle\tau\rangle^{-\mu-\nu}d\tau$$
$$+\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{a}{2}}\|\phi(\tau)\|_{\mathbf{L}^{1,a}}\tau^{-\mu}\langle\tau\rangle^{-\nu}d\tau$$
$$\leq \int_{\frac{t}{2}}^t \tau^{-1-\nu-n+\frac{n}{2p}}d\tau\sup_{t>0}\{t\}^{\frac{n}{2p}\sigma}\langle t\rangle^{n\sigma+n-\frac{n}{2p}}\|\phi(t)\|_{\mathbf{L}^p}$$
$$+t^{-n+\frac{n}{2p}-\frac{a-n}{2}}\int_0^{\frac{t}{2}}\{\tau\}^{-\frac{n}{2p}\sigma-\mu}\langle\tau\rangle^{\frac{a-n}{2}-1-\nu}d\tau$$
$$\times\sup_{t>0}\{t\}^{\frac{n}{2p}\sigma}\langle t\rangle^{n\sigma-\frac{a-n}{2}}\|\phi(\tau)\|_{\mathbf{L}^{1,a}}$$
$$\leq C\langle t\rangle^{-\nu-n+\frac{n}{2p}}\|\phi\|_{\mathbf{Y}}$$

for all $t>1$. As above we find (5.30) for all $0<t\leq 1$, and

$$\left\|\int_0^t \mathcal{G}(t-\tau)\phi(\tau)\tau^{-\mu}\langle\tau\rangle^{-\nu}d\tau\right\|_{\mathbf{L}^\infty} \leq \int_{\frac{t}{2}}^t \|\phi(\tau)\|_{\mathbf{L}^\infty}\langle\tau\rangle^{-\mu-\nu}d\tau$$
$$+\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{n}{2}-\frac{a}{2}}\|\phi(\tau)\|_{\mathbf{L}^{1,a}}\tau^{-\mu}\langle\tau\rangle^{-\nu}d\tau$$
$$\leq \int_{\frac{t}{2}}^t \tau^{-1-\nu-n}d\tau\sup_{t>0}\{t\}^{\frac{n}{2p}(\sigma+1)}\langle t\rangle^{nn\sigma+n}\|\phi(t)\|_{\mathbf{L}^\infty}$$
$$+t^{-\frac{n}{2}-\frac{a}{2}}\int_0^{\frac{t}{2}}\{\tau\}^{-\frac{n}{2p}\sigma-\mu}\langle\tau\rangle^{\frac{a-n}{2}-1-\nu}d\tau\sup_{t>0}\{t\}^{\frac{n}{2p}\sigma}\langle t\rangle^{n\sigma-\frac{a-n}{2}}\|\phi(\tau)\|_{\mathbf{L}^{1,a}}$$
$$\leq C\langle t\rangle^{-\nu-n}\|\phi\|_{\mathbf{Y}}$$

for all $t>1$.

In the same manner we obtain estimates in the norms $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{Y}}$

$$\left\||\cdot|^a\int_0^t \mathcal{G}(t-\tau)\phi(\tau)\tau^{-\mu}d\tau\right\|_{\mathbf{L}^{1,n}}$$
$$\leq \int_0^t \tau^{\frac{a-n}{2}-n\sigma-\mu}d\tau\sup_{t>0}t^{n\sigma-\frac{a-n}{2}}\||\cdot|^a\phi(t)\|_{\mathbf{L}^1}\leq Ct^{\frac{a-n}{2}}\|\phi\|_{\widetilde{\mathbf{Y}}}$$

for all $t>0$, since $n\sigma+\mu-\frac{a-n}{2}=1-\frac{a-n}{2}<1$. Also in view of estimates (5.35) with $r=\infty$ we find

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}(t-\tau)\,\phi(\tau)\,\tau^{-\mu} d\tau \right\|_{\mathbf{L}^\infty} \\
&\le \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}-\frac{a}{2}} \left\| |\cdot|^a \phi(\tau) \right\|_{\mathbf{L}^1} \tau^{-\mu} d\tau + \int_{\frac{t}{2}}^t \|\phi(\tau)\|_{\mathbf{L}^\infty} \tau^{-\mu} d\tau \\
&\le C \|\phi\|_{\widetilde{\mathbf{Y}}} \left(\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}-\frac{a}{2}} \tau^{-n\sigma-\mu+\frac{a-n}{2}} d\tau + \int_{\frac{t}{2}}^t \tau^{-n(\sigma+1)-\mu} d\tau \right) \\
&\le C t^{-n} \|\phi\|_{\widetilde{\mathbf{Y}}}
\end{aligned}$$

for all $t > 0$, since by changing the variable of integration $\tau = tz$ we get

$$\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}-\frac{a}{2}} \tau^{\frac{a-n}{2}-1} d\tau = t^{-n} \int_0^{\frac{1}{2}} (1-z)^{-\frac{n}{2}-\frac{a}{2}} z^{\frac{a-n}{2}-1} dz \le C t^{-n}.$$

Hence, the result of the lemma follows, and Lemma 5.11 is proved.

Now by interpolation inequality (1.4) we get (5.31); hence condition (5.4) with $\alpha = \frac{n}{2p}\sigma < 1$ is true. By estimates (1.24) as above we have (5.32). Therefore by Lemma 5.8 we see that conditions (5.5), (5.14), (5.15) and (5.16) are fulfilled. Consider now the rest condition (5.12). It is a consequence of the second estimate of Lemma 5.8 and the estimates

$$\begin{aligned}
&\|\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau))\|_{\mathbf{L}^{1,a}} \le C \|w-v\|_{\mathbf{L}^{1,a}} \left(\|w\|_{\mathbf{L}^\infty}^\sigma + \|v\|_{\mathbf{L}^\infty}^\sigma \right) \\
&\le C\tau^{\frac{a-n}{2}-n\sigma} \|w-v\|_{\widetilde{\mathbf{X}}} \left(\|w\|_{\widetilde{\mathbf{X}}}^\sigma + \|v\|_{\widetilde{\mathbf{X}}}^\sigma \right)
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau))\|_{\mathbf{L}^\infty} \le C \|w-v\|_{\mathbf{L}^\infty} \left(\|w\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^\infty} \right)^\sigma \\
&\le C\tau^{-n(\sigma+1)} \|w-v\|_{\widetilde{\mathbf{X}}} \left(\|w\|_{\widetilde{\mathbf{X}}}^\sigma + \|v\|_{\widetilde{\mathbf{X}}}^\sigma \right).
\end{aligned}$$

Thus we obtain (5.33); hence estimate (5.34) follows. Now (5.12) follows from the second estimate of Lemma 5.8. Applying Theorems 5.2 - 5.5 we get the result of Theorem 5.10 which is then proved.

Example 5.12. **Burgers type equations with initial data having zero mean value**

Now we consider the Cauchy problem for the Burgers type equation with initial data having zero mean value $\int_{\mathbf{R}^n} u_0(x)\,dx = 0$. By the rotation in the x - plane we can always transform the Burgers type equation to the form, where the vector $\lambda = (-1, 0, ..., 0)$. Therefore, we consider the problem

$$\begin{cases} u_t - \Delta u - \partial_{x_1}\left(|u|^{\sigma+1}\right) = 0,\ x \in \mathbf{R}^n,\ t > 0, \\ u(0,x) = u_0(x),\ x \in \mathbf{R}^n, \end{cases} \tag{5.36}$$

where $\sigma > 0$.

We choose the asymptotic kernel

$$G_0(t,x) = \frac{x_1}{(4\pi)^{\frac{n}{2}}(t+1)^{1+\frac{n}{2}}} e^{-\frac{|x|^2}{4(t+1)}}$$

and the functional $f(\phi) \equiv \int_{\mathbf{R}^n} \phi(x) x_1 dx$. Denote $g(t) = 1+\theta^{\sigma+1}\eta \log(1+t)$,

$$\eta = |\lambda| (4\pi)^{-\frac{n}{2}(1+\sigma)} \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4}(1+\sigma)} |x_1|^{\sigma+1} dx > 0,$$

and

$$\theta = \int_{\mathbf{R}^n} u_0(x) x_1 dx.$$

Theorem 5.13. *Let $\sigma < \frac{1}{n+1}$ such that $\mu = 1-(n+1)\sigma > 0$ is sufficiently small. Assume that the initial data $u_0 \in \mathbf{L}^p(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$ with $a \in (1,2]$ and $p > 1$, have a small norm $\|u_0\|_{\mathbf{L}^{1,a}} + \|u_0\|_{\mathbf{L}^p} \le \varepsilon$ and the moment $\theta \ge C\varepsilon > 0$ with some $C > 0$. Also suppose that u_0 have zero mean value $\int_{\mathbf{R}^n} u_0(x) dx = 0$ and the moments $\int_{\mathbf{R}^n} x_j u_0(x) dx = 0$, $j \ne 1$, $\int_{\mathbf{R}^n} x_1 u_0(x) dx > 0$. Then the Cauchy problem for the Burgers type equation (5.36) has a unique global solution $u(t,x) \in \mathbf{C}\left([0,\infty); \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^p(\mathbf{R}^n)\right) \cap \mathbf{C}((0,\infty); \mathbf{L}^\infty(\mathbf{R}^n))$ satisfying the asymptotics*

$$u(t,x) = At^{-\frac{1}{\sigma}} V\left(\frac{x}{\sqrt{t}}\right) + O\left(t^{-\frac{1}{\sigma}-\gamma}\right),$$

for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$, where $A \in \mathbf{R}$ and $V(\xi)$ is the solution of the integral equation

$$V(\xi) = \widetilde{G_0}(\xi) - \frac{\mu}{\beta}\int_0^1 \frac{dz}{z(1-z)^{\frac{n}{2}}}\int_{\mathbf{R}^n} \widetilde{G}\left(\frac{\xi - y\sqrt{z}}{\sqrt{1-z}}\right) K(y)\,dy,$$

where $\beta = \sigma\int_{\mathbf{R}^n} |V(y)|^{\sigma+1}\,dy,$

$$\widetilde{G_0}(x) = \frac{x_1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4}},\ \widetilde{G}(x) = \frac{1}{2^n \pi^{\frac{n}{2}}} e^{-\frac{|x|^2}{4}}$$

and

$$K(y) = |V(y)|^{\sigma+1} - V(y)\int_{\mathbf{R}^n} |V(\xi)|^{\sigma+1}\,d\xi.$$

Proof. To apply Theorems 5.2 - 5.5 we choose the space

$$\mathbf{Z} = \left\{\phi \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^p(\mathbf{R}^n)\right\},$$

where now $a \in (1,2]$, and $p > 1$ and the space

$$\mathbf{X} = \left\{\phi \in \mathbf{C}([0,\infty);\mathbf{Z}) \cap \mathbf{C}\left((0,\infty);\mathbf{W}^1_\infty(\mathbf{R}^n)\right) : \ \|\phi\|_{\mathbf{X}} < \infty\right\},$$

where the norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t\geq 0}\left(\langle t\rangle^{-\frac{a-1}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} + \langle t\rangle^{\frac{n+1}{2}-\frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^p}\right)$$
$$+ \sup_{t>0}\left(\{t\}^{\frac{n}{2p}} \langle t\rangle^{\frac{n+1}{2}} \|\phi(t)\|_{\mathbf{L}^\infty} + \{t\}^{\frac{n}{2p}+\frac{1}{2}} \langle t\rangle^{\frac{n}{2}+1} \|\nabla\phi(t)\|_{\mathbf{L}^\infty}\right).$$

Also we consider the norm

$$\|\phi\|_{\mathbf{Y}} = \sup_{t>0}\{t\}^{\frac{n\sigma}{2p}+\frac{1}{2}} \langle t\rangle^{\frac{(n+1)\sigma}{2}+\frac{1}{2}}\left(\langle t\rangle^{-\frac{a-1}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}}\right.$$
$$\left.+ \langle t\rangle^{\frac{n+1}{2}-\frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^p} + \{t\}^{\frac{n}{2p}} \langle t\rangle^{\frac{n+1}{2}} \|\phi(t)\|_{\mathbf{L}^\infty}\right).$$

We also define the space

$$\widetilde{\mathbf{X}} = \left\{\phi \in \mathbf{C}\left([0,\infty);\mathbf{L}^{1,a}(\mathbf{R}^n)\right) \cap \mathbf{C}((0,\infty);\mathbf{L}^\infty(\mathbf{R}^n)) :\right.$$
$$\left.\|\phi\|_{\widetilde{\mathbf{X}}} < \infty\right\},$$

where the norm

$$\|\phi\|_{\widetilde{\mathbf{X}}} = \sup_{t>0}\left(t^{-\frac{a-1}{2}} \left\||\cdot|^a \phi(t)\right\|_{\mathbf{L}^1} + t^{\frac{n+1}{2}} \|\phi(t)\|_{\mathbf{L}^\infty}\right).$$

Also we consider the norm for the nonlinearity $\|\phi\|_{\widetilde{\mathbf{Y}}} = \left\|t^{\frac{(n+1)\sigma}{2}+\frac{1}{2}}\phi(t)\right\|_{\widetilde{\mathbf{X}}}$.

By a direct computation we see that

$$\operatorname{Re} f\left(\mathcal{N}\left(\theta G_0\left(t\right)\right)\right) d\tau = \left|\lambda\right| \theta^{\sigma+1} \int_{\mathbf{R}^n} \left|G_0\left(t,x\right)\right|^{\sigma+1} dx$$
$$= \left|\lambda\right| \theta^{\sigma+1} \left(4\pi\right)^{-\frac{n}{2}\left(1+\sigma\right)} t^{-\left(1+\sigma\right)\left(1+\frac{n}{2}\right)} \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4t}\left(\sigma+1\right)} \left|x_1\right|^{\sigma+1} dx$$
$$= \eta \theta^{\sigma+1} t^{-\frac{1}{2}-\frac{n+1}{2}\sigma};$$

hence condition (5.2) with

$$\eta = \left|\lambda\right| \left(4\pi\right)^{-\frac{n}{2}\left(\sigma+1\right)} \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4t}\left(\sigma+1\right)} \left|x_1\right|^{\sigma+1} dx$$

and $\mu = \frac{1}{2} - \frac{n+1}{2}\sigma$ is fulfilled.

By Lemma 1.28 with $\delta = \nu = 2$ for the functions having zero mean value $\int_{\mathbf{R}^n} \phi\left(x\right) dx = 0$ and the moments $\int_{\mathbf{R}^n} x_j \phi\left(x\right) dx = 0$, $j \neq 1$, we get

$$\left\| \left|\cdot\right|^b \left(\mathcal{G}\left(t\right)\phi - \vartheta G_0\left(t\right)\right)\right\|_{\mathbf{L}^q} \leq C t^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-\frac{a-b}{2}} \left\| \left|\cdot\right|^a \phi\right\|_{\mathbf{L}^1} \tag{5.37}$$

for all $t > 0$, where $\vartheta = \int_{\mathbf{R}^n} \phi\left(x\right) x_1 dx$, $1 \leq r \leq q \leq \infty$, $0 \leq b \leq a$.

Now we prove the following auxiliary result.

Lemma 5.14. *Let the function $\phi\left(t,x\right)$ have a zero mean value $\int_{\mathbf{R}^n} \phi(t,x)dx = 0$ and zero first moments*

$$\int_{\mathbf{R}^n} x_j \phi\left(t,x\right) dx = 0.$$

Then the following inequalities are valid

$$\left\| \langle t\rangle^{\nu} \int_0^t \mathcal{G}\left(t-\tau\right)\phi\left(\tau\right)\tau^{-\mu}\langle\tau\rangle^{-\nu} d\tau \right\|_{\mathbf{X}} \leq C\left\|\phi\right\|_{\mathbf{Y}}$$

and

$$\left\| \int_0^t \mathcal{G}\left(t-\tau\right)\phi\left(\tau\right)\tau^{-\mu} d\tau \right\|_{\widetilde{\mathbf{X}}} \leq C\left\|\phi\right\|_{\widetilde{\mathbf{Y}}}$$

for all $t > 0$, $0 \leq \nu < \frac{a}{2}$, provided that the right-hand sides are finite.

Proof. Since $\vartheta = 0$, by estimates (5.37) and (1.2) we get

$$\left\| \int_0^t \mathcal{G}\left(t-\tau\right)\phi\left(\tau\right)\tau^{-\mu}\langle\tau\rangle^{-\nu} d\tau \right\|_{\mathbf{L}^{1,a}} \leq \int_0^t \left\|\phi\left(\tau\right)\right\|_{\mathbf{L}^{1,a}} \tau^{-\mu}\langle\tau\rangle^{-\nu} d\tau$$
$$\leq \int_0^t \{\tau\}^{-\frac{n}{2p}\sigma} \tau^{-\mu} \langle\tau\rangle^{\frac{a-1}{2}-\frac{(n+1)\sigma+1}{2}-\nu} d\tau$$
$$\times \sup_{t>0} \{t\}^{\frac{n}{2p}\sigma} \langle t\rangle^{\frac{(n+1)\sigma}{2}+\frac{1}{2}-\frac{a-1}{2}} \left\|\phi\left(t\right)\right\|_{\mathbf{L}^{1,a}}$$
$$\leq C\langle t\rangle^{\frac{a-1}{2}-\nu}\left\|\phi\right\|_{\mathbf{Y}}$$

for all $t > 0$, since $\frac{n}{2p}\sigma + \mu = \frac{n}{2p}\sigma + \frac{1}{2} - \frac{n+1}{2}\sigma < 1$. In the same manner by virtue of (5.37) we have

$$\left\| \int_0^t \mathcal{G}(t-\tau)\phi(\tau)\tau^{-\mu}\langle\tau\rangle^{-\nu} d\tau \right\|_{\mathbf{L}^p} \le \int_{\frac{t}{2}}^t \|\phi(\tau)\|_{\mathbf{L}^p}\langle\tau\rangle^{-\mu-\nu} d\tau$$

$$+ \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{a}{2}} \|\phi(\tau)\|_{\mathbf{L}^{1,a}} \tau^{-\mu}\langle\tau\rangle^{-\nu} d\tau$$

$$\le \int_{\frac{t}{2}}^t \tau^{-1-\nu-\frac{n+1}{2}+\frac{n}{2p}} d\tau \sup_{t>0}\{t\}^{\frac{n}{2p}\sigma}\langle t\rangle^{\frac{(n+1)\sigma+1}{2}+\frac{n+1}{2}-\frac{n}{2p}} \|\phi(t)\|_{\mathbf{L}^p}$$

$$+ t^{-\frac{n+1}{2}+\frac{n}{2p}-\frac{a-1}{2}} \int_0^{\frac{t}{2}} \{\tau\}^{-\frac{n}{2p}\sigma-\mu}\langle\tau\rangle^{\frac{a-1}{2}-1-\nu} d\tau$$

$$\times \sup_{t>0}\{t\}^{\frac{n}{2p}\sigma}\langle t\rangle^{\frac{(n+1)\sigma+1}{2}-\frac{a-1}{2}} \|\phi(\tau)\|_{\mathbf{L}^{1,a}}$$

$$\le C\langle t\rangle^{-\nu-\frac{n+1}{2}+\frac{n}{2p}} \|\phi\|_{\mathbf{Y}}$$

for all $t > 1$. As above we find

$$\left\| \int_0^t \mathcal{G}(t-\tau)\phi(\tau)\tau^{-\mu}\langle\tau\rangle^{-\nu} d\tau \right\|_{\mathbf{L}^\infty}$$

$$\le \int_{\frac{t}{2}}^t \tau^{-1-\nu-\frac{n+1}{2}} d\tau \sup_{t>0}\{t\}^{\frac{n}{2p}(\sigma+1)}\langle t\rangle^{\frac{(n+1)\sigma+1}{2}+\frac{n+1}{2}} \|\phi(t)\|_{\mathbf{L}^\infty}$$

$$+ t^{-\frac{n}{2}-\frac{a}{2}} \int_0^{\frac{t}{2}} \{\tau\}^{-\frac{n}{2p}\sigma-\mu}\langle\tau\rangle^{\frac{a-1}{2}-1-\nu} d\tau \sup_{t>0}\{t\}^{\frac{n}{2p}\sigma}\langle t\rangle^{\frac{(n+1)\sigma+1}{2}-\frac{a-1}{2}} \|\phi(\tau)\|_{\mathbf{L}^{1,a}}$$

$$\le C\langle t\rangle^{-\nu-\frac{n+1}{2}} \|\phi\|_{\mathbf{Y}}$$

for all $t > 1$.

Furthermore we obtain estimates in the norms $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{Y}}$

$$\left\| |\cdot|^a \int_0^t \mathcal{G}(t-\tau)\phi(\tau)\tau^{-\mu} d\tau \right\|_{\mathbf{L}^{1,n}}$$

$$\le \int_0^t \tau^{\frac{a-1}{2}-\frac{(n+1)\sigma+1}{2}-\mu} d\tau \sup_{t>0} t^{\frac{(n+1)\sigma+1}{2}-\frac{a-1}{2}} \left\| |\cdot|^a \phi(t) \right\|_{\mathbf{L}^1} \le Ct^{\frac{a-1}{2}} \|\phi\|_{\widetilde{\mathbf{Y}}}$$

for all $t > 0$, since $\frac{(n+1)\sigma+1}{2} - \frac{a-1}{2} + \mu = 1 - \frac{a-1}{2} < 1$. In addition by estimates (5.37) with $r = \infty$ we find

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}(t-\tau)\,\phi(\tau)\,\tau^{-\mu} d\tau \right\|_{\mathbf{L}^\infty} \\
&\le \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}-\frac{a}{2}} \left\| |\cdot|^a \phi(\tau) \right\|_{\mathbf{L}^1} \tau^{-\mu} d\tau + \int_{\frac{t}{2}}^t \|\phi(\tau)\|_{\mathbf{L}^\infty} \tau^{-\mu} d\tau \\
&\le C \|\phi\|_{\widetilde{\mathbf{Y}}} \left(\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}-\frac{a}{2}} \tau^{\frac{a-1}{2}-\frac{(n+1)\sigma+1}{2}-\mu} d\tau + \int_{\frac{t}{2}}^t \tau^{-\frac{(n+1)(\sigma+1)}{2}-\frac{1}{2}-\mu} d\tau \right) \\
&\le C t^{-\frac{n+1}{2}} \|\phi\|_{\widetilde{\mathbf{Y}}}
\end{aligned}$$

for all $t > 0$. Hence, the result of the lemma follows, and Lemma 5.14 is proved.

Now by interpolation inequality (1.4) we get (5.31); hence condition (5.4) with $\alpha = \frac{n}{2p}\sigma < 1$ is true. By estimates (1.24) in the same manner as above we have (5.32). Therefore by Lemma 5.8 we see that conditions (5.5), (5.14), (5.15) and (5.16) are fulfilled. The rest condition (5.12) is a consequence of the second estimate of Lemma 5.14 and the estimates

$$\begin{aligned}
&\|\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau))\|_{\mathbf{L}^{1,a}} \\
&\le C\tau^{\frac{a-1}{2}-\frac{(n+1)\sigma}{2}-\frac{1}{2}} \|w-v\|_{\widetilde{\mathbf{X}}} \left(\|w\|_{\widetilde{\mathbf{X}}}^\sigma + \|v\|_{\widetilde{\mathbf{X}}}^\sigma \right)
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau))\|_{\mathbf{L}^\infty} \\
&\le C\tau^{-\frac{(n+1)(\sigma+1)}{2}+\frac{1}{2}} \|w-v\|_{\widetilde{\mathbf{X}}} \left(\|w\|_{\widetilde{\mathbf{X}}}^\sigma + \|v\|_{\widetilde{\mathbf{X}}}^\sigma \right).
\end{aligned}$$

Thus (5.33) is true; hence we get (5.34). Now (5.12) follows from the second estimate of Lemma 5.14. Applying Theorems 5.2 - 5.5 we get the result of Theorem 5.13 which is then proved.

5.2 Fractional heat equations

In this section we study the Cauchy problem for the nonlinear dissipative equations with fractional power of the negative Laplacian and complex coefficients

$$\begin{cases} \partial_t u + \alpha(-\Delta)^{\frac{\rho}{2}} u + \beta |u|^\sigma u + \gamma |u|^\kappa u = 0, \ x \in \mathbf{R}^n, t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^n, \end{cases} \tag{5.38}$$

where $\alpha, \beta, \gamma \in \mathbf{C}$, $\operatorname{Re}\alpha > 0$, $\rho > 0$, $\kappa > \sigma > 0$. Furthermore we assume that $\operatorname{Re}\beta\delta(\alpha,\rho,\sigma) > 0$, where

$$\delta(\alpha,\rho,\sigma) = \int_{\mathbf{R}^n} |G(x)|^\sigma G(x)\,dx, G(x) = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-\alpha|\xi|^\rho}\right).$$

5.2.1 Small initial data

In this subsection we concentrate our attention on the subcritical case $0 < \sigma < \frac{\rho}{n}$ and prove global in time existence of small solutions to the Cauchy problem (5.38). Denote $\eta = \rho \operatorname{Re} \beta\delta(\alpha, \rho, \sigma)$,

$$\chi_\sigma(t) = 1 + \frac{n\sigma |\theta|^\sigma \eta}{\rho - n\sigma} t^{1 - \frac{n\sigma}{\rho}}.$$

Theorem 5.15. *Assume that* $\operatorname{Re}\alpha > 0$, $0 < \sigma < \kappa \le \frac{\rho}{n}$ *and*

$$u_0 \in \mathbf{L}^\infty(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n), \ a \in (0, \min(1, \rho)).$$

Furthermore suppose that

$$\operatorname{Re}\beta\delta(\alpha,\rho,\sigma) > 0, \ |\hat{u}_0(0)| = \theta (2\pi)^{-\frac{n}{2}} > 0.$$

Then there exists a positive ε *such that if* $\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^{1,a}} \le \varepsilon$, $|\hat{u}_0(0)| \ge C\varepsilon$ *and the value* σ *is close to* $\frac{\rho}{n}$, *so that* $\frac{\rho}{n} - \sigma \le C\varepsilon^\sigma$, *then the Cauchy problem (5.38) has a unique global solution* $u(t,x) \in \mathbf{C}\left([0,\infty); \mathbf{L}^\infty(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)\right)$ *satisfying the asymptotics*

$$u(t,x) = At^{-\frac{1}{\sigma}} V\left(t^{-\frac{1}{\rho}}(\cdot)\right) e^{i\psi(t)} + O\left(t^{-\frac{1}{\sigma} - \gamma}\right),$$

for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}^n$, *where* $A \in \mathbf{R}$, $\gamma > 0$. *Here* $V(\xi)$ *is the solution of the integral equation*

$$V(\xi) = G(\xi) - \frac{\mu}{\beta} \int_0^1 \frac{dz}{z(1-z)^{\frac{n}{\rho}}} \int_{\mathbf{R}^n} G\left(\frac{\xi - yz^{\frac{1}{\rho}}}{(1-z)^{\frac{1}{\rho}}}\right) K(y)\, dy,$$

where

$$\beta = \sigma \operatorname{Re} \int_{\mathbf{R}^n} |V(y)|^\sigma V(y)\, dy,$$

$\mu = 1 - \frac{n}{\rho}\sigma$, *and*

$$K(y) = |V(y)|^\sigma V(y) - V(y) \int_{\mathbf{R}^n} |V(\xi)|^\sigma V(\xi)\, d\xi.$$

The function $\psi(t)$ *satisfies the estimate*

$$\left| \psi(t) - \arg \widehat{u}_0(0) + |\theta|^\sigma \widetilde{\eta} \int_0^t \chi_\sigma^{-\frac{1}{\sigma}}(\tau)(1+\tau)^{-\frac{\sigma}{\rho}n}\, d\tau \right|$$
$$\le C\varepsilon^{1+2\sigma} \int_0^t \chi_\sigma^{-\frac{1}{\sigma}}(\tau)(1+\tau)^{-\frac{\sigma}{\rho}n}\, d\tau,$$

where $\widetilde{\eta} = \operatorname{Im} \beta\delta(\alpha,\rho,\sigma)$.

Remark 5.16. Our proof of Theorem 5.15 depends on the positivity of the value $\operatorname{Re} \beta\delta(\alpha, \rho, \sigma)$, which we need to derive better time decay properties of solutions (see Lemma 5.17 below). If $\rho = 2$ then we can calculate explicitly the value of $\delta(\alpha, \rho, \sigma)$ (see also Hayashi et al. [2003a], Hayashi et al. [2003b])

$$\delta(\alpha, 2, \sigma) = \frac{2^{\frac{n}{2}}}{(4\pi)^{\frac{n}{2}\sigma}} \left(\frac{|\alpha|^{n-\frac{n}{2}\sigma}}{\left((2+\sigma)|\alpha|^2 + \sigma\alpha^2\right)^{\frac{n}{2}}} \right).$$

We do not need the positivity of the kernel $G(x) = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-\alpha|\xi|^\rho}\right)$ of the Green operator which was essentially used in previously proving the blow-up results. The condition that the value σ should be close to $\frac{\rho}{n}$, so that $\frac{\rho}{n} - \sigma \le C\varepsilon^\sigma$ is rather technical and is caused by the application of the contraction mapping principle for proving global existence of solutions.

Proof of Theorem 5.15

Define the Green operator $\mathcal{G}(t)$ by

$$\mathcal{G}(t)\phi = \overline{\mathcal{F}}_{\xi\to x} e^{-\alpha|\xi|^\rho t}\hat{\phi}(\xi) = t^{-\frac{n}{\rho}} \int_{\mathbf{R}^n} G\left(t^{-\frac{1}{\rho}}(x-y)\right)\phi(y)dy \tag{5.39}$$

with a kernel $G(x) = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-\alpha|\xi|^\rho}\right)$.

As in the proof of Theorem 5.2 we change the dependent variable $u(t,x) = v(t,x)e^{-\varphi(t)+i\psi(t)}$ as above. Then for the new function $v(t,x)$ we get the following equation

$$\partial_t v + \alpha(-\Delta)^{\frac{\rho}{2}} v + \beta e^{-\sigma\varphi}|v|^\sigma v + \gamma e^{-\kappa\varphi}|v|^\kappa v - (\varphi' - i\psi')v = 0.$$

We assume that

$$\int_{\mathbf{R}^n} \left(\beta e^{-\sigma\varphi}|v|^\sigma v + \gamma e^{-\kappa\varphi}|v|^\kappa v - (\varphi' - i\psi')v\right) dx = 0,$$

then the mean value of new function $v(t,x)$ satisfies a conservation law:

$$\frac{d}{dt}\int_{\mathbf{R}^n} v(t,x)\,dx = 0.$$

Hence $\widehat{v}(t,0) = \widehat{v}_0(0)$ for all $t > 0$. We can choose $\varphi(0) = 0$ and $\psi(0)$ such that $\hat{v}_0(0) = |\hat{u}_0(0)| = \theta(2\pi)^{-\frac{n}{2}} > 0$. Thus we consider the Cauchy problem for the new dependent variables $(v(t,x), \varphi(t))$

$$\begin{cases} \partial_t v + \alpha(-\Delta)^{\frac{\rho}{2}} v = -\beta e^{-\sigma\varphi}\left(|v|^\sigma - \frac{1}{\theta}\int_{\mathbf{R}^n}|v|^\sigma v dx\right) v \\ \quad -\gamma e^{-\kappa\varphi}\left(|v|^\kappa - \frac{1}{\theta}\int_{\mathbf{R}^n}|v|^\kappa v dx\right) v, \\ \partial_t\varphi(t) = \frac{1}{\theta}e^{-\sigma\varphi}\left(\operatorname{Re}\beta\int_{\mathbf{R}^n}|v|^\sigma v dx + e^{(\sigma-\kappa)\varphi}\operatorname{Re}\gamma\int_{\mathbf{R}^n}|v|^\kappa v dx\right), \\ v(0,x) = v_0(x),\ \varphi(0) = 0. \end{cases} \tag{5.40}$$

We denote $h(t) = e^{\sigma\varphi(t)}$ and write (5.40) as

$$\begin{cases} \partial_t v + \alpha (-\Delta)^{\frac{\rho}{2}} v = F(v,h), \ v(0,x) = v_0(x), \\ \partial_t h = \frac{\sigma}{\theta} \left(\operatorname{Re}\beta \int_{\mathbf{R}^n} |v|^\sigma v dx + h^{1-\frac{\kappa}{\sigma}} \operatorname{Re}\gamma \int_{\mathbf{R}^n} |v|^\kappa v dx \right), \ h(0) = 1, \end{cases} \tag{5.41}$$

where

$$F(v,h) = -\beta h^{-1} \left(|v|^\sigma - \frac{1}{\theta} \int_{\mathbf{R}^n} |v|^\sigma v dx \right) v$$
$$-\gamma h^{-\frac{\kappa}{\sigma}} \left(|v|^\kappa - \frac{1}{\theta} \int_{\mathbf{R}^n} |v|^\kappa v dx \right) v.$$

We note that the mean value of the nonlinearity $\widehat{F(v,h)}(t,0) = 0$ for all $t > 0$. It is expected that the second summand $\gamma h^{-\frac{\kappa}{\sigma}} |v|^\kappa v$ of the nonlinearity $F(v,h)$ decays in time more rapidly than the first one $\beta h^{-1} |v|^\sigma v$. Denote by

$$\mathbf{X} = \left\{ \phi \in C([0,\infty), \mathbf{L}^\infty(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)) \right\},$$

with norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0} \left((1+t)^{\frac{n}{\rho}} \|\phi(t)\|_{\mathbf{L}^\infty} + \|\phi(t)\|_{\mathbf{L}^1} + (1+t)^{-\frac{a}{\rho}} \|\phi(t)\|_{\mathbf{L}^{1,a}} \right),$$

where $a \in (0, \min(1,\rho))$. Also we define

$$\mathbf{Z} = \left\{ \phi \in \mathbf{L}^\infty \cap \mathbf{L}^{1,a} \right\},$$

with norm

$$\|\phi\|_{\mathbf{Z}} = \|\phi(t)\|_{\mathbf{L}^\infty} + \|\phi(t)\|_{\mathbf{L}^{1,a}}.$$

As in the proof of Lemma 5.8 we get the following result

Lemma 5.17. *Let the function $\phi(t,x)$ have a zero mean value*

$$\int_{\mathbf{R}^n} \phi(t,x)\, dx = 0.$$

Then the following inequalities are valid

$$\left\| \langle t \rangle^\nu \int_0^t \mathcal{G}(t-\tau)\phi(\tau)\tau^{-\mu} \langle \tau \rangle^{-\nu} d\tau \right\|_{\mathbf{X}} \le C \|\phi\|_{\mathbf{Y}}$$

for all $t > 0$, $0 \le \nu < \frac{a}{\rho}$, provided that the right-hand sides are finite.

By virtue of Lemma 3.12 we get that $G \in \mathbf{X}$ is the asymptotic kernel for the Green operator $\mathcal{G}$ in spaces $\mathbf{X}$, $\mathbf{Z}$ and the operator $\mathcal{G}$ is a self-similar asymptotic operator in spaces $\mathbf{X}$, $\mathbf{Z}$ with $\gamma = \frac{a}{\rho}$ and

$$f(\phi) = \int_{\mathbf{R}^n} \phi dx.$$

Denote

$$\chi_\sigma(t) = 1 + \frac{n\sigma |\theta|^\sigma \eta}{\rho - n\sigma} t^{1-\frac{\sigma}{\rho}n} .$$

By a direct calculation we have

$$\begin{aligned}
&\|F(v(t), \chi_\sigma(t)) - F(w(t), \chi_\sigma(t))\|_{\mathbf{L}^\infty} \\
&\le C\chi_\sigma^{-1}(t) \|v - w\|_{\mathbf{L}^\infty} \left(\|v(t)\|_{\mathbf{L}^\infty}^\sigma + \|w(t)\|_{\mathbf{L}^\infty}^\sigma\right) \\
&\times \left(1 + \frac{\|v(t)\|_{\mathbf{L}^1} + \|w(t)\|_{\mathbf{L}^1}}{\theta}\right) \\
&\le C\left(\frac{\rho}{n} - \sigma\right)(1+t)^{-1-\frac{n}{\rho}} \|v - w\|_{\mathbf{X}} \left(\|v(t)\|_{\mathbf{X}}^\sigma + \|w(t)\|_{\mathbf{X}}^\sigma\right) \\
&\times \left(1 + \frac{\|v(t)\|_{\mathbf{X}} + \|w(t)\|_{\mathbf{X}}}{\theta}\right),
\end{aligned} \tag{5.42}$$

$$\begin{aligned}
&\|F(v(t), \chi_\sigma(t)) - F(w(t), \chi_\sigma(t))\|_{\mathbf{L}^1} \\
&\le C\chi_\sigma^{-1}(t) \|v - w\|_{\mathbf{L}^\infty} \left(\|v(t)\|_{\mathbf{L}^\infty}^\sigma + \|w(t)\|_{\mathbf{L}^\infty}^\sigma\right) \\
&\times \left(\|v(t)\|_{\mathbf{L}^1} + \|w(t)\|_{\mathbf{L}^1} + \frac{\|v(t)\|_{\mathbf{L}^1}^2 + \|w(t)\|_{\mathbf{L}^1}^2}{\theta}\right) \\
&\le C\left(\frac{\rho}{n} - \sigma\right)(1+t)^{-1} \|v - w\|_{\mathbf{X}} \left(\|v(t)\|_{\mathbf{X}}^\sigma + \|w(t)\|_{\mathbf{X}}^\sigma\right) \\
&\times \left(\|v(t)\|_{\mathbf{X}} + \|w(t)\|_{\mathbf{X}} + \frac{\|v(t)\|_{\mathbf{X}}^2 + \|w(t)\|_{\mathbf{X}}^2}{\theta}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\|F(v(t), \chi_\sigma(t)) - F(w(t), \chi_\sigma(t))\|_{\mathbf{L}^{1,a}} \\
&\le C\chi_\sigma^{-1}(t) \|v - w\|_{\mathbf{L}^{1,a}} \left(\|v(t)\|_{\mathbf{L}^\infty}^\sigma + \|w(t)\|_{\mathbf{L}^\infty}^\sigma\right) \\
&\times \left(1 + \frac{\|v(t)\|_{\mathbf{L}^1} + \|w(t)\|_{\mathbf{L}^1}}{\theta}\right) \\
&\le C\left(\frac{\rho}{n} - \sigma\right)(1+t)^{-1+\frac{\alpha}{\rho}} \\
&\times \|v - w\|_{\mathbf{X}} \left(\|v(t)\|_{\mathbf{X}}^\sigma + \|w(t)\|_{\mathbf{X}}^\sigma\right) \\
&\times \left(1 + \frac{\|v(t)\|_{\mathbf{X}} + \|w(t)\|_{\mathbf{X}}}{\theta}\right)
\end{aligned}$$

for all $t > 0$. This yields the estimate

$$\begin{aligned}
&\|\langle t\rangle \left(F(v(t), \chi_\sigma(t)) - F(w(t), \chi_\sigma(t))\right)\|_{\mathbf{X}} \\
&\le C\|v - w\|_{\mathbf{X}} \left(\|v(t)\|_{\mathbf{X}}^\sigma + \|w(t)\|_{\mathbf{X}}^\sigma\right) \left(1 + \frac{\|v(t)\|_{\mathbf{X}} + \|w(t)\|_{\mathbf{X}}}{\theta}\right).
\end{aligned} \tag{5.43}$$

Therefore via Lemma 5.17 we get

$$\left\| \int_0^t \mathcal{G}(t-\tau) \left(F\left(v\left(\tau\right), \chi_\sigma\left(\tau\right)\right) - F\left(w\left(\tau\right), \chi_\sigma\left(\tau\right)\right)\right) d\tau \right\|_{\mathbf{X}}$$
$$\leq C \left\| v - w \right\|_{\mathbf{X}} \left(\left\| v\left(t\right)\right\|_{\mathbf{X}}^{\sigma} + \left\| w\left(t\right)\right\|_{\mathbf{X}}^{\sigma}\right) \left(1 + \frac{\left\| v\left(t\right)\right\|_{\mathbf{X}} + \left\| w\left(t\right)\right\|_{\mathbf{X}}}{\theta}\right).$$

Also by a direct calculation we find

$$\left\| \mathcal{N}(v) - \mathcal{N}(w)\right\|_{\mathbf{L}^1}$$
$$\leq C \left\{t\right\}^{-\frac{\sigma n + a}{\rho}} \left\langle t\right\rangle^{-\frac{n\sigma}{\rho}} \left\| v\left(t\right) - w(t)\right\|_{\mathbf{X}} \left(\left\| v\left(t\right)\right\|_{\mathbf{X}}^{\sigma} + \left\| w(t)\right\|_{\mathbf{X}}^{\sigma}\right).$$

Hence if we suppose that $\rho - n\sigma \leq C\varepsilon^{\sigma}$ as in the prove of Theorem 5.2 we obtain a unique solution $v\left(t,x\right) \in \mathbf{X},\ h\left(t\right) = e^{\sigma\varphi(t)} \in \mathbf{C}\left(0,\infty\right)$ satisfying equations

$$\begin{cases} v\left(t\right) = \mathcal{G}\left(t\right) v_0 + \int_0^t \mathcal{G}(t-\tau) F\left(v\left(\tau\right), h\left(\tau\right)\right) d\tau, \\ h\left(t\right) = 1 + \frac{\sigma}{\theta} \int_0^t d\tau \left(\operatorname{Re}\beta \int_{\mathbf{R}^n} \left|v\right|^{\sigma} v dx + h^{1-\frac{\kappa}{\sigma}} \operatorname{Re}\gamma \int_{\mathbf{R}^n} \left|v\right|^{\kappa} v dx\right), \end{cases} \tag{5.44}$$

and estimates

$$\left\| v\left(t\right) - \mathcal{G}\left(t\right) v_0\right\|_{\mathbf{L}^1} \leq C\varepsilon^{1+\sigma}$$

and

$$\left| h\left(t\right) - \chi_\sigma\left(t\right)\right| \leq C\varepsilon^{2\sigma} \chi_\sigma\left(t\right). \tag{5.45}$$

We also have by applying (5.42) to (5.44)

$$\left\| v\left(t\right) - \mathcal{G}\left(t\right) v_0\right\|_{\mathbf{L}^\infty} \leq C\varepsilon^{1+\sigma} \left(1+t\right)^{-\frac{n}{\rho}}, \tag{5.46}$$

and by the definition of ψ we see that

$$\psi\left(t\right) = \arg \widehat{u_0\left(0\right)}$$
$$- \frac{1}{\theta} \int_0^t e^{-\sigma\varphi} \left(\operatorname{Im}\beta \int_{\mathbf{R}^n} \left|v\right|^{\sigma} v dx + e^{(\sigma-\kappa)\varphi} \operatorname{Im}\gamma \int_{\mathbf{R}^n} \left|v\right|^{\kappa} v dx\right).$$

By the time decay property of the solution v we have

$$\left| \psi\left(t\right) - \arg \widehat{u_0\left(0\right)} + \left|\theta\right|^{\sigma} \widetilde{\eta} \int_0^t h^{-\frac{1}{\sigma}}\left(\tau\right) \left(1+\tau\right)^{-\frac{\sigma}{\rho}n} d\tau \right| \leq C\varepsilon^{\kappa} t^{-\frac{\kappa-\sigma}{\rho}n},$$

where $\widetilde{\eta} = \operatorname{Im}\beta\delta\left(\alpha,\rho,\sigma\right)$. Hence by (5.45)

$$\left| \psi\left(t\right) - \arg \widehat{u_0\left(0\right)} + \left|\theta\right|^{\sigma} \widetilde{\eta} \int_0^t \chi_\sigma^{-\frac{1}{\sigma}}\left(\tau\right) \left(1+\tau\right)^{-\frac{\sigma}{\rho}n} d\tau \right|$$
$$\leq C \left|\theta\right|^{\sigma} \widetilde{\eta} \int_0^t \chi_\sigma^{-1-\frac{1}{\sigma}}\left(\tau\right) \left| h\left(\tau\right) - \chi_\sigma\left(\tau\right)\right| \left(1+\tau\right)^{-\frac{\sigma}{\rho}n} d\tau$$
$$\leq C \int_0^t \chi_\sigma^{-\frac{1}{\sigma}}\left(\tau\right) \left(1+\tau\right)^{-\frac{\sigma}{\rho}n} d\tau \tag{5.47}$$

for large $t > 0$. Then via formulas

$$u(t,x) = e^{-\varphi(t)+i\psi(t)}v(t,x) = h^{-\frac{1}{\sigma}}(t)\, e^{i\psi(t)}v(t,x)$$

we find the estimates

$$\begin{aligned}
&\left\| u(t) - \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right) e^{-\varphi(t)+i\psi(t)} \right\|_{\mathbf{L}^\infty} \\
&\leq \left\| u(t) - (\mathcal{G}(t)\, v_0)\, e^{-\varphi(t)+i\psi(t)} \right\|_{\mathbf{L}^\infty} + \\
&\left\| \left(\mathcal{G}(t)\, v_0 - \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right)\right) e^{-\varphi(t)+i\psi(t)} \right\|_{\mathbf{L}^\infty} \\
&\leq C\varepsilon^{1+\sigma}(1+t)^{-\frac{1}{\sigma}}, \qquad (5.48)
\end{aligned}$$

where we have used the estimate

$$\left\| \left(\mathcal{G}(t)\, v_0 - \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right)\right) e^{-\varphi(t)+i\psi(t)} \right\|_{\mathbf{L}^\infty} \leq C t^{-\frac{1}{\sigma}-\frac{a}{\rho}} \|\phi\|_{\mathbf{L}^{1,a}}$$

and (5.46). We also have by (5.45)

$$\begin{aligned}
&\left\| \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right) h^{-\frac{1}{\sigma}}(t)\, e^{i\psi(t)} - \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right) \chi_\sigma^{-\frac{1}{\sigma}}(t)\, e^{i\psi(t)} \right\|_{\mathbf{L}^\infty} \\
&\leq C\varepsilon t^{-\frac{n}{\rho}} \chi_\sigma^{-1-\frac{1}{\sigma}}(t)\, |h(t) - \chi_\sigma(t)|;
\end{aligned}$$

hence, via (5.48) it follows that

$$\begin{aligned}
&\left\| u(t) - \theta t^{-\frac{n}{\rho}} G\left(t^{-\frac{1}{\rho}}(\cdot)\right) \chi_\sigma^{-\frac{1}{\sigma}}(t)\, e^{i\psi(t)} \right\|_{\mathbf{L}^\infty} \\
&\leq C\varepsilon^{1+\sigma}(1+t)^{-\frac{1}{\sigma}}. \qquad (5.49)
\end{aligned}$$

Also by applying Theorem 5.4 and Theorem 5.5, because of estimate (5.43) and Lemma 5.17 we get the result of Theorem 5.15.

5.2.2 Large initial data

This subsection is devoted to the study of the Cauchy problem (5.38), when the initial data are not small. For simplicity we take $\alpha = 1$, $\beta = 1$ and $\gamma = 0$, and therefore, we consider the following problem

$$\begin{cases} u_t + (-\Delta)^{\frac{\rho}{2}} u + |u|^\sigma u = 0, \ x \in \mathbf{R}^n, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^n, \end{cases} \qquad (5.50)$$

where $\rho \in (0,2]$. Let $u_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n)$, $u_0(x) \geq 0$ almost everywhere in $\mathbf{R}^n$, then $\theta = \int_{\mathbf{R}^n} u_0(x)\, dx > 0$. We are interested in the global in time existence of solutions to the Cauchy problem (5.50) with subcritical powers of the nonlinearity $\sigma \in \left(0, \frac{\rho}{n}\right)$. We also suppose that σ is sufficiently close to $\frac{\rho}{n}$.

Theorem 5.18. *We assume that the initial data* $u_0 \in \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n)$, $0 < a < \min(1,\rho)$ *are such that* $u_0(x) \geq 0$ *almost everywhere. in* $\mathbf{R}^n$, $\theta = \int_{\mathbf{R}^n} u_0(x)\,dx > 0$. *Suppose that* $\frac{\rho}{n} - \varepsilon < \sigma < \frac{\rho}{n}$, *where* $\varepsilon > 0$ *is sufficiently small. Then there exists a unique global solution*

$$u \in \mathbf{C}\left([0,\infty);\mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n)\right)$$

of the Cauchy problem (5.50), satisfying the following time decay estimates

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-\frac{1}{\sigma}}$$

for large $t > 0$. *Furthermore there exist a number* A *and a function* $V \in \mathbf{L}^{1,a} \cap \mathbf{L}^\infty$ *such that the asymptotic formula is valid*

$$u(t,x) = At^{-\frac{1}{\sigma}}V\left(xt^{-\frac{1}{\rho}}\right) + O\left(t^{-\frac{1}{\sigma}-\gamma}\right) \tag{5.51}$$

for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}^n$, *where* $\gamma = \frac{1}{2}\min\left(a, 1 - \frac{n\sigma}{\rho}\right)$, *and* $V(\xi)$ *is the solution of the integral equation*

$$V(\xi) = G(\xi) - \frac{1}{\beta}\int_0^1 \frac{dz}{z(1-z)^{\frac{n}{\rho}}}\int_{\mathbf{R}^n} G\left(\left(\xi - yz^{\frac{1}{\rho}}\right)(1-z)^{-\frac{1}{\rho}}\right)K(y)\,dy \tag{5.52}$$

with $\beta = \frac{\sigma}{1-\frac{n\sigma}{\rho}}\int_{\mathbf{R}^n} V^{1+\sigma}(y)\,dy$ *and*

$$K(y) = V^{1+\sigma}(y) - V(y)\int_{\mathbf{R}^n} V^{1+\sigma}(\xi)\,d\xi.$$

Proof of Theorem 5.18

Considering problems (3.31) and (3.32), we get via Lemma 3.14 and the result of Section 5.2

$$\|u(t)\|_{\mathbf{L}^p} \leq C\langle t\rangle^{-\frac{n}{\rho}\left(1-\frac{1}{p}\right)}\left(1 + C\varepsilon\left(\frac{\rho}{n} - \sigma\right)^{-\frac{1}{\sigma}} t^{\frac{1}{\sigma}-\frac{n}{\rho}}\right)^{-1} \tag{5.53}$$

and

$$\|u(t)\|_{\mathbf{L}^{1,a}} \leq C\langle t\rangle^{\frac{a}{\rho}}\left(1 + C\varepsilon\left(\frac{\rho}{n} - \sigma\right)^{-\frac{1}{\sigma}} t^{\frac{1}{\sigma}-\frac{n}{\rho}}\right)^{-1} \tag{5.54}$$

for all $t > 0$ and $1 \leq p \leq \infty$. As in Chapter 3, we consider the Cauchy problem (3.35) for the new dependent variables $(v(t,x), \varphi(t))$. We now prove the estimate

$$\|v(t)\|_{\mathbf{L}^{1,a}} \leq C\langle t\rangle^{\frac{a}{\rho}}$$

for all $t > 0$. By estimate (5.53) we have

$$\|u(t)\|_{\mathbf{L}^\infty}^{\sigma} \le C\langle t\rangle^{-\sigma\frac{n}{\rho}}\left(1+C\varepsilon^{\sigma}\left(\frac{\rho}{n}-\sigma\right)^{-1}t^{1-\frac{n\sigma}{\rho}}\right)^{-1},$$

then

$$\varphi'(t) \le \frac{1}{\theta}\int_{\mathbf{R}^n} u^{\sigma} v dx \le C\langle t\rangle^{-\sigma\frac{n}{\rho}};$$

hence $\varphi(t) \le C\langle t\rangle^{1-\frac{n\sigma}{\rho}}$. Thus by virtue of (5.53) and (5.54) we get

$$\|v(t)\|_{\mathbf{L}^{1,a}} \le e^{\varphi(t)}\|u(t)\|_{\mathbf{L}^{1,a}} \le C(T)\langle t\rangle^{\frac{a}{\rho}}$$

and

$$\|v(t)\|_{\mathbf{L}^p} \le e^{\varphi(t)}\|u(t)\|_{\mathbf{L}^p} \le C(T)\langle t\rangle^{-\frac{n}{\rho}\left(1-\frac{1}{p}\right)}$$

for all $0 < t \le T$. Now we consider $t > T$. We use the integral equation (3.37). By virtue of estimates (5.54) and Lemma 3.12 we get

$$\left\|\int_T^t \mathcal{G}(t-\tau)\left(u^{\sigma}(\tau)v(\tau)-\frac{v(\tau)}{\theta}\int_{\mathbf{R}^n}u^{\sigma}(\tau)v(\tau)dx\right)d\tau\right\|_{\mathbf{L}^{1,a}}$$
$$\le C\int_T^t\left\|u^{\sigma}(\tau)v(\tau)-\frac{v(\tau)}{\theta}\int_{\mathbf{R}^n}u^{\sigma}(\tau)v(\tau)dx\right\|_{\mathbf{L}^{1,a}}d\tau$$
$$\le C\varepsilon^{-\sigma}\left(\frac{\rho}{n}-\sigma\right)\int_T^t\langle\tau\rangle^{-1}\|v(\tau)\|_{\mathbf{L}^{1,a}}d\tau$$

for all $t > T$. Therefore in view of (3.37) we obtain

$$\|v(t)\|_{\mathbf{L}^{1,a}} \le \|\mathcal{G}(t-T)v(T)\|_{\mathbf{L}^{1,a}} + C\varepsilon^{-\sigma}\left(\frac{\rho}{n}-\sigma\right)\int_T^t\langle\tau\rangle^{-1}\|v(\tau)\|_{\mathbf{L}^{1,a}}d\tau$$
$$\le C(T)\langle t\rangle^{\frac{a}{\rho}} + \epsilon\int_T^t\langle\tau\rangle^{-1}\|v(\tau)\|_{\mathbf{L}^{1,a}}d\tau$$

for all $t > T$. Here $\epsilon > 0$ is small enough, and $T > 0$ is sufficiently large (remember that $\sigma < \frac{\rho}{n}$ is close to $\frac{\rho}{n}$). The application of the Gronwall's lemma yields the estimate

$$\|v(t)\|_{\mathbf{L}^{1,a}} \le C\langle t\rangle^{\frac{a}{\rho}} \tag{5.55}$$

for all $t > 0$. In the same manner by virtue of estimates (5.53) and Lemma 3.12 we get

$$\|v(t)\|_{\mathbf{L}^p} \leq \|\mathcal{G}(t-T)v(T)\|_{\mathbf{L}^p}$$
$$+\left\|\int_T^t \mathcal{G}(t-\tau)\left(u^\sigma(\tau)-\frac{1}{\theta}\int_{\mathbf{R}^n} u^\sigma(\tau)v(\tau)\,dx\right)v(\tau)\,d\tau\right\|_{\mathbf{L}^p}$$
$$\leq C(T)\langle t\rangle^{-\frac{n}{\rho}\left(1-\frac{1}{p}\right)} + C\int_T^{\frac{t}{2}}(t-\tau)^{-\frac{n}{\rho}\left(1-\frac{1}{p}\right)-\frac{a}{\rho}}\tau^{\frac{a}{\rho}-1}d\tau$$
$$+C\varepsilon^{-\sigma}\left(\frac{\rho}{n}-\sigma\right)\int_{\frac{t}{2}}^t\langle\tau\rangle^{\cdot-1}\|v(\tau)\|_{\mathbf{L}^p}\,d\tau$$
$$\leq C(T)\langle t\rangle^{-\frac{n}{\rho}\left(1-\frac{1}{p}\right)}+\epsilon\int_{\frac{t}{2}}^t\langle\tau\rangle^{-1}\|v(\tau)\|_{\mathbf{L}^p}\,d\tau$$

for all $t > T$. The further application of the Gronwall's lemma yields the estimate

$$\|v(t)\|_{\mathbf{L}^p} \leq C\langle t\rangle^{-\frac{n}{\rho}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$. Therefore in the same way as in the proof of Theorems 5.4-5.5 we get the result of Theorem 5.18 .

5.3 Whitham type equations

This section is devoted to the investigation of large time asymptotics for solutions to the Cauchy problem for nonconvective type nonlinear dissipative equations in the subcritical case

$$\begin{cases} u_t+\mathcal{N}(u)+\mathcal{L}u=0,\ x\in\mathbf{R},\ t>0,\\ u(0,x)=u_0(x),\ x\in\mathbf{R}. \end{cases} \tag{5.56}$$

The linear part of equation (5.56) is a pseudodifferential operator defined by the Fourier transformation

$$\mathcal{L}u=\overline{\mathcal{F}}_{\xi\to x}\left(L(\xi)\,\widehat{u}(\xi)\right),$$

and the nonlinearity $\mathcal{N}(u)$ is a quadratic pseudodifferential operator

$$\mathcal{N}(u)=\overline{\mathcal{F}}_{\xi\to x}\int_{\mathbf{R}} a(t,\xi,y)\,\widehat{u}(t,\xi-y)\,\widehat{u}(t,y)\,dy,$$

defined by the symbol $a(t,\xi,y)$. Nonconvective type nonlinearity means that

$$a(t,0,y)\neq 0.$$

We consider here the real valued solutions $u(t,x)$.

Suppose that the linear operator $\mathcal{L}$ satisfies the dissipation condition which in terms of the symbol $L(\xi)$ has the form

$$\operatorname{Re}L(\xi)\geq\mu\{\xi\}^\delta\langle\xi\rangle^\nu \tag{5.57}$$

for all $\xi \in \mathbf{R}$, where $\mu > 0$, $\nu \geq 0$, $\delta > 0$. Also we suppose that the symbol is smooth $L(\xi) \in \mathbf{C}^1(\mathbf{R}^n)$ and has the estimate

$$\left|\partial_\xi^l L(\xi)\right| \leq C\{\xi\}^{\delta-l}\langle\xi\rangle^\nu \tag{5.58}$$

for all $\xi \in \mathbf{R} \setminus \{0\}$, $l = 0, 1$.

To find the asymptotic formulas for solutions we assume that the symbol $L(\xi)$ has the following asymptotic representation in the origin

$$L(\xi) = L_0(\xi) + O\left(|\xi|^{\delta+\gamma}\right) \tag{5.59}$$

for all $|\xi| \leq 1$, where $L_0(\xi) = \mu_1 |\xi|^\delta + i\mu_2 |\xi|^{\delta-1}\xi$, μ_1, $\mu_2 \in \mathbf{R}$, $\gamma \in (0,1)$.

We suppose that the symbol of the nonlinear operator $\mathcal{N}$ is continuous with respect to time $t > 0$ and

$$\left|\partial_\xi^l a(t,\xi,y)\right| \leq C\{\xi - y\}^{-l}\left(\{\xi - y\}^\alpha \langle\xi - y\rangle^\sigma + \{y\}^\alpha \langle y\rangle^\sigma\right) \tag{5.60}$$

for all $\xi, y \in \mathbf{R}$, $t > 0$, $l = 0, 1$, where $\alpha \geq 0$, $\sigma = 0$ if $\nu = 0$ and $\sigma \in [0,\nu]$ if $\nu > 0$.

The subcritical case with respect to the large time asymptotic behavior of solutions means that

$$\delta > \delta_c = 1 + \alpha.$$

We assume that the symbols of the nonlinearity have the asymptotics

$$a(t,0,y) = a_0(y) + O\left(\{y\}^{\alpha+\gamma}\langle y\rangle^\sigma\right) \tag{5.61}$$

for all $y, z \in \mathbf{R}$, $t > 0$, where $\gamma \in (0,1)$, $a_0(y)$ is homogeneous of order α. Also we suppose the total mass of the initial data is not zero

$$\widehat{u}_0(0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} u_0(x)\,dx \neq 0.$$

Denote

$$\kappa \equiv \theta^2 \int_{\mathbf{R}} a_0(y)\,e^{-L_0(-y)-L_0(y)}dy.$$

Define

$$\|\varphi\|_{\mathbf{A}^{0,\infty}} = \|\widehat{\varphi}(\cdot)\|_{\mathbf{L}^\infty_\xi(|\xi|\leq 1)} \text{ and } \|\varphi\|_{\mathbf{B}^{0,1}} = \|\widehat{\varphi}(\cdot)\|_{\mathbf{L}^1_\xi(|\xi|\geq 1)}$$

and

$$\|\varphi\|_{\mathbf{D}^{0,0}} = \left\||\partial_\xi|^\gamma \widehat{\varphi}(\cdot)\right\|_{\mathbf{L}^\infty_\xi},$$

where $\gamma \in (0, \min(1,\delta))$ is such that $\gamma < \alpha$ if $\alpha \neq 0$. Denote as well

$$G_0(x) = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-L_0(\xi)}\right),$$

$$\mathcal{N}_0(\varphi) = \overline{\mathcal{F}}_{\xi\to x}\int_{\mathbf{R}} a_0(y)\,\widehat{\varphi}(\xi - y)\,\widehat{\varphi}(y)\,dy.$$

In this section we prove the following result.

Theorem 5.19. *Assume that* $u_0 \in \mathbf{A}^{0,\infty} \cap \mathbf{B}^{0,1} \cap \mathbf{D}^{0,0}$ *with a sufficiently small norm*

$$\|u_0\|_{\mathbf{A}^{0,\infty}} + \|u_0\|_{\mathbf{B}^{0,1}} + \|u_0\|_{\mathbf{D}^{0,0}} = \varepsilon.$$

Suppose that $\kappa > 0$ *and*

$$\theta = \widehat{u_0}(0) > 0.$$

We assume that $1 - \varepsilon^3 < \omega < 1$, *where* $\omega = \frac{\delta_c}{\delta}$. *Then there exists a unique solution* $u \in \mathbf{L}^\infty((0,\infty) \times \mathbf{R}) \cap \mathbf{C}\left([0,\infty); \mathbf{A}^{0,\infty} \cap \mathbf{B}^{0,1} \cap \mathbf{D}^{0,0}\right)$ *to the Cauchy problem (5.56) satisfying the following time decay estimate*

$$\|u(t)\|_\infty \leq C t^{\frac{\alpha}{\delta}-1}$$

for all $t > 0$. *Furthermore there exist a number* A *and a function* $V \in \mathbf{L}^\infty \cap \mathbf{L}^1$ *such that the asymptotic formula is valid*

$$u(t,x) = A t^{\frac{\alpha}{\delta}-1} V\left(x t^{-\frac{1}{\delta}}\right) + O\left(t^{\frac{\alpha}{\delta}-1-\widetilde{\gamma}}\right)$$

for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}^n$, *where* $\widetilde{\gamma} = \frac{1}{2}\min(1-\omega,\gamma)$. *Here the function* $V(\xi)$ *is the solution of the integral equation*

$$V(\xi) = G_0(\xi) - \frac{1}{\eta}\int_0^1 \frac{dz}{z(1-z)^{\frac{1}{\delta}}} \int_{\mathbf{R}} G_0\left(\left(\xi - y z^{\frac{1}{\delta}}\right)(1-z)^{-\frac{1}{\delta}}\right) F(y)\, dy,$$

where

$$F(y) = \mathcal{N}_0(V)(y) - V(y)\widehat{\mathcal{N}_0(V)}(0)$$

and

$$\eta = \frac{1}{1-\omega}\widehat{\mathcal{N}_0}(V)(0).$$

Remark 5.20. The conditions of the theorem on the initial data u_0 can also be expressed in terms of the standard weighted Sobolev spaces as follows

$$\|u_0\|_{\mathbf{H}^{\varrho,0}} + \|u_0\|_{\mathbf{H}^{0,\varrho}} \leq \varepsilon,$$

where $\varrho > \frac{1}{2}$. However, the conditions on the initial data u_0 are described more precisely in the norm $\mathbf{A}^{0,\infty} \cap \mathbf{B}^{0,1} \cap \mathbf{D}^{0,0}$.

Remark 5.21. We give an example of the application of Theorem 5.19. For the potential Whitham equation

$$u_t + (u_x)^2 + \mu_1 |\partial_x|^\delta u + \mu_2 |\partial_x|^{\delta-1} u_x = 0, \; x \in \mathbf{R}, \; t > 0, \tag{5.62}$$

we have $a(t,\xi,y) = -(\xi - y)y$, $L(\xi) = \mu_1|\xi|^\delta - i\mu_2\xi|\xi|^{\delta-1}$, $a_0(y) = y^2$, $L_0(\xi) = \mu_1|\xi|^\delta - i\mu_2\xi|\xi|^{\delta-1}$, $\mu_1 > 0$, $\mu_2 \in \mathbf{R}$. The conditions (5.60) and (5.61) are fulfilled with $\sigma = \alpha = 2$, $\nu = \delta$, $\delta > \delta_c = 3$ so that $\omega = \frac{\delta_c}{\delta} < 1$ is sufficiently close to 1. Then for small initial data u_0 such that $\theta > 0$ and the norm

$$\|u_0\|_{\mathbf{H}^{\varrho,0}} + \|u_0\|_{\mathbf{H}^{0,\varrho}} \le \varepsilon$$

with $\varrho > \frac{1}{2}$, the asymptotics

$$u(t,x) = t^{\frac{1}{\delta}-1} V\left(xt^{-\frac{1}{\delta}}\right) + O\left(t^{\frac{1}{\delta}-1-\widetilde{\gamma}}\right)$$

is valid for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$.

5.3.1 Preliminary Lemmas

The Green operator $\mathcal{G}$ is given by

$$\mathcal{G}(t)\phi = \overline{\mathcal{F}}_{\xi \to x}\left(e^{-L(\xi)t}\hat{\phi}(\xi)\right).$$

From Lemmas 1.38 we obtain some preliminary estimates for the Green operator $\mathcal{G}(t)$ in the norms

$$\begin{aligned}
\|\varphi(t)\|_{\mathbf{A}^{\rho,p}} &= \||\cdot|^{\rho}\,\widehat{\varphi}(t,\cdot)\|_{\mathbf{L}^p_{\xi}(|\xi|\le 1)}, \\
\|\varphi(t)\|_{\mathbf{B}^{s,p}} &= \||\cdot|^{s}\,\widehat{\varphi}(t,\cdot)\|_{\mathbf{L}^p_{\xi}(|\xi|\ge 1)}, \\
\|\varphi(t)\|_{\mathbf{D}^{\rho,s}} &= \||\partial_\xi|^{\gamma}\{\cdot\}^{\rho}\langle\cdot\rangle^{s}\,\widehat{\varphi}(t,\cdot)\|_{\mathbf{L}^\infty_{\xi}}
\end{aligned}$$

where $\rho, s \in \mathbf{R}$, $\gamma \in (0,1)$.

Lemma 5.22. *Let the linear operator $\mathcal{L}$ satisfy dissipation conditions (5.57) and (5.58). Suppose that $\widehat{\varphi}(0) = 0$. Then the estimates are valid for all $t > 0$*

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{A}^{\rho,p}} \le C\langle t\rangle^{-\frac{1}{\delta}\left(\rho+\frac{1}{p}-\frac{1}{q}\right)}\|\varphi\|_{\mathbf{A}^{0,q}},$$

where $\rho \ge 0$ if $p = q$ and $\rho + \frac{1}{p} - \frac{1}{q} > 0$ if $1 \le p < q \le \infty$,

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{A}^{\rho,p}} \le C\langle t\rangle^{-\frac{1}{\delta}\left(\rho+\gamma+\frac{1}{p}\right)}\|\varphi\|_{\mathbf{D}^{0,0}},$$

where $\rho + \gamma \ge 0$ if $p = \infty$ and $\rho + \gamma + \frac{1}{p} > 0$ if $1 \le p < \infty$,

$$\|\mathcal{G}(t)\varphi\|_{\mathbf{B}^{s,p}} \le Ce^{-\frac{\mu}{2}t}\{t\}^{-\frac{s}{\nu}}\|\varphi\|_{\mathbf{B}^{0,p}}$$

where $1 \le p \le \infty$, $s \ge 0$, if $\nu > 0$ and $s = 0$ if $\nu = 0$ and

$$\begin{aligned}
\|\mathcal{G}(t)\varphi\|_{\mathbf{D}^{\rho,s}} &\le C\langle t\rangle^{-\frac{\rho}{\delta}}\{t\}^{-\frac{s}{\nu}}\left(\|\varphi\|_{\mathbf{D}^{0,0}} + \|\varphi\|_{\mathbf{A}^{0,\infty}} + \|\varphi\|_{\mathbf{B}^{0,\infty}}\right) \\
&+ C\langle t\rangle^{-\frac{\rho}{\delta}+\frac{\gamma}{2\delta}}\{t\}^{-\frac{s}{\nu}}\left(\|\varphi\|_{\mathbf{A}^{0,\infty}} + \|\varphi\|_{\mathbf{B}^{0,\infty}}\right)^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}}_{\mathbf{D}^{0,0}}
\end{aligned}$$

where $1 \le p \le q \le \infty$, $\rho \ge 0$, $s \ge 0$, if $\nu > 0$ and $s = 0$ if $\nu = 0$, $0 \le \gamma < \min(1,\delta)$ if $\rho = 0$ and $0 \le \gamma < \min(1,\delta,\rho)$ if $\rho > 0$.

Define $\mathcal{G}_0(t)$ by

$$\mathcal{G}_0(t)\,\phi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-L_0(\xi)t}\hat{\phi}(\xi)\right),$$

where the symbol $L_0(\xi) = \mu_1 |\xi|^{\delta} + i\mu_2 |\xi|^{\delta-1}\xi$ is homogeneous of order $\delta > 0$, $\mu_1,\ \mu_2 \in \mathbf{R}$ and

$$G_0(x) = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-L_0(\xi)}\right).$$

Using the method of Lemma 1.39 we estimate a difference $\mathcal{G}(t) - \mathcal{G}_0(t)$.

Lemma 5.23. *Let the linear operator $\mathcal{L}$ satisfy conditions (5.57) and (5.59), then the estimates*

$$\|(\mathcal{G}(t) - \mathcal{G}_0(t))\,\phi\|_{\mathbf{A}^{\rho,p}} \le C\langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{1}{\delta}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\phi\|_{\mathbf{A}^{0,q}}$$

for $1 \le p \le q \le \infty$, and

$$\left\|\mathcal{G}_0(t)\,\phi - t^{-\frac{1}{\delta}}\widehat{\phi}(0)\,G_0\left(t^{-\frac{1}{\delta}}(\cdot)\right)\right\|_{\mathbf{A}^{\rho,p}} \le C\langle t\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{1}{\delta p}}\|\phi\|_{\mathbf{D}^{0,0}}$$

are valid for all $t > 0$, where $1 \le p \le \infty$, $\rho + \gamma \ge 0$.

In the next lemma we estimate the Green operator $\mathcal{G}(t)$ in our basic norms

$$\begin{aligned}\|\phi\|_{\mathbf{X}} =& \sup_{\rho\in[-\gamma,\alpha+\gamma]}\sup_{t>0}\langle t\rangle^{\frac{\rho+1}{\delta}}\|\phi(t)\|_{\mathbf{A}^{\rho,1}} + \sup_{\rho\in[0,\alpha+\gamma]}\sup_{t>0}\langle t\rangle^{\frac{\rho}{\delta}}\|\phi(t)\|_{\mathbf{A}^{\rho,\infty}}\\ &+ \sup_{s\in[0,\sigma]}\sup_{1\le p\le\infty}\sup_{t>0}\{t\}^{\frac{s}{\nu}}\langle t\rangle^{1+\frac{\gamma}{\delta}+\frac{1}{\delta p}}\|\phi(t)\|_{\mathbf{B}^{s,p}}\\ &+ \sup_{\rho=0,\alpha}\sup_{s\in[0,\sigma]}\sup_{t>0}\{t\}^{\frac{s}{\nu}}\langle t\rangle^{\frac{\rho-\gamma}{\delta}}\|\phi(t)\|_{\mathbf{D}^{\rho,s}},\end{aligned}$$

and

$$\begin{aligned}\|\phi\|_{\mathbf{Y}} =& \sup_{1\le p\le\infty}\sup_{t>0}\langle t\rangle^{\omega+\frac{1}{\delta p}}\{t\}^{\frac{\sigma}{\nu}}\|\phi(t)\|_{\mathbf{A}^{0,p}}\\ &+ \sup_{1\le p\le\infty}\sup_{t>0}\langle t\rangle^{\omega+\frac{\gamma}{\delta}+\frac{1}{\delta p}}\{t\}^{\frac{\sigma}{\nu}}\|\phi(t)\|_{\mathbf{B}^{0,p}}\\ &+ \sup_{t>0}\langle t\rangle^{\omega-\frac{\gamma}{\delta}}\{t\}^{\frac{\sigma}{\nu}}\|\phi(t)\|_{\mathbf{D}^{0,0}};\end{aligned}$$

here $\gamma \in (0, \min(1,\delta))$ is such that $\gamma < \alpha$ if $\alpha > 0$, $\omega \in (0,1)$. We take $\omega = \frac{\delta_c}{\delta} < 1$ below. Define the function $g(t) = \langle t\rangle^{1-\omega}$.

Lemma 5.24. *Let the function $f(t,x)$ have a zero mean value $\widehat{f}(t,0) = 0$. Then the following inequality*

$$\left\|\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau) f(\tau)\,d\tau\right\|_{\mathbf{X}} \le C\|f\|_{\mathbf{Y}}$$

is valid, provided that the right-hand side is finite.

Proof. By virtue of the first two estimates of Lemma 5.22 we obtain for $\rho \in [-\gamma, \alpha+\gamma]$ if $p=1$ and for $\rho \in [0, \alpha+\gamma]$ if $p=\infty$

$$\left\| \int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{A}^{\rho,p}}$$
$$\leq C \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{\rho+\gamma}{\delta}-\frac{1}{\delta p}} \langle\tau\rangle^{\frac{\gamma}{\delta}-1} d\tau \sup_{\tau>0} \langle\tau\rangle^{\omega-\frac{\gamma}{\delta}} \|f(\tau)\|_{\mathbf{D}^{0,0}}$$
$$+C\int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-\frac{\rho}{\delta}} \langle\tau\rangle^{-\frac{1}{\delta p}-1} \{\tau\}^{-\frac{\sigma}{\nu}} d\tau \sup_{\tau>0} \langle\tau\rangle^{\omega+\frac{1}{\delta p}} \{\tau\}^{\frac{\sigma}{\nu}} \|f(\tau)\|_{\mathbf{A}^{0,p}}$$
$$\leq C\langle t\rangle^{-\frac{\rho}{\delta}-\frac{1}{\delta p}} \|f\|_{\mathbf{Y}}.$$

Similarly, by the third estimate of Lemma 5.22 we get for $s \in [0,\sigma]$, $1 \leq p \leq \infty$, $t>0$

$$\left\| \int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{B}^{s,p}}$$
$$\leq C\int_0^t e^{-\frac{\mu}{2}(t-\tau)} \langle\tau\rangle^{-1-\frac{\gamma}{\delta}-\frac{1}{\delta p}} \{t-\tau\}^{-\frac{s}{\nu}} \{\tau\}^{-\frac{\sigma}{\nu}} d\tau$$
$$\times \sup_{\tau>0} \langle\tau\rangle^{1+\frac{\gamma}{\delta}+\frac{1}{\delta p}} \{\tau\}^{\frac{\sigma}{\nu}} \|f(\tau)\|_{\mathbf{B}^{0,p}} \leq C\langle t\rangle^{-1-\frac{\gamma}{\delta}-\frac{1}{\delta p}} \{t\}^{-\frac{s}{\nu}} \|f\|_{\mathbf{Y}}.$$

Finally by the fourth estimate of Lemma 5.22 we find for $\rho = 0, \alpha$, $s \in [0,\sigma]$, $t>0$

$$\left\| \int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{D}^{\rho,s}}$$
$$\leq C\int_0^t d\tau \langle t-\tau\rangle^{-\frac{\rho}{\delta}} \{t-\tau\}^{-\frac{s}{\nu}} g^{-1}(\tau)$$
$$\times \left(\|f(\tau)\|_{\mathbf{D}^{0,0}} + \|f(\tau)\|_{\mathbf{A}^{0,\infty}} + \|f(\tau)\|_{\mathbf{B}^{0,\infty}}\right)$$
$$+C\int_0^t d\tau \langle t-\tau\rangle^{-\frac{\rho}{\delta}+\frac{\gamma}{2\delta}} \{t-\tau\}^{-\frac{s}{\nu}} g^{-1}(\tau)$$
$$\times \left(\|f(\tau)\|_{\mathbf{A}^{0,\infty}} + \|f(\tau)\|_{\mathbf{B}^{0,\infty}}\right)^{\frac{1}{2}} \|f(\tau)\|_{\mathbf{D}^{0,0}}^{\frac{1}{2}}.$$

Using the norm $\mathbf{Y}$ we have

$$\left\| \int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{D}^{\rho,s}}$$
$$\leq C\|f\|_{\mathbf{Y}} \int_0^t \langle t-\tau\rangle^{-\frac{\rho}{\delta}} \langle\tau\rangle^{\frac{\gamma}{\delta}-1} \{t-\tau\}^{-\frac{s}{\nu}} \{\tau\}^{-\frac{\sigma}{\nu}} d\tau$$
$$+C\|f\|_{\mathbf{Y}} \int_0^t \langle t-\tau\rangle^{-\frac{\rho}{\delta}+\frac{\gamma}{2\delta}} \langle\tau\rangle^{\frac{\gamma}{2\delta}-1} \{t-\tau\}^{-\frac{s}{\nu}} \{\tau\}^{-\frac{\sigma}{\nu}} d\tau$$
$$\leq C\|f\|_{\mathbf{Y}} \{t\}^{-\frac{s}{\nu}} \langle t\rangle^{\frac{\gamma-\rho}{\delta}}.$$

Hence, the results of the lemma follow, and Lemma 5.24 is proved.

Now we estimate the nonlinearity $\mathcal{N}(u)$ in the norms $\mathbf{A}^{0,p}$, $\mathbf{B}^{0,p}$ and $\mathbf{D}^{0,0}$.

Lemma 5.25. *Let the nonlinear operator $\mathcal{N}$ satisfy condition (5.60). Then the inequalities*

$$\begin{aligned}\|\mathcal{N}(\varphi)\|_{\mathbf{A}^{0,p}} \le\ & C\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\\ &\times\left(\|\varphi(t)\|_{\mathbf{A}^{0,p}} + \|\varphi(t)\|_{\mathbf{B}^{0,\infty}}\right)\end{aligned}$$

$$\begin{aligned}&\|\mathcal{N}(\varphi)\|_{\mathbf{B}^{0,p}} \le C\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\\ &\times\left(\|\varphi(t)\|_{\mathbf{A}^{0,p}} + \|\varphi(t)\|_{\mathbf{B}^{0,p}}\right)\\ &+ C\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\gamma,p}} + \|\varphi(t)\|_{\mathbf{B}^{0,p}}\right)\end{aligned}$$

and

$$\begin{aligned}&\|\mathcal{N}(\varphi)\|_{\mathbf{D}^{0,0}} \le C\|\varphi(t)\|_{\mathbf{D}^{\alpha,\sigma}}\left(\|\varphi(t)\|_{\mathbf{A}^{0,1}} + \|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\\ &+ C\|\varphi(t)\|_{\mathbf{D}^{0,0}}\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\\ &+ C\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha-\gamma,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{0,\infty}} + \|\varphi(t)\|_{\mathbf{B}^{0,\infty}}\right)\\ &+ C\left(\|\varphi(t)\|_{\mathbf{A}^{-\gamma,1}} + \|\varphi(t)\|_{\mathbf{B}^{0,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,\infty}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,\infty}}\right)\end{aligned}$$

are valid for $1 \le p \le \infty$, provided that the right-hand sides are bounded.

Proof. By virtue of condition (5.60)

$$\begin{aligned}&\|\mathcal{N}(\varphi)\|_{\mathbf{A}^{0,p}} \le \left\|\int_{\mathbf{R}} |a(t,\cdot,y)|\left|\widehat{\varphi}(t,\cdot-y)\,\widehat{\varphi}(t,y)\right| dy\right\|_{\mathbf{L}^p_\xi(|\xi|\le 1)}\\ &\le C\left\|\int_{\mathbf{R}}\left(\langle\cdot-y\rangle^{\sigma}\{\cdot-y\}^{\alpha} + \langle y\rangle^{\sigma}\{y\}^{\alpha}\right)\left|\widehat{\varphi}(t,\cdot-y)\,\widehat{\varphi}(t,y)\right| dy\right\|_{\mathbf{L}^p_\xi(|\xi|\le 1)};\end{aligned}$$

hence, by the Young inequality we obtain

$$\begin{aligned}&\|\mathcal{N}(\varphi)\|_{\mathbf{A}^{0,p}} \le C\left\|\langle\cdot\rangle^{\sigma}\{\cdot\}^{\alpha}\widehat{\varphi}(t)\right\|_{\mathbf{L}^1}\Big(\|\widehat{\varphi}(t)\|_{\mathbf{L}^p_\xi(|\xi|\le 1)}\\ &+\|\widehat{\varphi}(t)\|_{\mathbf{L}^\infty_\xi(|\xi|>1)}\Big)\\ &\le C\left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{0,p}} + \|\varphi(t)\|_{\mathbf{B}^{0,\infty}}\right).\end{aligned}$$

In the same manner

$$\begin{aligned}\|\mathcal{N}(\varphi)\|_{\mathbf{B}^{0,p}} \le\ & \left\|\int_{|y|\le\frac{1}{2}} |a(t,\xi,y)|\left|\widehat{\varphi}(t,\cdot-y)\,\widehat{\varphi}(t,y)\right| dy\right\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)}\\ +& \left\|\int_{|y|\ge\frac{1}{2}} |a(t,\cdot,y)|\left|\widehat{\varphi}(t,\cdot-y)\,\widehat{\varphi}(t,y)\right| dy\right\|_{\mathbf{L}^p_\xi(|\xi|\ge 1)}.\end{aligned}$$

Thus

$$\|\mathcal{N}(\varphi)\|_{\mathbf{B}^{0,p}} \le C \left\| \int_{|y|\le\frac{1}{2}} \{\cdot - y\}^{\gamma} \left(\langle \cdot - y\rangle^{\sigma} \{\cdot - y\}^{\alpha} + \{y\}^{\alpha}\right) \right.$$
$$\times \left. |\widehat{\varphi}(t, \cdot - y)\,\widehat{\varphi}(t,y)|\, dy \right\|_{\mathbf{L}^p_{\xi}(|\xi|\ge 1)}$$
$$+ C \left\| \int_{|y|\ge\frac{1}{2}} \{y\}^{\gamma} \left(\langle \cdot - y\rangle^{\sigma} \{\cdot - y\}^{\alpha} + \langle y\rangle^{\sigma}\{y\}^{\alpha}\right) \right.$$
$$\times \left. |\widehat{\varphi}(t, \cdot - y)\,\widehat{\varphi}(t,y)|\, dy \right\|_{\mathbf{L}^p_{\xi}(|\xi|\ge 1)},$$

and therefore,

$$\|\mathcal{N}(\varphi)\|_{\mathbf{B}^{0,p}} \le C \left\| \langle\cdot\rangle^{\sigma} \{\cdot\}^{\alpha+\gamma}\,\widehat{\varphi}(t) \right\|_{\mathbf{L}^1} \|\widehat{\varphi}(t)\|_{\mathbf{L}^p}$$
$$+ C \left\|\{\cdot\}^{\gamma}\,\widehat{\varphi}(t)\right\|_{\mathbf{L}^p} \left\|\langle\cdot\rangle^{\sigma}\{\cdot\}^{\alpha}\,\widehat{\varphi}(t)\right\|_{\mathbf{L}^1}$$
$$\le C \left(\|\varphi(t)\|_{\mathbf{A}^{\alpha+\gamma,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{0,p}} + \|\varphi(t)\|_{\mathbf{B}^{0,p}}\right)$$
$$+ C \left(\|\varphi(t)\|_{\mathbf{A}^{\alpha,1}} + \|\varphi(t)\|_{\mathbf{B}^{\sigma,1}}\right)\left(\|\varphi(t)\|_{\mathbf{A}^{\gamma,p}} + \|\varphi(t)\|_{\mathbf{B}^{0,p}}\right).$$

The first two estimates of the lemma then follow. Denote

$$\widetilde{a}(t,\xi,y) = a(t,\xi,y)\left(\{\xi-y\}^{\alpha}\langle\xi-y\rangle^{\sigma} + \{y\}^{\alpha}\langle y\rangle^{\sigma}\right)^{-1},$$
$$\varPhi(t,\xi,y) = \left(\{\xi-y\}^{\alpha}\langle\xi-y\rangle^{\sigma} + \{y\}^{\alpha}\langle y\rangle^{\sigma}\right)\widehat{\varphi}(t,\xi-y)\,\widehat{\varphi}(t,y).$$

We have

$$\|\mathcal{N}(\varphi)\|_{\mathbf{D}^{0,0}} = \left\| |\partial_{\xi}|^{\gamma} \int_{\mathbf{R}} \widetilde{a}(t,\cdot,y)\,\varPhi(t,\cdot,y)\,dy \right\|_{\mathbf{L}^{\infty}}$$
$$\le C \left\| \int_{\mathbf{R}} \varPhi(t,\cdot,y)\,|\partial_{\xi}|^{\gamma}\,\widetilde{a}(t,\cdot,y)\,dy \right\|_{\mathbf{L}^{\infty}}$$
$$+ C \left\| \int_{\mathbf{R}} \left[|\partial_{\xi}|^{\gamma}, \varPhi(t,\cdot,y)\right]\widetilde{a}(t,\cdot,y)\,dy \right\|_{\mathbf{L}^{\infty}},$$

where the commutator

$$\left[|\partial_{\xi}|^{\gamma}, \varPhi(t,\xi,y)\right]\widetilde{a}(t,\xi,y)$$
$$\equiv \int_{\mathbf{R}} |\varPhi(t,\xi-\eta,y) - \varPhi(t,\xi,y)|\,\widetilde{a}(t,\xi-\eta,y)\,|\eta|^{-1-\gamma}\,d\eta.$$

By virtue of condition (5.60) we estimate the commutator

$$
\begin{aligned}
&\left\| \int_{\mathbf{R}} \left[|\partial_\xi|^\gamma , \Phi(t,\cdot,y) \right] \widetilde{a}(t,\cdot,y)\, dy \right\|_{\mathbf{L}^\infty} \\
&\le C \left\| \int_{\mathbf{R}} \int_{\mathbf{R}} \left| \Phi(t,\cdot-\eta,y) - \Phi(t,\cdot,y) \right| |\eta|^{-1-\gamma}\, d\eta dy \right\|_{\mathbf{L}^\infty} \\
&\le C \left\| \int_{\mathbf{R}} \left(|\partial_\xi|^\gamma \{\cdot - y\}^\alpha \langle \cdot - y\rangle^\sigma \widehat{\varphi}(t,\cdot-y) \right) \widehat{\varphi}(t,y)\, dy \right\|_{\mathbf{L}^\infty} \\
&+ C \left\| \int_{\mathbf{R}} \left(|\partial_\xi|^\gamma \widehat{\varphi}(t,\cdot-y) \right) \{y\}^\alpha \langle y\rangle^\sigma \widehat{\varphi}(t,y)\, dy \right\|_{\mathbf{L}^\infty} \\
&\le C \left\| |\partial_\xi|^\gamma \{\cdot\}^\alpha \langle\cdot\rangle^\sigma \widehat{\varphi}(t) \right\|_{\mathbf{L}^\infty} \left\| \widehat{\varphi}(t) \right\|_{\mathbf{L}^1} \\
&+ C \left\| |\partial_\xi|^\gamma \widehat{\varphi}(t) \right\|_{\mathbf{L}^\infty} \left\| \{\cdot\}^\alpha \langle\cdot\rangle^\sigma \widehat{\varphi}(t) \right\|_{\mathbf{L}^1} \\
&\le C \left\| \varphi(t) \right\|_{\mathbf{D}^{\alpha,\sigma}} \left(\left\| \varphi(t) \right\|_{\mathbf{A}^{0,1}} + \left\| \varphi(t) \right\|_{\mathbf{B}^{0,1}} \right) \\
&+ C \left\| \varphi(t) \right\|_{\mathbf{D}^{0,0}} \left(\left\| \varphi(t) \right\|_{\mathbf{A}^{\alpha,1}} + \left\| \varphi(t) \right\|_{\mathbf{B}^{\sigma,1}} \right). \qquad (5.63)
\end{aligned}
$$

Via (5.60) we have

$$
\begin{aligned}
&\int_{|\eta| \ge \frac{1}{2}\{\xi - y\}} \left| \widetilde{a}(t,\xi-\eta,y) - \widetilde{a}(t,\xi,y) \right| \frac{d\eta}{|\eta|^{1+\gamma}} \\
&\le C \int_{|\eta| \ge \frac{1}{2}\{\xi - y\}} \frac{d\eta}{|\eta|^{1+\gamma}} \le C \{\xi - y\}^{-\gamma}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{|\eta| \le \frac{1}{2}\{\xi - y\}} \left| \widetilde{a}(t,\xi-\eta,y) - \widetilde{a}(t,\xi,y) \right| \frac{d\eta}{|\eta|^{1+\gamma}} \\
&\le C \int_{|\eta| \le \frac{1}{2}\{\xi - y\}} \left| \int_{\xi-y}^{\xi-y-\eta} \{\zeta\}^{-1} d\zeta \right| \frac{d\eta}{|\eta|^{1+\gamma}} \\
&\le C \{\xi - y\}^{-1} \int_{|\eta| \le \frac{1}{2}\{\xi - y\}} \frac{d\eta}{|\eta|^{\gamma}} \le C \{\xi - y\}^{-\gamma}.
\end{aligned}
$$

Thus we have the estimate

$$
\begin{aligned}
|\partial_\xi|^\gamma \widetilde{a}(t,\xi,y) &= \int_{\mathbf{R}} \left| \widetilde{a}(t,\xi-\eta,y) - \widetilde{a}(t,\xi,y) \right| |\eta|^{-1-\gamma}\, d\eta \\
&\le C \{\xi - y\}^{-\gamma}
\end{aligned}
$$

for all $\xi, y \in \mathbf{R}$. Therefore,

$$\begin{aligned}
&\left\| \int_{\mathbf{R}} \Phi\left(t, \cdot, y\right) \left|\partial_{\xi}\right|^{\gamma} \widetilde{a}\left(t, \cdot, y\right) dy \right\|_{\mathbf{L}^{\infty}} \\
&\leq C \left\| \int_{\mathbf{R}} \left\{\cdot - y\right\}^{\alpha-\gamma} \left\langle \cdot - y\right\rangle^{\sigma} \widehat{\varphi}\left(t, \cdot - y\right) \widehat{\varphi}\left(t, y\right) dy \right\|_{\mathbf{L}^{\infty}} \\
&+ C \left\| \int_{\mathbf{R}} \left\{\cdot - y\right\}^{-\gamma} \widehat{\varphi}\left(t, \cdot - y\right) \left\{y\right\}^{\alpha} \left\langle y\right\rangle^{\sigma} \widehat{\varphi}\left(t, y\right) dy \right\|_{\mathbf{L}^{\infty}} \\
&\leq C \left\| \left\{\cdot\right\}^{\alpha-\gamma} \left\langle \cdot \right\rangle^{\sigma} \widehat{\varphi}\left(t\right) \right\|_{\mathbf{L}^{1}} \left\| \widehat{\varphi}\left(t\right) \right\|_{\mathbf{L}^{\infty}} \\
&+ C \left\| \left\{\cdot\right\}^{-\gamma} \widehat{\varphi}\left(t\right) \right\|_{\mathbf{L}^{1}} \left\| \left\{\cdot\right\}^{\alpha} \left\langle \cdot \right\rangle^{\sigma} \widehat{\varphi}\left(t\right) \right\|_{\mathbf{L}^{\infty}} \\
&\leq C \left(\left\|\varphi\left(t\right)\right\|_{\mathbf{A}^{\alpha-\gamma,1}} + \left\|\varphi\left(t\right)\right\|_{\mathbf{B}^{\sigma,1}} \right) \left(\left\|\varphi\left(t\right)\right\|_{\mathbf{A}^{0,\infty}} + \left\|\varphi\left(t\right)\right\|_{\mathbf{B}^{0,\infty}} \right) \\
&+ C \left(\left\|\varphi\left(t\right)\right\|_{\mathbf{A}^{-\gamma,1}} + \left\|\varphi\left(t\right)\right\|_{\mathbf{B}^{0,1}} \right) \left(\left\|\varphi\left(t\right)\right\|_{\mathbf{A}^{\alpha,\infty}} + \left\|\varphi\left(t\right)\right\|_{\mathbf{B}^{\sigma,\infty}} \right). \qquad (5.64)
\end{aligned}$$

In view of (5.63) - (5.64) we get

$$\begin{aligned}
&\left\|\mathcal{N}\left(\varphi\right)\right\|_{\mathbf{D}^{0,0}} \leq C \left\|\varphi\left(t\right)\right\|_{\mathbf{D}^{\alpha,\sigma}} \left(\left\|\varphi\left(t\right)\right\|_{\mathbf{A}^{0,1}} + \left\|\varphi\left(t\right)\right\|_{\mathbf{B}^{0,1}} \right) \\
&+ C \left\|\varphi\left(t\right)\right\|_{\mathbf{D}^{0,0}} \left(\left\|\varphi\left(t\right)\right\|_{\mathbf{A}^{\alpha,1}} + \left\|\varphi\left(t\right)\right\|_{\mathbf{B}^{\sigma,1}} \right) \\
&+ C \left(\left\|\varphi\left(t\right)\right\|_{\mathbf{A}^{\alpha-\gamma,1}} + \left\|\varphi\left(t\right)\right\|_{\mathbf{B}^{\sigma,1}} \right) \left(\left\|\varphi\left(t\right)\right\|_{\mathbf{A}^{0,\infty}} + \left\|\varphi\left(t\right)\right\|_{\mathbf{B}^{0,\infty}} \right) \\
&+ C \left(\left\|\varphi\left(t\right)\right\|_{\mathbf{A}^{-\gamma,1}} + \left\|\varphi\left(t\right)\right\|_{\mathbf{B}^{0,1}} \right) \left(\left\|\varphi\left(t\right)\right\|_{\mathbf{A}^{\alpha,\infty}} + \left\|\varphi\left(t\right)\right\|_{\mathbf{B}^{\sigma,\infty}} \right).
\end{aligned}$$

Thus the third estimate of the lemma is true and Lemma 5.25 is proved.

5.3.2 Proof of Theorem 5.19

For the local existence of classical solutions for the Cauchy problem (5.56) we refer to Chapter 2, Section 2.5. for all $t > 0$. We define

$$\mathbf{Z} = \left\{ \phi \in \mathbf{A}^{0,\infty} \cap \mathbf{B}^{0,1} \cap \mathbf{D}^{0,0} \right\}$$

with norm

$$\left\|\phi\right\|_{\mathbf{Z}} = \left\|\phi\right\|_{\mathbf{A}^{0,\infty}} + \left\|\phi\right\|_{\mathbf{B}^{0,1}} + \left\|\phi\right\|_{\mathbf{D}^{0,0}} = \varepsilon.$$

Denote

$$g(t) = \left(1 + \frac{\kappa\theta}{1-\omega}\right)^{1-\omega}, \omega = \frac{\delta_c}{\delta} < 1.$$

Applying Lemmas 5.22-5.23 we understand that $\mathcal{G}_0$ is a self-similar asymptotic operator in spaces $\mathbf{X}, \mathbf{Z}$ with functional

$$f(\phi) = \int_{\mathbf{R}} \phi(x)\, dx.$$

Also the function

$$G_0(x,t) = t^{-\frac{1}{\delta}} \overline{\mathcal{F}}_{\xi \to xt^{-\frac{1}{\delta}}} \left(e^{-L_0(\xi)}\right)$$

is the asymptotic kernel. By a direct calculation we have

$$\frac{\sigma}{\theta} \operatorname{Re} f\left(\mathcal{N}\left(\theta G_0(t)\right)\right) \geq \frac{\eta}{2} \theta^{\sigma} t^{-1+\omega},$$

so $\mathcal{N}$ is a subcritical nonconvective type. By Lemma 5.25

$$\|\mathcal{N}(v)\|_{\mathbf{Y}} \leq C \|v\|_{\mathbf{X}}^2$$

and

$$\left\| \frac{v(\tau)}{\theta} \widehat{\mathcal{N}(v(\tau))}(0) \right\|_{\mathbf{Y}} \leq C \|v\|_{\mathbf{X}}^2,$$

Hence

$$\|\mathcal{K}\|_{\mathbf{Y}} = \left\| \mathcal{N}(v(t)) - \frac{v(t)}{\theta} \widehat{\mathcal{N}(v(t))}(0) \right\|_{\mathbf{Y}} \leq C \|v\|_{\mathbf{X}}^2. \tag{5.65}$$

Applying Lemma 5.24 we get

$$\left\| \int_0^t g^{-1}(\tau)\, \mathcal{G}(t-\tau)\, \mathcal{K}(\tau)\, d\tau \right\|_{\mathbf{X}}$$
$$\leq C \|\mathcal{K}\|_{\mathbf{Y}} \leq C \|v\|_{\mathbf{X}}^2.$$

Via Lemma 5.25 we obtain

$$\left| \langle t \rangle^{\omega} \left(\widehat{\mathcal{N}(v(\tau))}(0) \right) \right| \leq C \|v\|_{\mathbf{X}}^2.$$

In the same manner we can estimate the differences $\mathcal{K}(v) - \mathcal{K}(w)$ and $\mathcal{N}(v) - \mathcal{N}(w)$ to see that all conditions of Theorem 5.2 are fulfilled, and therefore, there exists a unique solution $u \in \mathbf{X}$.

By analogy we easily get

$$\begin{aligned} &\left\| \int_0^t \left| \mathcal{G}_0 (t-\tau) \left(\mathcal{K} (v(\tau)) - \mathcal{K} (w(\tau)) \right) \right| \tau^{-\omega} d\tau \right\|_{\mathbf{X}} \\ &\leq C \|v - w\|_{\mathbf{X}} \left(1 + \frac{\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}}{\theta} \right), \end{aligned}$$

$$\begin{aligned} &\left\| \langle t \rangle^{\gamma} \int_0^t \left| \mathcal{G} (t-\tau) \left(\mathcal{K} (v(\tau)) - \mathcal{K} (w(\tau)) \right) \right| \tau^{-\omega} d\tau \right\|_{\mathbf{X}} \\ &\leq C \left\| \langle t \rangle^{\gamma} (v(t) - w(t)) \right\|_{\mathbf{X}} \left(1 + \frac{\|v\|_{\mathbf{X}} + \|w\|_{\mathbf{X}}}{\theta} \right), \end{aligned}$$

$$\left\| \langle t \rangle^{\gamma} \int_0^t \left| \mathcal{G} (t-\tau) \mathcal{K} (v(\tau)) \right| \tau^{-\mu-\gamma} d\tau \right\|_{\mathbf{X}} \leq C \|v\|_{\mathbf{X}}^2$$

and

$$\left\| \langle t \rangle^{\gamma} \int_0^t \left(\mathcal{G} (t-\tau) - \mathcal{G}_0 (t-\tau) \right) \mathcal{K} (v(\tau)) \tau^{-\mu} d\tau \right\|_{\mathbf{X}} \leq C \|v\|_{\mathbf{X}}^2 .$$

Therefore via Theorem 5.4 and Theorem 5.5 we have the result of Theorem 5.19 which is then proved.

5.4 Damped wave equation

5.4.1 Small initial data

First we study the one dimensional nonlinear damped wave equation

$$\begin{cases} v_{tt} + v_t - v_{xx} + v^{1+\sigma} = 0, \ x \in \mathbf{R}, \ t > 0, \\ v(0, x) = \varepsilon v_0(x), \ v_t(0, x) = \varepsilon v_1(x), \ x \in \mathbf{R} \end{cases} \tag{5.66}$$

in the subcritical case $\sigma \in (0, 2)$, with small $\varepsilon > 0$. For simplicity we consider the one dimensional case. The higher dimensional damped wave equation can also be considered by our method.

Taking $v = u_1$ and $(1 + \partial_x)^{-1} v_t = u_2$ we rewrite equation (5.66) in the form of a system of nonlinear evolutionary equations

$$u_t + \mathcal{N}(u) + \mathcal{L} u = 0 \tag{5.67}$$

for the vector $u(t, x) = \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix}$, with the initial data

$$u(0,x)=\widetilde{u}(x)\equiv\begin{pmatrix}\varepsilon v_0(x)\\ \varepsilon(1+\partial_x)^{-1}v_1(x)\end{pmatrix},$$

where the linear part of system (5.67) is a pseudodifferential operator defined by the Fourier transformation as follows

$$\mathcal{L}u=\overline{\mathcal{F}}_{\xi\to x}L(\xi)\mathcal{F}_{x\to\xi}u,$$

with a matrix - symbol

$$L(\xi)=\left.\{L_{jk}(\xi)\}\right|_{j,k=1,2}=\begin{pmatrix}0 & -(1+i\xi)\\ \frac{\xi^2}{1+i\xi} & 1\end{pmatrix}$$

and the nonlinearity is defined by

$$\mathcal{N}(u)=(1+\partial_x)^{-1}\begin{pmatrix}0\\ u_1^{1+\sigma}\end{pmatrix},$$

with

$$(1+\partial_x)^{-1}=\overline{\mathcal{F}}_{\xi\to x}(1+i\xi)^{-1}\mathcal{F}_{x\to\xi}=e^{-x}\int_{-\infty}^{x}dx'e^{x'}.$$

Denote by

$$\lambda_1(\xi)=\frac{1}{2}+\frac{1}{2}\sqrt{1-4\xi^2},\lambda_2(\xi)=\frac{1}{2}-\frac{1}{2}\sqrt{1-4\xi^2}$$

the eigenvalues of the matrix $L(\xi)$. Note that the matrix

$$Q(\xi)=\begin{pmatrix}Q_{11}(\xi) & Q_{12}(\xi)\\ Q_{21}(\xi) & Q_{22}(\xi)\end{pmatrix}=\begin{pmatrix}1+i\xi & 1+i\xi\\ -\lambda_1(\xi) & -\lambda_2(\xi)\end{pmatrix}$$

and

$$Q^{-1}(\xi)=\frac{1}{(1+i\xi)(\lambda_1(\xi)-\lambda_2(\xi))}\begin{pmatrix}-\lambda_2(\xi) & -(1+i\xi)\\ \lambda_1(\xi) & 1+i\xi\end{pmatrix}$$

diagonalize the matrix $L(\xi)$, that is

$$Q^{-1}(\xi)L(\xi)Q(\xi)=\begin{pmatrix}\lambda_1(\xi) & 0\\ 0 & \lambda_2(\xi)\end{pmatrix}.$$

Consider the system of ordinary differential equations with constant coefficients depending on the parameter $\xi\in\mathbf{R}$

$$\frac{d}{dt}\widehat{u}(t,\xi)+L(\xi)\widehat{u}(t,\xi)=0. \tag{5.68}$$

Multiplying system (5.68) by $Q^{-1}(\xi)$ from the left and changing $\widehat{u}(t,\xi)=Q(\xi)w(t,\xi)$ we diagonalize system (5.68)

$$\frac{d}{dt}\begin{pmatrix} w_1(t,\xi) \\ w_2(t,\xi) \end{pmatrix} = -\begin{pmatrix} \lambda_1(\xi) & 0 \\ 0 & \lambda_2(\xi) \end{pmatrix}\begin{pmatrix} w_1(t,\xi) \\ w_2(t,\xi) \end{pmatrix};$$

hence by integrating with respect to time $t \geq 0$ we find

$$\begin{pmatrix} w_1(t,\xi) \\ w_2(t,\xi) \end{pmatrix} = \begin{pmatrix} e^{-t\lambda_1(\xi)} & 0 \\ 0 & e^{-t\lambda_2(\xi)} \end{pmatrix}\begin{pmatrix} w_1(0,\xi) \\ w_2(0,\xi) \end{pmatrix}.$$

Returning to the solution $\widehat{u}(t,\xi)$ we get

$$\begin{aligned} \widehat{u}(t,\xi) &= \begin{pmatrix} \widehat{u}_1(t,x) \\ \widehat{u}_2(t,x) \end{pmatrix} = Q(\xi)\begin{pmatrix} w_1(t,\xi) \\ w_2(t,\xi) \end{pmatrix} \\ &= Q(\xi)\begin{pmatrix} e^{-t\lambda_1(\xi)} & 0 \\ 0 & e^{-t\lambda_2(\xi)} \end{pmatrix} Q^{-1}(\xi)\begin{pmatrix} \widehat{u}_0(\xi) \\ \widehat{u}_1(\xi) \end{pmatrix} \\ &= e^{-tL(\xi)}\begin{pmatrix} \widehat{u}_0(\xi) \\ \widehat{u}_1(\xi) \end{pmatrix}, \end{aligned}$$

where the fundamental Cauchy matrix has the form

$$\begin{aligned} e^{-tL(\xi)} &= Q(\xi)\begin{pmatrix} e^{-t\lambda_1(\xi)} & 0 \\ 0 & e^{-t\lambda_2(\xi)} \end{pmatrix} Q^{-1}(\xi) \\ &= \frac{1}{\sqrt{1-4\xi^2}}\begin{pmatrix} -\lambda_2(\xi) & -(1+i\xi) \\ \frac{\xi^2}{1+i\xi} & \lambda_1(\xi) \end{pmatrix} e^{-t\lambda_1(\xi)} \\ &+ \frac{1}{\sqrt{1-4\xi^2}}\begin{pmatrix} \lambda_1(\xi) & 1+i\xi \\ -\frac{\xi^2}{1+i\xi} & -\lambda_2(\xi) \end{pmatrix} e^{-t\lambda_2(\xi)}. \end{aligned}$$

We rewrite the Cauchy problem (5.67) in the form of the integral equation

$$u(t) = \mathcal{G}(t)\widetilde{u} - \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau, \tag{5.69}$$

where the Green operator

$$\mathcal{G}(t)\psi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-tL(\xi)}\hat{\psi}(\xi)\right).$$

In this section we prove the following result.

Theorem 5.26. *Assume that the initial data* $\widetilde{u} \in \left(\mathbf{L}^\infty(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})\right)^2$, $a \in (0,1)$, *are such that the mean value*

$$\theta = \varepsilon \int_{\mathbf{R}} (\widetilde{u}_1(x) + \widetilde{u}_2(x))\,dx > 0.$$

Then there exists a positive ε *such that the Cauchy problem for equation (5.67) has a unique mild solution* $u(t,x) \in \left(\mathbf{C}\left([0,\infty);\mathbf{L}^\infty(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})\right)\right)^2$ *satisfying the following time decay estimate*

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon \langle t \rangle^{-\frac{1}{\sigma}}$$

for large $t > 0$ *and any* $\sigma \in \left(2 - \varepsilon^3, 2\right)$. *Furthermore the asymptotic formula is valid*

$$u(t,x) = e_1 \left((t\eta)^{-\frac{1}{\sigma}} V\left(\frac{x}{\sqrt{t}}\right) + O\left(t^{-\frac{1}{\sigma}-\gamma}\right)\right),$$

for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}$, *where* $\gamma = \frac{1}{2}\min\left(a, 1 - \frac{\sigma}{2}\right)$, $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. *Here* $V \in \mathbf{L}^{1,a}(\mathbf{R}) \cap \mathbf{L}^\infty(\mathbf{R})$ *is the solution of the integral equation*

$$V(\xi) = \frac{1}{\sqrt{4\pi}} e^{-\frac{\xi^2}{4}} - \frac{1}{\eta\sqrt{4\pi}} \int_0^1 \frac{dz}{z(1-z)^{\frac{1}{2}}} \int_{\mathbf{R}} e^{-\frac{(\xi - y\sqrt{z})^2}{4(1-z)}} F(y)\, dy, \qquad (5.70)$$

where

$$\eta = \frac{\sigma}{1 - \frac{\sigma}{2}} \int_{\mathbf{R}} V^{1+\sigma}(y)\, dy$$

and

$$F(y) = V^{1+\sigma}(y) - V(y) \int_{\mathbf{R}} V^{1+\sigma}(\xi)\, d\xi.$$

Remark 5.27. As a consequence of Theorem 5.26 we have the following asymptotics for the damped wave equation (5.66)

$$v(t,x) = (t\eta)^{-\frac{1}{\sigma}} V\left(\frac{x}{\sqrt{t}}\right) + O\left(t^{-\frac{1}{\sigma}-\gamma}\right),$$

for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$ if the initial data

$$\left(v_0, (1 + \partial_x)^{-1} v_1\right) \in \left(\mathbf{L}^\infty(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})\right)^2$$

with $a \in (0,1)$ are such that the mean value

$$\int_{\mathbf{R}} (v_0(x) + v_1(x))\, dx > 0.$$

Before proving Theorem 5.26 we obtain in Subsection 5.4.1 some preliminary estimates of the Green operator solving the linearized Cauchy problem corresponding to (5.67).

Preliminary Lemmas

First we collect some preliminary estimates for the Green operator

$$\mathcal{G}(t)\psi = \overline{\mathcal{F}}_{\xi \to x}\left(e^{-tL(\xi)} \hat{\psi}(\xi)\right)$$

in the weighted Lebesgue norms $\|\phi\|_{\mathbf{L}^p}$ and $\|\phi\|_{\mathbf{L}^{1,a}}$, where $a \in (0,1)$, $1 \leq p \leq \infty$. Also we show that $\mathcal{G}(t)$ asymptotically behaves as the Green function $\mathcal{G}_0(t)$ for the heat equation

$$\mathcal{G}_0(t)\psi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-t\xi^2}\hat{\psi}(\xi)\right) = \int_{\mathbf{R}} G_0(t, x-y)\psi(y)\,dy$$

with a kernel

$$G_0(t,x) = (2\pi)^{-\frac{1}{2}}\overline{\mathcal{F}}_{\xi\to x}\left(e^{-t\xi^2}\right) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}.$$

Denote a matrix

$$A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and a vector

$$\vartheta = \int_{\mathbf{R}} \phi(x)\,dx.$$

From Lemma 1.33 we have the following result.

Lemma 5.28. *Suppose that the vector-function $\phi \in \left(\mathbf{L}^\infty(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})\right)^2$, where $a \in (0,1)$. Then the estimates*

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^p} \leq C\|\phi\|_{\mathbf{L}^p}$$

and

$$\left\||\cdot|^\omega\left(\mathcal{G}(t)\phi - (4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} A_0\vartheta\right)\right\|_{\mathbf{L}^p} \leq Ct^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{a-\omega}{2}}\|\phi\|_{\mathbf{L}^{1,a}}$$

are valid for all $t > 0$, $1 \leq p \leq \infty$, $0 \leq \omega \leq a$.

The following lemma will be used for estimates of the nonlinearity in Theorems 5.2 - 5.5

Lemma 5.29. *Let the vector-function $\psi(t,x)$ be such that $A_0\int_{\mathbf{R}}\psi(t,x)\,dx = 0$ and the norm*

$$\sup_{t>0}\langle t\rangle^{\nu+\frac{1}{2}}\|\psi(t)\|_{\mathbf{L}^\infty} + \sup_{t>0}\langle t\rangle^{\nu-\frac{a}{2}}\|\psi(t)\|_{\mathbf{L}^{1,a}} \equiv \|\psi\|_{\mathbf{F}}$$

be finite, where $a \in (0,1)$, $\nu \in (0,1)$. Also suppose that the function $g(t)$ is such that $g(t) \geq \langle t\rangle^\mu$ for all $t > 0$, where $\mu > 0$ is such that $\mu + \nu - \frac{a}{2} < 1$. Then the following inequalities are valid

$$\left\|\int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)\psi(\tau)\,d\tau\right\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{\frac{1}{2}-\nu-\mu}\|\psi\|_{\mathbf{F}}$$

and

$$\left\|\int_0^t g^{-1}(\tau)\mathcal{G}(t-\tau)\psi(\tau)\,d\tau\right\|_{\mathbf{L}^{1,a}} \leq C\langle t\rangle^{1+\frac{a}{2}-\nu-\mu}\|\psi\|_{\mathbf{F}}$$

for all $t > 0$.

Proof. Since $A_0 \int_{\mathbf{R}} \psi(t,x)\,dx = 0$, we have by the estimates of Lemma 5.28

$$\|\mathcal{G}(t-\tau)\psi(\tau)\|_{\mathbf{L}^\infty} \le C\|\psi(\tau)\|_{\mathbf{L}^\infty}, \tag{5.71}$$

$$\|\mathcal{G}(t-\tau)\psi(\tau)\|_{\mathbf{L}^\infty} \le C(t-\tau)^{-\frac{1+a}{2}}\|\psi(\tau)\|_{\mathbf{L}^{1,a}} \tag{5.72}$$

and

$$\|\mathcal{G}(t-\tau)\psi(\tau)\|_{\mathbf{L}^{1,a}} \le C\|\psi(\tau)\|_{\mathbf{L}^{1,a}} \tag{5.73}$$

for all $0 < \tau < t$. Therefore, by virtue of (5.71) and (5.72) we get

$$\begin{aligned}
&\left\|\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)\psi(\tau)\,d\tau\right\|_{\mathbf{L}^\infty} \\
&\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{a}{2}-\frac{1}{2}}\|\psi(\tau)\|_{\mathbf{L}^{1,a}}\langle\tau\rangle^{-\mu}d\tau + C\int_{\frac{t}{2}}^t\|\psi(\tau)\|_{\mathbf{L}^\infty}\langle\tau\rangle^{-\mu}d\tau \\
&\le C\|\psi\|_{\mathbf{F}}\left(\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{a}{2}-\frac{1}{2}}\langle\tau\rangle^{\frac{a}{2}-\nu-\mu}d\tau + \int_{\frac{t}{2}}^t\langle\tau\rangle^{-\nu-\mu-\frac{1}{2}}d\tau\right) \\
&\le C\langle t\rangle^{\frac{1}{2}-\nu-\mu}\|\psi\|_{\mathbf{F}}
\end{aligned}$$

for all $t > 0$. In the same manner via (5.73) we find

$$\begin{aligned}
&\left\|\int_0^t g^{-1}(\tau)\,\mathcal{G}(t-\tau)\psi(\tau)\,d\tau\right\|_{\mathbf{L}^{1,a}} \\
&\le C\int_0^t\|\psi(\tau)\|_{\mathbf{L}^{1,a}}\langle\tau\rangle^{-\mu}d\tau \le C\|\psi\|_{\mathbf{F}}\int_0^t\langle\tau\rangle^{\frac{a}{2}-\nu-\mu}d\tau \\
&\le C\langle t\rangle^{1+\frac{a}{2}-\nu-\mu}\|\psi\|_{\mathbf{F}}
\end{aligned}$$

for all $t > 0$. Lemma 5.29 is proved.

Denote

$$\zeta = \frac{\theta^{1+\sigma}}{(4\pi)^{\frac{\sigma}{2}}\left(1-\frac{\sigma}{2}\right)(1+\sigma)^{\frac{1}{2}}}.$$

Since $2 - \varepsilon^3 < \sigma < 2$ and $\theta \ge C\varepsilon$, we may suppose that $\zeta \ge 1$.

Lemma 5.30. *Assume that* $\widetilde{w} \in \left(\mathbf{L}^{1,a}(\mathbf{R}) \cap \mathbf{L}^\infty(\mathbf{R})\right)^2$ *with* $a \in (0,1)$ *and*

$$\theta \equiv \varepsilon\left(A_0\int_{\mathbf{R}}\widetilde{w}(x)\,dx\right)_1 \ge C\varepsilon > 0.$$

Let the function $(w(t,x))_1 = w_1(t,x)$ *satisfy the estimates*

$$\|w_1(t)\|_{\mathbf{L}^{1+\sigma}} \le C\varepsilon\langle t\rangle^{-\frac{\sigma}{2(1+\sigma)}}$$

and

$$\|w_1(t) - (\mathcal{G}(t)\varepsilon\widetilde{w})_1\|_{\mathbf{L}^{1+\sigma}} \le C\varepsilon^{1+\sigma}\langle t\rangle^{-\frac{\sigma}{2(1+\sigma)}}$$

for all $t > 0$. Then there exists $\varepsilon > 0$ such that the following inequality is valid

$$1 + \int_0^t d\tau \int_{\mathbf{R}} w_1^{1+\sigma}(\tau, x)\, dx \ge \frac{1}{2}\langle t\rangle^{1-\frac{\sigma}{2}} \tag{5.74}$$

for all $t > 0$.

Proof. Via Lemma 5.28 we get

$$\|(\mathcal{G}(t)\varepsilon\widetilde{w})_1 - \theta G_0(t,x)\|_{\mathbf{L}^{1+\sigma}} \le C\varepsilon\langle t\rangle^{-\frac{\sigma}{2(1+\sigma)}-\frac{a}{2}},$$

and via the assumption

$$\|w_1(t) - (\mathcal{G}(t)\varepsilon\widetilde{w})_1\|_{\mathbf{L}^{1+\sigma}} \le C\varepsilon^{1+\sigma}\langle t\rangle^{-\frac{\sigma}{2(1+\sigma)}}$$

we have applying the Hölder inequality

$$\begin{aligned}
&\left\|w_1^{1+\sigma} - \theta^{1+\sigma}G_0^{1+\sigma}\right\|_{\mathbf{L}^1} \le \left\|w_1^{1+\sigma}(t) - (\mathcal{G}(t)\varepsilon\widetilde{w})_1^{1+\sigma}\right\|_{\mathbf{L}^1} \\
&+ \left\|(\mathcal{G}(t)\varepsilon\widetilde{w})_1^{1+\sigma} - \theta^{1+\sigma}G_0^{1+\sigma}(t,x)\right\|_{\mathbf{L}^1} \\
&\le C\left(\|w_1\|_{\mathbf{L}^{1+\sigma}}^{\sigma} + \|(\mathcal{G}(t)\varepsilon\widetilde{w})_1\|_{\mathbf{L}^{1+\sigma}}^{\sigma}\right)\|w_1(t) - (\mathcal{G}(t)\varepsilon\widetilde{w})_1\|_{\mathbf{L}^{1+\sigma}} \\
&+ C\left(\|(\mathcal{G}(t)\varepsilon\widetilde{w})_1\|_{\mathbf{L}^{1+\sigma}}^{\sigma} + \theta^{\sigma}\|G_0\|_{\mathbf{L}^{1+\sigma}}^{\sigma}\right)\|(\mathcal{G}(t)\varepsilon\widetilde{w})_1 - \theta G_0(t,x)\|_{\mathbf{L}^{1+\sigma}} \\
&\le C\varepsilon^{2+\sigma}\langle t\rangle^{-\frac{\sigma}{2}} + C\varepsilon^{1+\sigma}\langle t\rangle^{-\frac{\sigma}{2}-\frac{a}{2}}
\end{aligned}$$

for all $t > 0$. By a direct computation we obtain for the heat kernel $G_0(t,x) = (4\pi t)^{-\frac{1}{2}}e^{-\frac{x^2}{4t}}$

$$\int_{\mathbf{R}} (G_0(t,x))^{1+\sigma} dx = (1+\sigma)^{-\frac{1}{2}}(4\pi t)^{-\frac{\sigma}{2}}.$$

Therefore, we get

$$\begin{aligned}
&\left|\int_{\mathbf{R}} w_1^{1+\sigma}(t,x)\, dx - \frac{\theta^{1+\sigma}t^{-\frac{\sigma}{2}}}{(4\pi)^{\frac{\sigma}{2}}(1+\sigma)^{\frac{1}{2}}}\right| \\
&= \left|\int_{\mathbf{R}} w_1^{1+\sigma}(t,x)\, dx - \theta^{1+\sigma}\int_{\mathbf{R}} (G_0(t,x))^{1+\sigma} dx\right| \\
&\le C\left\|w_1^{1+\sigma} - \theta^{1+\sigma}G_0^{1+\sigma}\right\|_{\mathbf{L}^1} \le C\varepsilon^{2+\sigma}\langle t\rangle^{-\frac{\sigma}{2}} + C\varepsilon^{1+\sigma}\langle t\rangle^{-\frac{\sigma}{2}-\frac{a}{2}}
\end{aligned}$$

for all $t > 0$; hence

$$
\begin{aligned}
&\left|\int_0^t d\tau \int_{\mathbf{R}} w_1^{1+\sigma}(t,x)\,dx - \frac{\theta^{1+\sigma} t^{1-\frac{\sigma}{2}}}{(4\pi)^{\frac{\sigma}{2}}\left(1-\frac{\sigma}{2}\right)(1+\sigma)^{\frac{1}{2}}}\right| \\
&= \left|\int_0^t d\tau \int_{\mathbf{R}} v^{\sigma}(\tau,x)\,dx - \zeta t^{1-\frac{\sigma}{2}}\right| \\
&\le C\varepsilon^{2+\sigma}\int_0^t \langle\tau\rangle^{-\frac{\sigma}{2}}\,d\tau + C\varepsilon^{1+\sigma}\int_0^t \langle\tau\rangle^{-\frac{\sigma}{2}-\frac{a}{2}}\,d\tau \\
&\le C\varepsilon\zeta t^{1-\frac{\sigma}{2}} + C\varepsilon^{1+\sigma} t^{1-\frac{\sigma}{2}-\frac{a}{2}} \le \frac{1}{2}\zeta t^{1-\frac{\sigma}{2}}
\end{aligned}
$$

for all $t > 0$. Thus we get

$$
\int_0^t d\tau \int_{\mathbf{R}} w_1^{1+\sigma}(t,x)\,dx \ge \zeta t^{1-\frac{\sigma}{2}}. \tag{5.75}
$$

Estimate (5.75) implies (5.74), since $\zeta \ge 1$ and Lemma 5.30 is proved.

Proof of Theorem 5.26

We use the method of Theorems 5.2 - 5.5. We change the dependent variable $u(t,x) = e^{-\varphi(t)}w(t,x)$, then we get from (5.67)

$$
w_t + \mathcal{L}w + e^{-\sigma\varphi}\mathcal{N}(w) - \varphi' w = 0 \tag{5.76}
$$

or in the integral form

$$
w(t) = \mathcal{G}(t)\widetilde{w} - \int_0^t \mathcal{G}(t-\tau)\left(e^{-\sigma\varphi}\mathcal{N}(w) - \varphi' w\right)d\tau. \tag{5.77}
$$

According to the large time asymptotic behavior described by Lemma 5.28 we need to demand that the real-valued function $\varphi(t)$ satisfies the zero total mass condition

$$
A_0\int_{\mathbf{R}}\left(e^{-\sigma\varphi}\mathcal{N}(w) - \varphi' w\right)dx = 0,
$$

which is reduced to one equation

$$
\begin{aligned}
&e^{-\sigma\varphi(t)}\int_{\mathbf{R}}(1+\partial_x)^{-1} w_1^{1+\sigma}(t,x)\,dx - \varphi'(t)\int_{\mathbf{R}}(A_0 w(t,x))_1\,dx \\
&= e^{-\sigma\varphi(t)}\int_{\mathbf{R}} w_1^{1+\sigma}(t,x)\,dx - \varphi'(t)\int_{\mathbf{R}}(A_0 w(t,x))_1\,dx = 0.
\end{aligned}
$$

We also can assume that $\varphi(0) = 0$. Since, $A_0 L(0) = 0$ via equation (5.76) we get

$$
\frac{d}{dt}\int_{\mathbf{R}}(A_0 w(t,x))_1\,dx = 0:
$$

$$\int_{\mathbf{R}} (A_0 w(t,x))_1 \, dx = \varepsilon \int_{\mathbf{R}} \left(A_0 \widetilde{u}(x)\right)_1 dx$$
$$= \varepsilon \int_{\mathbf{R}} \left(v_0(x) + (1+\partial_x)^{-1} v_1(x) \right) dx$$
$$= \varepsilon \int_{\mathbf{R}} (v_0(x) + v_1(x)) \, dx \equiv \theta$$

for all $t > 0$. Therefore we obtain the equation

$$\varphi'(t) = e^{-\sigma\varphi(t)} \frac{1}{\theta} \int_{\mathbf{R}} w_1^{1+\sigma}(t,x) \, dx. \tag{5.78}$$

Multiplying equation (5.78) by the factor $e^{\sigma\varphi(t)}$, then integrating with respect to time $t > 0$ and changing the dependent variable $e^{\sigma\varphi(t)} = g(t)$, we get

$$g(t) = 1 + \frac{\sigma}{\theta} \int_0^t \int_{\mathbf{R}} w_1^{1+\sigma}(\tau,x) \, dx d\tau. \tag{5.79}$$

Thus we need to solve the following integral equation

$$w = \mathcal{M}(w), \tag{5.80}$$

where the operator

$$\mathcal{M}(w)(t) = \mathcal{G}(t) \varepsilon \widetilde{u} - \int_0^t g^{-1}(\tau) \mathcal{G}(t-\tau) f(\tau) \, d\tau$$

and

$$f(t) = \mathcal{N}(w(t)) - w(t) \frac{1}{\theta} \int_{\mathbf{R}} w_1^{1+\sigma}(t,x) \, dx.$$

We define $w^{(0)} = \mathcal{G}(t) \widetilde{u}$ and successive approximations $w^{(k+1)} = \mathcal{M}\left(w^{(k)}\right)$ for $k = 0, 1, 2, ...$ We prove that $\mathcal{M}$ is a contraction mapping in the set

$$\mathbf{X} = \Big\{ w \in \mathbf{C}\left([0,\infty); \mathbf{L}^{1,a}(\mathbf{R}) \cap \mathbf{L}^{\infty}(\mathbf{R})\right) :$$
$$\sup_{t>0} \left(\langle t \rangle^{\frac{1}{2}} \|w(t)\|_{\mathbf{L}^{\infty}} + \langle t \rangle^{-\frac{a}{2}} \|\langle \cdot \rangle^{a} w(t)\|_{\mathbf{L}^1} \right) \le 2\varepsilon \Big\}.$$

We also prove both that the mapping $\mathcal{M}$ transforms the set $\mathbf{X}$ into itself, and that the estimates are valid

$$w^{(k)} \in \mathbf{X}; \ \left\| w^{(k)}(t) - \mathcal{G}(t) \varepsilon \widetilde{u} \right\|_{\mathbf{L}^{1+\sigma}} \le C \varepsilon^{1+\sigma} \langle t \rangle^{-\frac{\sigma}{2(1+\sigma)}}, \tag{5.81}$$

$$g^{(k)}(t) \ge \frac{1}{2} \langle t \rangle^{1-\frac{\sigma}{2}}, \tag{5.82}$$

$$\int_{\mathbf{R}} A_0 f^{(k)}(t,x) \, dx = 0, \ \int_{\mathbf{R}} \left(A_0 w^{(k)}(t,x) \right)_1 dx = \theta \tag{5.83}$$

for all $t > 0$ and $k \geq 0$, where

$$f^{(k)}(t) = \mathcal{N}\left(w^{(k)}(t)\right) - w^{(k)}(t)\frac{1}{\theta}\int_{\mathbf{R}}\left(w_1^{(k)}(t,x)\right)^{1+\sigma}dx$$

and

$$g^{(k)}(t) = 1 + \frac{\sigma}{\theta}\int_0^t\int_{\mathbf{R}}\left(w_1^{(k)}(\tau,x)\right)^{1+\sigma}dxd\tau.$$

For $k = 0$ the estimates (5.81) follow from Lemma 5.28. Then applying Lemma 5.30 we get the estimate (5.82) with $k = 0$. Equalities (5.83) are true due to the definition of $w^{(0)}$. Then by induction we suppose that (5.81) to (5.83) are valid for some $k \geq 0$. Applying the identity

$$(1+\partial_x)^{-1}\psi(x) = e^{-x}\int_{-\infty}^{x}e^{y}\psi(y)\,dy;$$

we see that

$$\left\|(1+\partial_x)^{-1}\psi\right\|_{\mathbf{L}^\infty} \leq C\|\psi\|_{\mathbf{L}^\infty},$$
$$\left\|(1+\partial_x)^{-1}\psi\right\|_{\mathbf{L}^{1,a}} \leq C\|\psi\|_{\mathbf{L}^{1,a}}.$$

Therefore, as a consequence of (5.81) via interpolation inequality (1.4), we get the estimates for the function $f^{(k)}(t)$

$$\begin{aligned}\left\|f^{(k)}(t)\right\|_{\mathbf{L}^\infty} &\leq C\left\|w^{(k)}(t)\right\|_{\mathbf{L}^\infty}^{1+\sigma}\left(1+\frac{1}{\theta}\left\|w^{(k)}(t)\right\|_{\mathbf{L}^1}\right)\\ &\leq C\varepsilon^{1+\sigma}\langle t\rangle^{-\frac{1+\sigma}{2}}\end{aligned}$$

and

$$\begin{aligned}\left\|f^{(k)}(t)\right\|_{\mathbf{L}^{1,a}} &\leq C\left\|w^{(k)}(t)\right\|_{\mathbf{L}^\infty}^{\sigma}\left\|w^{(k)}(t)\right\|_{\mathbf{L}^{1,a}}\left(1+\frac{1}{\theta}\left\|w^{(k)}(t)\right\|_{\mathbf{L}^1}\right)\\ &\leq C\varepsilon^{1+\sigma}\langle t\rangle^{\frac{a-\sigma}{2}}\end{aligned}$$

which imply

$$\sup_{t>0}\left(\langle t\rangle^{\frac{1+\sigma}{2}}\left\|f^{(k)}(t)\right\|_{\mathbf{L}^\infty}+\langle t\rangle^{\frac{\sigma-a}{2}}\left\|f^{(k)}(t)\right\|_{\mathbf{L}^{1,a}}\right)\leq C\varepsilon^{1+\sigma}. \tag{5.84}$$

By (5.82) - (5.84) we can apply Lemma 5.29 with $\mu = 1-\frac{\sigma}{2}$, $\nu = \frac{\sigma}{2}$, to obtain

$$\begin{aligned}&\left\|w^{(k+1)}(t)-\mathcal{G}(t)\varepsilon\widetilde{u}\right\|_{\mathbf{L}^\infty} \leq C\left\|\int_0^t\frac{1}{g^{(k)}(\tau)}\mathcal{G}(t-\tau)f^{(k)}(\tau)\,d\tau\right\|_{\mathbf{L}^\infty}\\ &\leq C\langle t\rangle^{-\frac{1}{2}}\sup_{t>0}\left(\langle t\rangle^{\frac{1+\sigma}{2}}\left\|f^{(k)}(t)\right\|_{\mathbf{L}^\infty}+\langle t\rangle^{\frac{\sigma-a}{2}}\left\|f^{(k)}(t)\right\|_{\mathbf{L}^{1,a}}\right)\leq C\varepsilon^{1+\sigma}\langle t\rangle^{-\frac{1}{2}}\end{aligned}$$

and

$$\left\| w^{(k+1)}(t) - \mathcal{G}(t)\,\varepsilon\widetilde{u} \right\|_{\mathbf{L}^{1,a}} \le C \left\| \int_0^t \frac{1}{g^{(k)}(\tau)} \mathcal{G}(t-\tau)\, f^{(k)}(\tau)\, d\tau \right\|_{\mathbf{L}^{1,a}}$$
$$\le C \langle t \rangle^{\frac{a}{2}} \sup_{t>0} \left(\langle t \rangle^{\frac{1+\sigma}{2}} \left\| f^{(k)}(t) \right\|_{\mathbf{L}^\infty} + \langle t \rangle^{\frac{\sigma-a}{2}} \left\| f^{(k)}(t) \right\|_{\mathbf{L}^{1,a}} \right) \le C\varepsilon^{1+\sigma} \langle t \rangle^{\frac{a}{2}}$$

for all $t > 0$. In particular, we see that the estimates

$$\left\| w^{(k)}(t) \right\|_{\mathbf{L}^{1+\sigma}} \le C\varepsilon \langle t \rangle^{-\frac{\sigma}{2(1+\sigma)}}$$

and

$$\left\| w^{(k)}(t) - \mathcal{G}(t)\,\varepsilon\widetilde{u} \right\|_{\mathbf{L}^{1+\sigma}} \le C\varepsilon^{1+\sigma} \langle t \rangle^{-\frac{\sigma}{2(1+\sigma)}}$$

are true. Then the application of Lemma 5.30 yields

$$g^{(k+1)}(t) = 1 + \frac{\sigma}{\theta} \int_0^t \int_{\mathbf{R}} \left(w_1^{(k+1)}(\tau, x) \right)^{1+\sigma} dx d\tau \ge \frac{1}{2} \langle t \rangle^{1-\frac{\sigma}{2}}$$

for all $t > 0$. Therefore (5.81) and (5.82) are valid with k replaced by $k+1$. Integrating

$$w_t^{(k+1)} + \mathcal{L}w^{(k+1)} + e^{-\sigma\varphi}\mathcal{N}\left(w^{(k)} \right) - \varphi' w^{(k)} = 0$$

with respect to $x \in \mathbf{R}$ we get

$$\frac{d}{dt} \int_{\mathbf{R}} w^{(k+1)} dx + \int_{\mathbf{R}} \left(e^{-\sigma\varphi}\mathcal{N}\left(w^{(k)} \right) - \varphi' w^{(k)} \right) dx = 0.$$

In view of (5.83)

$$A_0 \int_{\mathbf{R}} w^{(k+1)}(t, x)\, dx = A_0 \int_{\mathbf{R}} \widetilde{u}(x)\, dx = \theta e_1, \tag{5.85}$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore by (5.85)

$$A_0 \int_{\mathbf{R}} f^{(k+1)}(t, x)\, dx$$
$$= A_0 \int_{\mathbf{R}} \left(\mathcal{N}\left(w^{(k+1)}(t, x) \right) - w^{(k+1)}(t) \frac{1}{\theta} \int_{\mathbf{R}} \left(w_1^{(k+1)}(t, x) \right)^{1+\sigma} dx \right) dx$$
$$= A_0 \int_{\mathbf{R}} \mathcal{N}\left(w^{(k+1)}(t, x) \right) dx - \int_{\mathbf{R}} \left(w_1^{(k+1)}(t, x) \right)^{1+\sigma} dx = 0.$$

By induction we see that properties (5.81) - (5.83) are true for all $k \ge 0$. Thus the mapping $\mathcal{M}$ transforms the set $\mathbf{X}$ into itself. In the same manner we can estimate the differences $\mathcal{M}(w^{(k)}) - \mathcal{M}(w^{(k-1)})$ to see that $\mathcal{M}$ is the contraction mapping in $\mathbf{X}$. Therefore, there exists a unique solution w of integral equation (5.80) in the set $\mathbf{X}$.

Now let us compute the asymptotics of the solution. From the proof of Theorem 5.7 we see that there exists a unique solution to the integral equation

$$V(\xi) = (4\pi)^{-\frac{1}{2}} e^{-\frac{\xi^2}{4}} - \frac{1}{\eta (4\pi)^{\frac{1}{2}}} \int_0^1 \frac{dz}{z(1-z)^{\frac{1}{2}}} \int_{\mathbf{R}} e^{-\frac{(\xi - y\sqrt{z})^2}{4(1-z)}} F(y)\, dy, \quad (5.86)$$

where

$$\eta = \frac{\sigma}{1-\frac{\sigma}{2}} \int_{\mathbf{R}} V^{1+\sigma}(\xi)\, d\xi$$

and

$$F(y) = V^{1+\sigma}(y) - V(y) \int_{\mathbf{R}} V^{1+\sigma}(\xi)\, d\xi.$$

Next by using the method of the proof of Theorem 5.5 we get the estimate

$$\left\| w(t) - e_1 t^{-\frac{1}{2}} \theta V\left(\frac{\cdot}{\sqrt{t}}\right) \right\|_{\mathbf{L}^p} \le C\varepsilon t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\gamma} \quad (5.87)$$

for all $t > 0$. That is we have the asymptotics

$$w(t) = t^{-\frac{1}{2}} e_1 \theta V\left(\frac{\cdot}{\sqrt{t}}\right) + O\left(t^{-\frac{1}{2}-\gamma}\right) \quad (5.88)$$

and

$$g(t) = \theta^{\sigma} \eta t^{1-\frac{\sigma}{2}} \left(1 + O\left(t^{-\gamma}\right)\right) \quad (5.89)$$

for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$. Therefore via the formula $u(t,x) = e^{-\varphi(t)} w(t,x)$, (5.88) and (5.89) we obtain the asymptotics of the solution. Theorem 5.26 is proved.

5.4.2 Large data

Consider the one dimensional nonlinear damped wave equation of the form

$$\begin{cases} u_{tt} + u_t - u_{xx} = \lambda |u|^{\sigma} u, \ x \in \mathbf{R}, \ t > 0, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \ x \in \mathbf{R}, \end{cases} \quad (5.90)$$

where $\sigma > 0$, $\lambda < 0$. Our aim is to prove the large time asymptotic formulas for the solutions of the Cauchy problem (5.90) without any restriction on the size of the initial data $u_0(x)$ and $u_1(x)$. We study the one dimensional case for simplicity, the higher dimensional cases also can be considered by our method.

We will prove the following result.

Theorem 5.31. *Let $\lambda < 0$. Assume that the initial data $u_0 \in \mathbf{W}_1^2 \cap \mathbf{H}^2 \cap \mathbf{L}^{1,a} \cap \mathbf{C}^3$, $u_1 \in \mathbf{W}_1^1 \cap \mathbf{H}^1 \cap \mathbf{L}^{1,a} \cap \mathbf{C}^2$, $a \in (0,1)$, are such that $\theta > 0$. Also suppose that the value $\sigma < 2$ is close to 2. Then the Cauchy problem (5.90)* *has a unique global solution* $u \in \mathbf{C}([0,\infty);\mathbf{W}_1^2 \cap \mathbf{H}^2 \cap \mathbf{L}^{1,a} \cap \mathbf{C}^3)$*, satisfying the following time decay estimates*

$$\|u(t)\|_{\mathbf{L}^\infty} \le C \langle t \rangle^{-\frac{1}{\sigma}}$$

for large $t > 0$*. Furthermore there exist a constant* b *and a function* $V \in \mathbf{L}^\infty \cap \mathbf{L}^{1,a}$ *such that the asymptotic formula is valid*

$$u(t,x) = bt^{-\frac{1}{\sigma}} V\left(xt^{-\frac{1}{2}}\right) + O\left(t^{-\frac{1}{\sigma}-\gamma}\right),$$

for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}$*, where* $\gamma = \frac{1}{2}\min\left(a, 1 - \frac{\sigma}{2}\right)$*, and* $V \in \mathbf{L}^{1,a} \cap \mathbf{L}^\infty$ *is a solution of the integral equation (5.70).*

Remark 5.32. The restriction for the value $\sigma < 2$ to be close to 2 is rather technical. It comes from the application of the contraction mapping principle in the proof of Theorem 5.31 below. We believe that the asymptotic behavior of solutions obtained in Theorem 5.31 is general whenever $\sigma \in (0,2)$ with some decay restrictions on the initial data, as it happens for the nonlinear heat equation (see Escobedo et al. [1995]).

Proof of Theorem 5.31

By Lemma 2.40, Lemma 2.41, Lemma 3.42 we get the time decay estimate

$$\sup_{t \ge 0} \left(\langle t \rangle^{\gamma + \frac{3}{2}} \|u_{tt}(t)\|_{\mathbf{L}^\infty} + \langle t \rangle^{\gamma + 1 - \frac{a}{2}} \|u_{tt}(t)\|_{\mathbf{L}^{1,a}} \right) \le C,$$

where $\gamma > 0$. Denote $f(t,x) = u_{tt}$ and consider two auxiliary Cauchy problems

$$\begin{cases} U_t - U_{xx} + |\lambda| U^{1+\sigma} = \varepsilon^2 |f|, \ x \in \mathbf{R}, \ t > 0, \\ U(0,x) = \varepsilon |u_0(x)|, \ x \in \mathbf{R}, \end{cases} \tag{5.91}$$

and

$$\begin{cases} V_t - V_{xx} + \varepsilon^{2\sigma} |\lambda| V^{1+\sigma} = |f|, \ x \in \mathbf{R}, \ t > 0, \\ V(0,x) = \frac{1}{\varepsilon} |u_0(x)|, \ x \in \mathbf{R}. \end{cases} \tag{5.92}$$

with sufficiently small $\varepsilon > 0$. Thus we can apply the results of Section 5.2 to calculate the large time asymptotic behavior of solutions $U(t,x)$ and $V(t,x)$ to the problems (5.91) and (5.92) respectively. By Lemma 3.45 $|u(t,x)| \le C\varepsilon^{-2} |U(t,x)|$ and we get an optimal time decay estimate for the solution

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\varepsilon^{-1} \langle t \rangle^{-\frac{1}{2}} \left(1 + C\varepsilon (2-\sigma)^{-\frac{1}{\sigma}} t^{\frac{1}{\sigma} - \frac{1}{2}}\right)^{-1} \tag{5.93}$$

for all $t > 0$. Now we prove the asymptotics of solutions. As above we make a change of the dependent variable $u(t,x) = e^{-\varphi(t)} w(t,x)$, then we get the integral equation (5.77).

Let us prove the estimate

$$\|w(t)\|_{\mathbf{L}^{1,a}} \le C\langle t\rangle^{\frac{a}{2}} \tag{5.94}$$

for all $t > 0$. By estimate (5.93) we have

$$\|u(t)\|_{\mathbf{L}^\infty}^\sigma \le C\langle t\rangle^{-\frac{\sigma}{2}}\left(1 + C\varepsilon^\sigma (2-\sigma)^{-1} t^{1-\frac{1}{2}\sigma}\right)^{-1},$$

We use the integral equation (5.77). By virtue of estimate (5.93), Lemma 3.42 and Lemma 5.28 we get

$$\begin{aligned}
&\left\| \int_0^t d\tau \mathcal{J}(t-\tau)\left(\begin{pmatrix}0\\1\end{pmatrix}(1+\partial_x)^{-1}|u(\tau,x)|^\sigma w_1(\tau,x)\right.\right.\\
&\left.\left. - w(\tau)\frac{1}{\theta}\int_{\mathbf{R}} |u(\tau,x)|^\sigma w_1(\tau,x)\,dx\right) d\tau\right\|_{\mathbf{L}^{1,a}}\\
&\le C\varepsilon^{-\sigma}(2-\sigma)\int_0^t \langle\tau\rangle^{-1}\|w(\tau)\|_{\mathbf{L}^{1,a}}\,d\tau
\end{aligned}$$

for all $t > 0$. Therefore in view of equation (5.77) we obtain

$$\|w(t)\|_{\mathbf{L}^{1,a}} \le C\langle t\rangle^{\frac{a}{2}} + C\varepsilon^{-\sigma}(2-\sigma)\int_0^t \langle\tau\rangle^{-1}\|w(\tau)\|_{\mathbf{L}^{1,a}}\,d\tau$$

for all $t > 0$. Application of the Gronwall lemma yields the estimate

$$\|w(t)\|_{\mathbf{L}^{1,a}} \le C\langle t\rangle^{\frac{a}{2}} \tag{5.95}$$

for all $t > 0$. In the same manner by virtue of (5.93), Lemma 3.42 and Lemma 5.28 we find

$$\begin{aligned}
&\|w(t)\|_{\mathbf{L}^p} \le \|\mathcal{J}(t)\widetilde{w}\|_{\mathbf{L}^p}\\
&+\left\| \int_0^t d\tau \mathcal{J}(t-\tau)\left(\begin{pmatrix}0\\1\end{pmatrix}(1+\partial_x)^{-1}|u(\tau,x)|^\sigma w_1(\tau,x)\right.\right.\\
&\left.\left. - w(\tau)\frac{1}{\theta}\int_{\mathbf{R}} |u(\tau,x)|^\sigma w_1(\tau,x)\,dx\right) d\tau\right\|_{\mathbf{L}^p}\\
&\le C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} + C\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{a}{2}}\langle\tau\rangle^{\frac{a}{2}-1}\,d\tau\\
&+ C\varepsilon^{-\sigma}(2-\sigma)\int_{\frac{t}{2}}^t \langle\tau\rangle^{-1}\|w(\tau)\|_{\mathbf{L}^p}\,d\tau\\
&\le C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} + \epsilon\int_{\frac{t}{2}}^t \langle\tau\rangle^{-1}\|w(\tau)\|_{\mathbf{L}^p}\,d\tau
\end{aligned}$$

for all $t > 0$, since $\sigma < 2$ is close to 2. Application of the Gronwall lemma yields the estimate

$$\|w(t)\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}$$

for all $t>0$. The above estimates now imply that

$$\left\| w_1(t) - \left(\mathcal{J}(t)\widetilde{w}\right)_1 \right\|_{\mathbf{L}^p} \le C(2-\sigma)\langle t\rangle^{-\frac{\sigma}{2(1+\sigma)}}$$

for all $t>0$.

As in derivation of (5.79) making a change of the dependent variable $e^{\sigma\varphi(t)} = g(t)$, we get

$$g(t) = 1 + \frac{|\lambda|\sigma}{\theta}\int_0^t \int_{\mathbf{R}} |w_1(\tau,x)|^{\sigma} w_1(\tau,x)\,dxd\tau. \qquad (5.96)$$

Denote $\zeta = \frac{\theta^{1+\sigma}}{(4\pi)^{\frac{\sigma}{2}}\left(1-\frac{\sigma}{2}\right)(1+\sigma)^{\frac{1}{2}}}$. Since $0<\sigma<2$ is close to 2, we may suppose that $\zeta \ge 1$.

Lemma 5.33. *We assume that* $\widetilde{w} \in \left(\mathbf{L}^{1,a}\cap\mathbf{L}^{\infty}\right)^2$ *with* $a\in(0,1)$ *and* $\theta \equiv \left(A_0\int_{\mathbf{R}}\widetilde{w}(x)\,dx\right)_1 > 0$. *Let the function* $(w(t,x))_1 = w_1(t,x)$ *satisfy the estimates*

$$\|w_1(t)\|_{\mathbf{L}^{1+\sigma}} \le C\langle t\rangle^{-\frac{\sigma}{2(1+\sigma)}}$$

and

$$\left\| w_1(t) - \left(\mathcal{J}(t)\widetilde{w}\right)_1 \right\|_{\mathbf{L}^{1+\sigma}} \le C(2-\sigma)\langle t\rangle^{-\frac{\sigma}{2(1+\sigma)}}$$

for all $t>0$. *Then the following inequality is valid*

$$\left| \int_0^t d\tau \int_{\mathbf{R}} |w_1(t,x)|^{\sigma} w_1(t,x)\,dx - \zeta t^{1-\frac{\sigma}{2}} \right| \le Ct^{1-\frac{\sigma}{2}} \qquad (5.97)$$

for all $t>0$.

Proof. We have

$$\|\phi\|_{\mathbf{L}^p} \le C\|\phi\|_{\mathbf{L}^{\infty}}^{1-\frac{1}{p(1+a)}} \left\|\langle\cdot\rangle^a\phi\right\|_{\mathbf{L}^1}^{\frac{1}{p(1+a)}} \qquad (5.98)$$

for all $1\le p\le\infty$. Then via Lemma 5.28 we get

$$\left\| \left(\mathcal{J}(t)\widetilde{w}\right)_1 - \theta G_0(t,x) \right\|_{\mathbf{L}^{1+\sigma}} \le C\langle t\rangle^{-\frac{\sigma}{2(1+\sigma)}-\frac{a}{2}},$$

and via the assumption

$$\left\| w_1(t) - \left(\mathcal{J}(t)\widetilde{w}\right)_1 \right\|_{\mathbf{L}^{1+\sigma}} \le C(2-\sigma)\langle t\rangle^{-\frac{\sigma}{2(1+\sigma)}}$$

we have applying the Hölder inequality

$$\begin{aligned}
&\left\| |w_1(t)|^{\sigma} w_1(t) - \theta^{1+\sigma}G_0^{1+\sigma} \right\|_{\mathbf{L}^1} \le \left\| |w_1(t)|^{\sigma} w_1(t) - \left(\mathcal{J}(t)\widetilde{w}\right)_1^{1+\sigma} \right\|_{\mathbf{L}^1} \\
&+ \left\| \left(\mathcal{J}(t)\widetilde{w}\right)_1^{1+\sigma} - \theta^{1+\sigma}G_0^{1+\sigma}(t,x) \right\|_{\mathbf{L}^1} \\
&\le C\left(\|w_1\|_{\mathbf{L}^{1+\sigma}}^{\sigma} + \left\|\left(\mathcal{J}(t)\widetilde{w}\right)_1\right\|_{\mathbf{L}^{1+\sigma}}^{\sigma} \right) \left\| w_1(t) - \left(\mathcal{J}(t)\widetilde{w}\right)_1 \right\|_{\mathbf{L}^{1+\sigma}} \\
&+ C\left(\left\|\left(\mathcal{J}(t)\widetilde{w}\right)_1\right\|_{\mathbf{L}^{1+\sigma}}^{\sigma} + \theta^{\sigma}\|G_0\|_{\mathbf{L}^{1+\sigma}}^{\sigma} \right) \left\| \left(\mathcal{J}(t)\widetilde{w}\right)_1 - \theta G_0(t,x) \right\|_{\mathbf{L}^{1+\sigma}} \\
&\le C(2-\sigma)\langle t\rangle^{-\frac{\sigma}{2}} + C\langle t\rangle^{-\frac{\sigma}{2}-\frac{a}{2}}
\end{aligned}$$

for all $t > 0$. By a direct computation we obtain for the heat kernel $G_0(t,x) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}$

$$\int_{\mathbf{R}} (G_0(t,x))^{1+\sigma} dx = (1+\sigma)^{-\frac{1}{2}} (4\pi t)^{-\frac{\sigma}{2}}.$$

Therefore we get

$$\begin{aligned}
&\left| \int_{\mathbf{R}} |w_1(t,x)|^{\sigma} w_1(t,x)\, dx - \frac{\theta^{1+\sigma} t^{-\frac{\sigma}{2}}}{(4\pi)^{\frac{\sigma}{2}} (1+\sigma)^{\frac{1}{2}}} \right| \\
&= \left| \int_{\mathbf{R}} |w_1(t,x)|^{\sigma} w_1(t,x)\, dx - \theta^{1+\sigma} \int_{\mathbf{R}} (G_0(t,x))^{1+\sigma} dx \right| \\
&\le C \left\| |w_1(t)|^{\sigma} w_1(t) - \theta^{1+\sigma} G_0^{1+\sigma} \right\|_{\mathbf{L}^1} \\
&\le C(2-\sigma) \langle t \rangle^{-\frac{\sigma}{2}} + C \langle t \rangle^{-\frac{\sigma}{2}-\frac{a}{2}}
\end{aligned}$$

for all $t > 0$. Hence

$$\begin{aligned}
&\left| \int_0^t d\tau \int_{\mathbf{R}} |w_1(t,x)|^{\sigma} w_1(t,x)\, dx - \frac{\theta^{1+\sigma} t^{1-\frac{\sigma}{2}}}{(4\pi)^{\frac{\sigma}{2}} \left(1-\frac{\sigma}{2}\right) (1+\sigma)^{\frac{1}{2}}} \right| \\
&= \left| \int_0^t d\tau \int_{\mathbf{R}} |w_1(t,x)|^{\sigma} w_1(t,x)\, dx - \zeta t^{1-\frac{\sigma}{2}} \right| \\
&\le C(2-\sigma) \int_0^t \langle \tau \rangle^{-\frac{\sigma}{2}} d\tau + C \int_0^t \langle \tau \rangle^{-\frac{\sigma}{2}-\frac{a}{2}} d\tau \le C t^{1-\frac{\sigma}{2}}
\end{aligned}$$

for all $t > 0$. Lemma 5.33 is proved.

In view of estimates (5.95), using Lemma 5.33 we see that

$$\left| g(t) - 1 - \zeta t^{1-\frac{\sigma}{2}} \right| \le C t^{1-\frac{\sigma}{2}}$$

for all $t > 0$, where

$$\zeta = \frac{\theta^{1+\sigma}}{(4\pi)^{\frac{\sigma}{2}} \left(1-\frac{\sigma}{2}\right) (1+\sigma)^{\frac{1}{2}}}$$

is sufficiently large. Now the asymptotic formulas are proved in the same manner as in the proof of Theorem 5.26. Theorem 5.31 is proved.

5.5 Sobolev type equations

This section is devoted to the study of the Cauchy problem (3.172) in the subcritical case

$$\begin{cases} \partial_t (u - \Delta u) - \alpha \Delta u = \lambda |u|^{\sigma} u,\ x \in \mathbf{R}^n,\ t > 0, \\ u(0,x) = u_0(x),\ x \in \mathbf{R}^n, \end{cases} \tag{5.99}$$

where $\alpha > 0$, $0 < \sigma < \frac{2}{n}$, $\lambda < 0$. By using the Duhamel principle we can rewrite the Cauchy problem (5.99) in the form

$$u(t) = \mathcal{G}(t) u_0 + \lambda \int_0^t \mathcal{G}(t-\tau) \mathcal{B} |u|^\sigma u(\tau) d\tau,$$

where the Green operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t) \phi = e^{-\alpha t} \overline{\mathcal{F}}_{\xi \to x} e^{\frac{\alpha t}{1+|\xi|^2}} \hat{\phi}(\xi)$$

and

$$\mathcal{B}\phi = \int_{\mathbf{R}^n} B(x-y) \phi(y) dy$$

with the Bessel-Macdonald kernel

$$B(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{i\xi x} \left(1 + |\xi|^2\right)^{-1} d\xi = |x|^{1-\frac{n}{2}} K_{\frac{n}{2}-1}(|x|).$$

Here

$$K_\nu(|x|) = K_{-\nu}(|x|) = 2^{-\nu-1} |x|^\nu \int_0^\infty \xi^{-\nu-1} e^{-\xi - \frac{|x|^2}{4\xi}} d\xi$$

is the Macdonald function (or modified Bessel function) of order $\nu \in \mathbf{R}$.

5.5.1 Small data

We prove global in time existence of small solutions to the Cauchy problem (5.99) in the subcritical case $0 < \sigma < \frac{2}{n}$. Denote $\widetilde{G}(x) = (4\pi\alpha)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4\alpha}}$.

Theorem 5.34. *Let $0 < \sigma < \frac{2}{n}$. Assume that the initial data $u_0 \in \mathbf{L}^\infty(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})$, $a \in (0,1)$ are sufficiently small $\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^{1,a}} \le \varepsilon$, and $\lambda\theta \le -C\varepsilon < 0$, where $\theta = \int_{\mathbf{R}^n} u_0(x)dx$. Also suppose that the value σ is close to $\frac{2}{n}$ so that $\frac{2}{n} - \sigma \le C\varepsilon^\sigma$. Then the Cauchy problem (5.99) has a unique global solution $u(t,x) \in \mathbf{C}\left([0,\infty); \mathbf{L}^\infty(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})\right)$, satisfying the following time decay estimates*

$$\|u(t)\|_{\mathbf{L}^\infty} \le C \langle t \rangle^{-\frac{1}{\sigma}} \tag{5.100}$$

for large $t > 0$. Furthermore there exist a number A and a function $V \in \mathbf{L}^{1,a} \cap \mathbf{L}^\infty$ such that the asymptotic formula is valid

$$u(t,x) = At^{-\frac{1}{\sigma}} V\left(xt^{-\frac{1}{2}}\right) + O\left(t^{-\frac{1}{\sigma}-\gamma}\right) \tag{5.101}$$

for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$, where $\gamma = \frac{1}{2}\min\left(a, 1 - \frac{n\sigma}{2}\right)$, and $V(\xi)$ is the solution of the integral equation

$$V(\xi) = \widetilde{G}(\xi) - \frac{1}{\beta}\int_0^1 \frac{dz}{z(1-z)^{\frac{n}{2}}} \times \int_{\mathbf{R}^n} \widetilde{G}\left(\left(\xi - yz^{\frac{1}{2}}\right)(1-z)^{-\frac{1}{2}}\right) F(y) dy, \tag{5.102}$$

with

$$\beta = \frac{\sigma}{1 - \frac{n}{2}\sigma} \int_{\mathbf{R}^n} V^{1+\sigma}(y)\, dy$$

and

$$F(y) = V^{1+\sigma}(y) - V(y) \int_{\mathbf{R}^n} V^{1+\sigma}(\xi)\, d\xi.$$

Proof of Theorem 5.34. The Green operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t)\phi = e^{-\alpha t} \overline{\mathcal{F}}_{\xi \to x} e^{\frac{\alpha t}{1+|\xi|^2}} \hat{\phi}(\xi).$$

Denote

$$\mathbf{Z} = \left\{ \phi \in \mathbf{L}^{\infty}(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R}), a \in (0,1) \right\},$$

with norm

$$\|\phi\|_{\mathbf{Z}} = (\|\phi(t)\|_{\mathbf{L}^{\infty}} + \|\phi(t)\|_{\mathbf{L}^{1,a}})$$

and

$$\mathbf{X} = \left\{ \phi \in \mathbf{C}\left([0,\infty), \mathbf{L}^{\infty} \cap \mathbf{L}^{1,a}\right), a \in (0,1) \right\},$$

where the norm $\|\cdot\|_{\mathbf{X}}$ is defined as above by

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0} \left(\langle t \rangle^{\frac{n}{2}} \|\phi(t)\|_{\mathbf{L}^{\infty}} + \langle t \rangle^{-\frac{a}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a}} \right).$$

Applying Lemma 3.49 we get that $\mathcal{G}$ is a self-similar asymptotic operator in spaces $\mathbf{X}, \mathbf{Z}$ with functional

$$f(\phi) = \int_{\mathbf{R}^n} \phi dx.$$

Also the function

$$G(x,t) = t^{-\frac{n}{2}} \widetilde{G}\left(x t^{-\frac{1}{2}}\right)$$

is the asymptotic kernel. Denote

$$g(t) = 1 + \frac{2|\theta|^{\sigma}\eta}{2 - n\sigma} t^{1 - \frac{\sigma}{2}n},$$

and

$$\eta = \sigma\lambda (4\pi\alpha)^{-\frac{n}{2}\sigma} (1+\sigma)^{-\frac{n}{2}}.$$

By a direct calculation we have

$$\frac{\sigma}{\theta} \operatorname{Re} f(\mathcal{N}(\theta G(t))) \geq \frac{\eta}{2} \theta^{\sigma} t^{-\mu}, \mu = 1 - \frac{n}{2}\sigma$$

so $\mathcal{N}$ is subcritical nonconvective type nonlinearity. We have

$$\begin{aligned}\left\|g^{-1}\mathcal{K}(v)\right\|_{\mathbf{L}^\infty} &\le Cg^{-1}\|v\|_{\mathbf{L}^\infty}^{1+\sigma}\left(1+\frac{1}{|\theta|}\|v(t)\|_{\mathbf{L}^1}\right)\\ &\le C\langle t\rangle^{-1-\frac{n}{2}}\|v\|_{\mathbf{X}}^{1+\sigma}\left(1+\frac{1}{|\theta|}\|v(t)\|_{\mathbf{X}}\right),\end{aligned} \tag{5.103}$$

and

$$\begin{aligned}&\left\|g^{-1}\mathcal{K}(v)\right\|_{\mathbf{L}^{1,a}} \le Cg^{-1}(t)\|v(t)\|_{\mathbf{L}^\infty}^{\sigma}\|v(t)\|_{\mathbf{L}^{1,a}}\left(1+\frac{1}{|\theta|}\|v(t)\|_{\mathbf{L}^1}\right)\\ &\le C\langle t\rangle^{-1+\frac{\alpha}{2}}\|v\|_{\mathbf{X}}^{1+\sigma}\left(1+\frac{1}{|\theta|}\|v(t)\|_{\mathbf{X}}\right)\end{aligned}$$

for all $t>0$. This yields the estimate

$$\left\|\langle t\rangle g^{-1}\mathcal{K}(v)\right\|_{\mathbf{X}} \le C\|v\|_{\mathbf{X}}^{1+\sigma}\left(1+\frac{1}{|\theta|}\|v(t)\|_{\mathbf{X}}\right).$$

Therefore,

$$\left\|\int_0^t \mathcal{G}(t-\tau)g^{-1}\mathcal{K}(v)\,d\tau\right\|_{\mathbf{X}} \le C\|v\|_{\mathbf{X}}^{1+\sigma}\left(1+\frac{1}{|\theta|}\|v(t)\|_{\mathbf{X}}\right).$$

In the same way we can prove

$$\begin{aligned}&\left\|\langle t\rangle^{\gamma}\int_0^t \mathcal{G}(t-\tau)g^{-1}\left(\mathcal{K}(v)-\mathcal{K}(w)\right)d\tau\right\|_{\mathbf{X}}\\ &\le C\left(\|v\|_{\mathbf{X}}^{\sigma}+\|w\|_{\mathbf{X}}^{\sigma}\right)\|v-w\|_{\mathbf{X}}\left(1+\frac{\|v(t)\|_{\mathbf{X}}+\|w(t)\|_{\mathbf{X}}}{|\theta|}\right).\end{aligned}$$

Also we have

$$\begin{aligned}&\left|\int_{\mathbf{R}^n}\left(|v|^{\sigma}v(t,x)-|w|^{\sigma}w(t,x)\right)dx\right|\\ &\le Ct^{-\frac{n\sigma}{2}}\left(\|v\|_{\mathbf{X}}^{\sigma}+\|w\|_{\mathbf{X}}^{\sigma}\right)\|(v-w)\|_{\mathbf{X}}\left(1+\frac{\|v(t)\|_{\mathbf{X}}+\|w(t)\|_{\mathbf{X}}}{|\theta|}\right).\end{aligned}$$

Thus all conditions of Theorems 5.2 to 5.5 are fulfilled, and this completes the proof of Theorem 5.34.

5.5.2 Large data

In this section we remove the smallness condition of the initial data and prove global in time existence of solutions to the Cauchy problem (5.99) with subcritical $\sigma\in\left(0,\frac{2}{n}\right)$ powers of the nonlinearity. We suppose that $\sigma\in\left(0,\frac{2}{n}\right)$ is sufficiently close to $\frac{2}{n}$.

Theorem 5.35. *Let $\sigma > 1$ for $n = 1$ and $\sigma > \frac{3}{4}$ for $n = 2$. Suppose that the initial data $u_0 \in \mathbf{W}^2_\infty(\mathbf{R}^n) \cap \mathbf{W}^2_1(\mathbf{R}^n) \cap \mathbf{L}^{1,a}(\mathbf{R}^n)$, $0 < a \leq 1$, are such that $\lambda\theta < 0$, where $\theta = \int_{\mathbf{R}^n} u_0(x)\,dx$. Suppose that $\frac{2}{n} - \varepsilon < \sigma < \frac{2}{n}$, where $\varepsilon > 0$ is sufficiently small. Then there exists a unique global solution $u(t,x) \in \mathbf{C}\left([0,\infty); \mathbf{L}^{1,a}(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n)\right)$ of the Cauchy problem (5.99), satisfying time decay estimate (5.100) and asymptotic formulas (5.101) and (5.102).*

Proof of Theorem 5.35. As in the proof of Theorem 3.51 by Lemma 3.53 we have

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{1-\frac{n}{2}\sigma-\frac{n}{2}}$$

for all $t > 0$. In the same manner by Lemma 3.49

$$\begin{aligned}
&\|\Delta u(t)\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-1-\frac{n}{2}} + \int_0^t \|\Delta\mathcal{G}(t-\tau)\mathcal{B}|u|^\sigma u(\tau)\|_{\mathbf{L}^\infty}\,d\tau \\
&\leq C\langle t\rangle^{-1-\frac{n}{2}} + C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-1-\frac{n}{2}} \|u(\tau)\|^{\sigma+1}_{\mathbf{L}^{\sigma+1}}\,d\tau \\
&+ C\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-1} \|u(\tau)\|^{\sigma+1}_{\mathbf{L}^\infty}\,d\tau \\
&\leq C\langle t\rangle^{-1-\frac{n}{2}} + C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-1-\frac{n}{2}} \langle\tau\rangle^{-\frac{n}{2}\sigma}\,d\tau \\
&+ C\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-1} \langle t\rangle^{\left(1-\frac{n}{2}\sigma-\frac{n}{2}\right)(\sigma+1)}\,d\tau \leq C\langle t\rangle^{\left(1-\frac{n}{2}\sigma-\frac{n}{2}\right)(\sigma+1)}
\end{aligned}$$

for all $t > 0$. Denote $f(t,x) = \Delta u_t$. Then by estimates of Lemma 3.53 and Lemma 3.49 we have

$$\begin{aligned}
&\|f(t)\|_{\mathbf{L}^\infty} \leq \|\Delta\partial_t\mathcal{G}(t)u_0\|_{\mathbf{L}^\infty} + \|\Delta\mathcal{B}|u|^\sigma u(t)\|_{\mathbf{L}^\infty} \\
&+ C\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-1} \|\Delta\mathcal{B}|u|^\sigma u(\tau)\|_{\mathbf{L}^\infty}\,d\tau \\
&+ C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}-2} \left(\||u|^\sigma u(\tau)\|_{\mathbf{L}^1} + \|\mathcal{B}|u|^\sigma u(\tau)\|_{\mathbf{L}^\infty}\right)d\tau \\
&\leq C\langle t\rangle^{\left(1-\frac{n}{2}\sigma-\frac{n}{2}\right)(2\sigma+1)} + C\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-1} \langle\tau\rangle^{\left(1-\frac{n}{2}\sigma-\frac{n}{2}\right)(2\sigma+1)}\,d\tau \\
&+ C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{2}-2} \langle\tau\rangle^{-\frac{n}{2}\sigma}\,d\tau \leq C\langle t\rangle^{-1-\frac{1}{\sigma}}
\end{aligned}$$

for all $t > 0$, since $\left(1-\frac{n}{2}\sigma-\frac{n}{2}\right)(2\sigma+1) \leq -1-\frac{1}{\sigma}$ and $-\frac{n}{2}-1-\frac{n}{2}\sigma \leq -1-\frac{1}{\sigma}$, when $\sigma \in \left(0,\frac{2}{n}\right)$ is sufficiently close to $\frac{2}{n}$.

In the same manner we estimate the $\mathbf{L}^{1,a}$ norm of f. By Lemma 3.49 we find

$$\begin{aligned}
&\|f(t)\|_{\mathbf{L}^{1,a}} \leq \|\Delta\partial_t \mathcal{G}(t)\, u_0\|_{\mathbf{L}^{1,a}} + \|\Delta\mathcal{B}\,|u|^\sigma u(t)\|_{\mathbf{L}^{1,a}} \\
&+ C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-2}\left(\||u|^\sigma u(\tau)\|_{\mathbf{L}^{1,a}} + \langle t-\tau\rangle^{\frac{a}{2}}\||u|^\sigma u(\tau)\|_{\mathbf{L}^1}\right)d\tau \\
&+ C\int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-1}\left(\|\Delta\mathcal{B}\,|u|^\sigma u(\tau)\|_{\mathbf{L}^{1,a}} + \langle t-\tau\rangle^{\frac{a}{2}}\|\Delta\mathcal{B}\,|u|^\sigma u(\tau)\|_{\mathbf{L}^1}\right)d\tau \\
&\leq C\langle t\rangle^{\frac{a}{2}-1-\frac{1}{\sigma}+\frac{n}{2}} + C\int_0^{\frac{t}{2}}\left(\langle t-\tau\rangle^{-2}\langle\tau\rangle^{\frac{a}{2}-1} + \langle t-\tau\rangle^{\frac{a}{2}-2}\langle\tau\rangle^{-1}\right)d\tau \\
&+ C\int_{\frac{t}{2}}^{t}\left(\langle t-\tau\rangle^{-1}\langle\tau\rangle^{\frac{a}{2}-1-\frac{1}{\sigma}+\frac{n}{2}} + \langle t-\tau\rangle^{\frac{a}{2}-1}\langle\tau\rangle^{-1-\frac{1}{\sigma}+\frac{n}{2}}\right)d\tau \\
&\leq C\langle t\rangle^{\frac{a}{2}-1-\frac{1}{\sigma}+\frac{n}{2}},
\end{aligned}$$

since $\sigma \in \left(0, \frac{2}{n}\right)$ is sufficiently close to $\frac{2}{n}$.

We now consider two auxiliary Cauchy problems (5.91) and (5.92) with sufficiently small $\varepsilon > 0$. Applying the results of Section 5.2 to calculate the large time asymptotic behavior of solutions $U(t,x)$ and $V(t,x)$ to the problems (5.91) and (5.92), by Lemma 3.45 we find $|u(t,x)| \leq C\varepsilon^{-2}|U(t,x)|$. Thus we get an optimal time decay estimate for the solution

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-\frac{n}{2}}\left(1 + C\varepsilon\left(\frac{2}{n}-\sigma\right)^{-\frac{1}{\sigma}} t^{\frac{1}{\sigma}-\frac{n}{2}}\right)^{-1} \tag{5.104}$$

for all $t > 0$. As in the proof of Theorem 5.2, we consider the Cauchy problem for the new dependent variables $(v(t,x), \varphi(t))$

$$\begin{cases} v_t - \alpha\Delta v = \left(\lambda|u|^\sigma v + fe^\varphi - \frac{1}{\theta}\int_{\mathbf{R}^n}(\lambda|u|^\sigma v + fe^\varphi)\,dx\right)v \\ \varphi'(t) = -\frac{1}{\theta}\int_{\mathbf{R}^n}(\lambda|u|^\sigma v + fe^\varphi)\,dx, \\ v(0,x) = u_0(x),\ \varphi(0) = 0. \end{cases} \tag{5.105}$$

We now prove the estimate

$$\|v(t)\|_{\mathbf{L}^{1,a}} \leq C\langle t\rangle^{\frac{a}{2}}$$

for all $t > 0$. By estimate (5.104) we have

$$\|u(t)\|_{\mathbf{L}^\infty}^\sigma \leq C\langle t\rangle^{-\sigma\frac{n}{2}}\left(1 + C\varepsilon^\sigma\left(\frac{2}{n}-\sigma\right)^{-1} t^{1-\frac{n}{2}\sigma}\right)^{-1},$$

then

$$\varphi'(t) \leq -\frac{1}{\theta}\int_{\mathbf{R}^n}(\lambda|u|^\sigma v + fe^\varphi)\,dx \leq C\langle t\rangle^{-\sigma\frac{n}{2}};$$

hence $\varphi(t) \leq C\langle t\rangle^{1-\frac{n}{2}\sigma}$. Thus by virtue of (5.104) we get

$$\|v(t)\|_{\mathbf{L}^{1,a}} \leq e^{\varphi(t)}\|u(t)\|_{\mathbf{L}^{1,a}} \leq C(T)\langle t\rangle^{\frac{a}{2}}$$

and

$$\|v(t)\|_{\mathbf{L}^\infty} \le e^{\varphi(t)} \|u(t)\|_{\mathbf{L}^\infty} \le C(T) \langle t \rangle^{-\frac{n}{2}}$$

for all $0 < t \le T$. Now we consider $t > T$. By virtue of estimates (5.104) and Lemma 3.49 we get

$$\begin{aligned}
&\left\| \int_T^t \mathcal{G}_0(t-\tau) \left(\lambda |u(\tau)|^\sigma v(\tau) + f(\tau) e^{\varphi(\tau)} \right.\right. \\
&\left.\left. - \frac{v(\tau)}{\theta} \int_{\mathbf{R}^n} \left(\lambda |u(\tau)|^\sigma v(\tau) + f(\tau) e^{\varphi(\tau)} \right) dx \right) d\tau \right\|_{\mathbf{L}^{1,a}} \\
&\le C \int_T^t \left\| \lambda |u(\tau)|^\sigma v(\tau) + f(\tau) e^{\varphi(\tau)} \right. \\
&\left. - \frac{v(\tau)}{\theta} \int_{\mathbf{R}^n} \left(\lambda |u(\tau)|^\sigma v(\tau) + f(\tau) e^{\varphi(\tau)} \right) dx \right\|_{\mathbf{L}^{1,a}} \\
&\le C\varepsilon^{-\sigma} \left(\frac{2}{n} - \sigma \right) \int_T^t \langle \tau \rangle^{-1} \|v(\tau)\|_{\mathbf{L}^{1,a}} d\tau
\end{aligned}$$

for all $t > T$. Therefore we obtain

$$\begin{aligned}
\|v(t)\|_{\mathbf{L}^{1,a}} &\le \|\mathcal{G}_0(t-T) v(T)\|_{\mathbf{L}^{1,a}} \\
&\quad + C\varepsilon^{-\sigma} \left(\frac{2}{n} - \sigma \right) \int_T^t \langle \tau \rangle^{-1} \|v(\tau)\|_{\mathbf{L}^{1,a}} d\tau \\
&\le C(T) \langle t \rangle^{\frac{a}{2}} + \epsilon \int_T^t \langle \tau \rangle^{-1} \|v(\tau)\|_{\mathbf{L}^{1,a}} d\tau
\end{aligned}$$

for all $t > T$. Here $\epsilon > 0$ is small enough, and $T > 0$ is sufficiently large (note that $\sigma < \frac{2}{n}$ is close to $\frac{2}{n}$). The application of the Gronwall's lemma yields the estimate

$$\|v(t)\|_{\mathbf{L}^{1,a}} \le C \langle t \rangle^{\frac{a}{2}} \tag{5.106}$$

for all $t > 0$. Likewise by virtue of estimates (5.104) and Lemma 3. 3.49 we get

$$\begin{aligned}
&\|v(t)\|_{\mathbf{L}^\infty} \le \|\mathcal{G}_0(t-T) v(T)\|_{\mathbf{L}^\infty} \\
&+ C \left\| \int_T^t \mathcal{G}_0(t-\tau) \left(\lambda |u(\tau)|^\sigma v(\tau) + f(\tau) e^{\varphi(\tau)} \right.\right. \\
&\left.\left. - \frac{v(\tau)}{\theta} \int_{\mathbf{R}^n} \left(\lambda |u(\tau)|^\sigma v(\tau) + f(\tau) e^{\varphi(\tau)} \right) dx \right) d\tau \right\|_{\mathbf{L}^\infty} \\
&\le C(T) \langle t \rangle^{-\frac{n}{2}} + C \int_T^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}-\frac{a}{2}} \tau^{\frac{a}{2}-1} d\tau \\
&+ C\varepsilon^{-\sigma} \left(\frac{2}{n} - \sigma \right) \int_{\frac{t}{2}}^t \langle \tau \rangle^{\cdot -1} \|v(\tau)\|_{\mathbf{L}^\infty} d\tau \\
&\le C(T) \langle t \rangle^{-\frac{n}{2}} + \epsilon \int_{\frac{t}{2}}^t \langle \tau \rangle^{-1} \|v(\tau)\|_{\mathbf{L}^\infty} d\tau
\end{aligned}$$

for all $t > T$. The application of the Gronwall's lemma yields the estimate

$$\|v(t)\|_{\mathbf{L}^\infty} \leq C \langle t \rangle^{-\frac{n}{2}}$$

for all $t > 0$. Now the asymptotic formulas are proved in the same manner as at the end of the proof of Theorem 5.34, and Theorem 5.35 is proved.

5.6 Oscillating solutions to nonlinear heat equation

This section is devoted to the study of global existence and large time asymptotic behavior of small solutions to the Cauchy problem for the nonlinear heat equation

$$\begin{cases} u_t - u_{xx} + |u|^\sigma u = 0, \ x \in \mathbf{R}, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R} \end{cases} \tag{5.107}$$

in the subcritical case $\sigma \in (0, 2)$.

In the present section we are interested in the asymptotic behavior of nonpositive solutions. First we prove the existence of a unique self-similar solution for equation (5.107) of the form $(1+t)^{-\frac{1}{\sigma}} S_{\rho,\omega}\left(\frac{x}{\sqrt{1+t}}\right)$ such that

$$\widehat{S_{\rho,\omega}}(\xi) = \chi_{\rho,\omega}(\xi) + O\left(\varepsilon^{1+\sigma}\right)$$

for all $\xi \in \mathbf{R}$, where $\chi_{\rho,\omega}(\xi) = (\rho + \omega \operatorname{sign}\xi) |\xi|^{2\lambda} e^{-\xi^2}$, $\lambda = \frac{1}{\sigma} - \frac{1}{2}$, and $\varepsilon = |\rho| + |\omega| > 0$ is sufficiently small. Note that these solutions change a sign, since the main part $\mathcal{F}^{-1}_{\xi \to x} \chi_{\rho,\omega}(\xi)$ is nonpositive.

Then in the next theorem we prove asymptotic stability of these self-similar solutions.

Theorem 5.36. *Let $\frac{4}{3} < \sigma < 2$. Assume that the initial data $u_0 \in \mathbf{H}^{1,1}(\mathbf{R})$ have the mean value $\int_{\mathbf{R}} u_0(x)dx = \int_{\mathbf{R}} S_{\rho,\omega}(x)dx$ and are close to the self-similar solution $S_{\rho,\omega}$ in the sense*

$$\|u_0 - S_{\rho,\omega}\|_{\mathbf{H}^{1,1}} \leq C\varepsilon^{1+\sigma},$$

where $\varepsilon = |\rho| + |\omega| > 0$ is sufficiently small. Then the Cauchy problem (5.107) has a unique global solution $u(t,x) \in \mathbf{C}\left([0,\infty); \mathbf{H}^{1,1}(\mathbf{R})\right)$, satisfying the following time decay estimates

$$\left\| u(t) - (1+t)^{-\frac{1}{\sigma}} S_{\rho,\omega}\left(\frac{(\cdot)}{\sqrt{1+t}}\right) \right\|_{\mathbf{L}^\infty} \leq C \langle t \rangle^{-\frac{1}{\sigma} - \gamma}$$

for all $t \geq 0$, where $\gamma > 0$.

5.6.1 Lemmas

First we give the estimates of the norm $\|\phi\|_{\mathbf{A}^\alpha} = \left\| |\xi|^{-\alpha} \widehat{\phi}(\xi) \right\|_{\mathbf{L}^2}$.

Lemma 5.37. *Let the moments $\int_{\mathbf{R}} x^j \phi(x)\, dx = 0$ for $0 \le j \le n$. Then the inequality is true*

$$\|\phi\|_{\mathbf{A}^\alpha} \le C \|\phi\|_{\mathbf{L}^2}^{1-\frac{\alpha}{n+\frac{1}{2}+\frac{1}{p}}} \left\| |x|^{n+1} \phi \right\|_{\mathbf{L}^p}^{\frac{\alpha}{n+\frac{1}{2}+\frac{1}{p}}}$$

for $\frac{1}{2} < \alpha < n + \frac{1}{2} + \frac{1}{p}, 1 \le p \le 2$.

Proof. Choosing ν such that $\max\left(\alpha, n+\frac{1}{2}\right) < \nu < n + \frac{1}{2} + \frac{1}{p}$, and applying the Cauchy inequality with $\delta = \|\phi\|_{\mathbf{L}^2}^{\frac{1}{n+\frac{1}{2}+\frac{1}{p}}} \left\| |x|^{n+1} \phi \right\|_{\mathbf{L}^p}^{-\frac{1}{n+\frac{1}{2}+\frac{1}{p}}}$, we find

$$\begin{aligned}
\|\phi\|_{\mathbf{A}^\alpha} &= \left\| |\xi|^{-\alpha} \widehat{\phi}(\xi) \right\|_{\mathbf{L}^2} \\
&\le \left(\int_{\mathbf{R}} dx \phi(x) \int_{\mathbf{R}} dy \phi(y) \int_{|\xi| \le \delta} d\xi\, |\xi|^{-2\alpha} \left(e^{i\xi x} - \sum_{k=0}^{n} \frac{(i\xi x)^k}{k!} \right) \right. \\
&\times \left. \left(e^{-i\xi y} - \sum_{k=0}^{n} \frac{(-i\xi y)^k}{k!} \right) \right)^{\frac{1}{2}} + \left(\int_{|\xi| > \delta} |\xi|^{-2\alpha} \left| \widehat{\phi}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}};
\end{aligned}$$

hence by choosing $\mu = \frac{1}{\delta}$, we get

$$\begin{aligned}
\|\phi\|_{\mathbf{A}^\alpha} &\le C \int_{|x| \le \mu} |\phi(x)|\, |x|^{\nu - \frac{1}{2}}\, dx \left(\int_{|\xi| \le \delta} d\xi\, |\xi|^{2\nu - 2\alpha - 1} \right)^{\frac{1}{2}} \\
&+ C \int_{|x| > \mu} |\phi(x)|\, |x|^{n+1} \frac{dx}{|x|^{n-\nu+\frac{3}{2}}} \left(\int_{|\xi| \le \delta} d\xi\, |\xi|^{2\nu - 2\alpha - 1} \right)^{\frac{1}{2}} + C\delta^{-\alpha} \|\phi\|_{\mathbf{L}^2} \\
&\le C\delta^{\nu-\alpha} \left(\mu^\nu \|\phi\|_{\mathbf{L}^2} + \mu^{\nu - n - \frac{1}{2} - \frac{1}{p}} \left\| |x|^{n+1} \phi \right\|_{\mathbf{L}^p} \right) + C\delta^{-\alpha} \|\phi\|_{\mathbf{L}^2} \\
&\le C \|\phi\|_{\mathbf{L}^2}^{1-\frac{\alpha}{n+\frac{1}{2}+\frac{1}{p}}} \left\| |x|^{n+1} \phi \right\|_{\mathbf{L}^p}^{\frac{\alpha}{n+\frac{1}{2}+\frac{1}{p}}}.
\end{aligned}$$

Thus the estimate of the lemma is true, and Lemma 5.37 is proved.

Next we prove the existence of self-similar solutions for equation (5.107) of the form $u(t,x) = (1+t)^{-\frac{1}{\sigma}} S\left(\frac{x}{\sqrt{1+t}}\right)$. By equation (5.107) we get the following ordinary differential equation for the function $S(x)$

$$-\frac{1}{2} \frac{d}{dx}(xS) - S'' - \lambda S + |S|^\sigma S = 0, \tag{5.108}$$

where $\lambda = \frac{1}{\sigma} - \frac{1}{2}$. Denote $\chi_{\rho,\omega}(\xi) = (\rho + \omega \operatorname{sign}\xi) |\xi|^{2\lambda} e^{-\xi^2}$.

Lemma 5.38. *There exists a unique solution of equation (5.108) such that*

$$\widehat{S_{\rho,\omega}}(\xi) - \chi_{\rho,\omega}(\xi) = O\left(\varepsilon^{1+\sigma}\right) \tag{5.109}$$

for all $\xi \in \mathbf{R}$, *where* $\varepsilon = |\rho| + |\omega| > 0$ *is sufficiently small.*

Proof. Applying the Fourier transformation to (5.108) we obtain for $\widehat{S_{\rho,\omega}}(\xi) = \mathcal{F}_{x\to\xi} S_{\rho,\omega}$ the equation (ρ and ω we will omit below)

$$\widehat{S}' + 2\xi\widehat{S} = \frac{2}{\xi}\left(\lambda\widehat{S} - \mathcal{F}_{x\to\xi}\left(|S|^{\sigma} S\right)\right). \tag{5.110}$$

Note that the linear part of equation (5.110) has a general solution of the form $(C_1 + C_2 \mathrm{sign}\xi)\,|\xi|^{2\lambda} e^{-\xi^2}$ with arbitrary constants C_1 and C_2. We look for the solutions in the form

$$\widehat{S} = \chi + \phi + w, \tag{5.111}$$

where

$$\phi(\xi) = \sum_{j=0}^{n} a_j \xi^j e^{-\xi^2},$$

here integer n is such that $n > 2\lambda$. In the case when $2\lambda = l$ is integer, we take a modified representation

$$\phi(\xi) = \sum_{j=0, j\neq l}^{n} a_j \xi^j e^{-\xi^2} + a_l \xi^l e^{-\xi^2} \log|\xi|.$$

The constants a_j we will define later by the condition

$$w(\xi) = o\left(|\xi|^n\right) \tag{5.112}$$

for $\xi \to 0$. Substituting representation (5.111) into (5.110) we find

$$w' + 2\xi w = \frac{1}{\xi}\left(2\lambda w - 2\mathcal{F}_{x\to\xi}\left(|S|^{\sigma} S\right) + \psi(\xi)\right), \tag{5.113}$$

where

$$\psi(\xi) = 2\lambda\phi - 2\xi^2\phi - \xi\phi'.$$

That is, we have

$$\psi(\xi) = \sum_{j=0}^{n} a_j (2\lambda - j)\,\xi^j e^{-\xi^2}$$

and in the case of $2\lambda = l$ we obtain

$$\psi(\xi) = \sum_{j=0, j\neq l}^{n} (2\lambda - j)\, a_j \xi^j e^{-\xi^2} - a_l \xi^l e^{-\xi^2}.$$

Now the integration of (5.113) with respect to ξ yields

$$w(\xi) = \int_0^{\xi} e^{\eta^2-\xi^2} \left(2\lambda w - 2\mathcal{F}_{x\to\eta}\left(|S|^{\sigma} S\right) + \psi(\eta)\right) \frac{d\eta}{\eta}. \tag{5.114}$$

We write the Taylor expansion

$$2e^{\xi^2}\mathcal{F}_{x\to\xi}\left(|S|^{\sigma} S\right) = \sum_{j=0}^{n} A_j \xi^j + O\left(\xi^{n+1}\right),$$

where

$$\begin{aligned} A_j &= \frac{1}{j!}\,\partial_{\xi}^{j}\left(2e^{\xi^2}\mathcal{F}_{x\to\xi}\left(|S|^{\sigma} S\right)\right)\Big|_{\xi=0} \\ &= 2\sum_{l=0}^{j} \frac{1}{(j-l)!l!}\left(\partial_{\xi}^{l} e^{\xi^2}\right)\left(\partial_{\xi}^{l-j}\mathcal{F}_{x\to\xi}\left(|S|^{\sigma} S\right)\right)\Big|_{\xi=0} \\ &= (2\pi)^{-\frac{1}{2}} \sum_{k=0}^{\left[\frac{j}{2}\right]} \frac{2}{(2k)!\,(j-2k)!} \int_{\mathbf{R}} (ix)^{j-2k} |S(x)|^{\sigma} S(x)\, dx. \end{aligned}$$

Therefore the condition (5.112) implies that

$$a_j = \frac{A_j}{2\lambda - j} \tag{5.115}$$

for $0 \le j \le n$, and in the case of $2\lambda = l$ we have relations (5.115) for $0 \le j \le n$, $j \ne l$, whereas $a_l = -A_l$.

We now solve equation (5.114) by the successive approximations. Let $w_0(\xi) = 0$, $\phi_{-1}(\xi) = 0$ and $w_{m+1}(\xi)$ for $m \ge 0$ is defined by the recurrent relations

$$w_{m+1}(\xi) = \int_0^{\xi} e^{\eta^2-\xi^2} \left(2\lambda w_{m+1}(\eta) - 2\mathcal{F}_{x\to\eta}\left(|S_m|^{\sigma} S_m\right) + \psi_m(\eta)\right) \frac{d\eta}{\eta}, \tag{5.116}$$

where

$$\widehat{S}_m = \chi + \phi_{m-1} + w_m,$$

$$\phi_{m-1}(\xi) = \sum_{j=0}^{n} a_j^{(m-1)} \xi^j e^{-\xi^2}$$

and

$$\psi_m(\xi) = \sum_{j=0}^{n} a_j^{(m)} (2\lambda - j)\, \xi^j e^{-\xi^2}.$$

Here integer $n > 2\lambda$,

$$a_j^{(m)} = \frac{A_j^{(m)}}{2\lambda - j}$$

and

$$A_j^{(m)} = (2\pi)^{-\frac{1}{2}} \sum_{k=0}^{\left[\frac{j}{2}\right]} \frac{2}{(2k)!\,(j-2k)!} \int_{\mathbf{R}} (ix)^{j-2k} \left|S_m(x)\right|^{\sigma} S_m(x)\,dx$$

for $0 \le j \le n$. Whereas in the case when $2\lambda = l$ is integer, we have

$$\phi_{m-1}(\xi) = \sum_{j=0, j\neq l}^{n} a_j^{(m-1)} \xi^j e^{-\xi^2} + a_l^{(m-1)} \xi^l e^{-\xi^2} \log|\xi|,$$

$$\psi_m(\xi) = \sum_{j=0, j\neq l}^{n} (2\lambda - j)\, a_j^{(m)} \xi^j e^{-\xi^2} - a_l^{(m)} \xi^l e^{-\xi^2}$$

and relations

$$a_j^{(m)} = \frac{A_j^{(m)}}{2\lambda - j}$$

for $0 \le j \le n$, $j \neq l$, and $a_l^{(m)} = -A_l^{(m)}$, $l = 2\lambda$.

Let us prove the estimates

$$\left\| |\xi|^{-n} w_m(\xi) \right\|_{\mathbf{L}^2} \le C_1 \varepsilon^{1+\sigma}, \tag{5.117}$$

$$\left\| \langle \xi \rangle w_m(\xi) \right\|_{\mathbf{L}^2} \le C_2 \varepsilon^{1+\sigma}, \tag{5.118}$$

$$\left\| \langle \xi \rangle \partial_\xi^n w_m \right\|_{\mathbf{L}^2} \le C_2 \varepsilon^{1+\sigma} \tag{5.119}$$

and

$$\sum_{j=0}^{n} \left| a_j^{(m)} \right| \le C_3 \varepsilon^{1+\sigma} \tag{5.120}$$

with $n > \max\left(2\lambda + \frac{1}{2}, \frac{1}{\sigma}\right)$.

For $m = 0$ estimates (5.117) - (5.120) are true. Then by induction we suppose that these estimates are valid for some $m > 0$.

Consider the first estimate (5.117). Note that changing the order of integration we get

$$\begin{aligned}
&\left\| |\xi|^{-n} \int_0^{\xi} \Phi(\eta) \frac{d\eta}{\eta} \right\|_{\mathbf{L}^2}^2 \\
&= \int_{\mathbf{R}} d\xi\, |\xi|^{-2n} \int_0^{\xi} \overline{\Phi(\zeta)} \frac{d\zeta}{\zeta} \int_0^{\xi} \Phi(\eta) \frac{d\eta}{\eta} \\
&= \int_0^{\infty} \frac{d\zeta}{\zeta} \overline{\Phi(\zeta)} \int_{\zeta}^{\infty} d\xi\, |\xi|^{-2n} \int_0^{\zeta} \Phi(\eta) \frac{d\eta}{\eta} \\
&+ \int_0^{\infty} \frac{d\eta}{\eta} \Phi(\eta) \int_{\eta}^{\infty} d\xi\, |\xi|^{-2n} \int_0^{\eta} \frac{d\zeta}{\zeta} \overline{\Phi(\zeta)} \\
&+ \int_{-\infty}^{0} \frac{d\zeta}{\zeta} \overline{\Phi(\zeta)} \int_{-\infty}^{\zeta} d\xi\, |\xi|^{-2n} \int_{\zeta}^{0} \Phi(\eta) \frac{d\eta}{\eta} \\
&+ \int_{-\infty}^{0} \frac{d\eta}{\eta} \Phi(\eta) \int_{-\infty}^{\eta} d\xi\, |\xi|^{-2n} \int_{\eta}^{0} \frac{d\zeta}{\zeta} \overline{\Phi(\zeta)},
\end{aligned}$$

then by applying the Cauchy inequality we obtain

$$\begin{aligned}
&\left\| |\xi|^{-n} \int_0^{\xi} \Phi(\eta) \frac{d\eta}{\eta} \right\|_{\mathbf{L}^2}^2 \\
&\le \frac{2}{2n-1} \int_{\mathbf{R}} d\zeta\, |\Phi(\zeta)|\, |\zeta|^{-2n} \left| \int_0^{\zeta} \Phi(\eta) \frac{d\eta}{\eta} \right| \\
&\le \frac{2}{2n-1} \left\| |\zeta|^{-n} \Phi \right\|_{\mathbf{L}^2} \left\| |\zeta|^{-n} \int_0^{\zeta} \Phi(\eta) \frac{d\eta}{\eta} \right\|_{\mathbf{L}^2};
\end{aligned}$$

hence the inequality follows

$$\left\| |\xi|^{-n} \int_0^{\xi} \Phi(\eta) \frac{d\eta}{\eta} \right\|_{\mathbf{L}^2} \le \frac{2}{2n-1} \left\| |\xi|^{-n} \Phi \right\|_{\mathbf{L}^2}. \tag{5.121}$$

The application of estimate (5.121) to equation (5.116) yields

$$\begin{aligned}
&\left\| |\xi|^{-n} w_{m+1} \right\|_{\mathbf{L}^2} \\
&\le 2\lambda \left\| |\xi|^{-n} \int_0^{\xi} e^{\eta^2-\xi^2} w_{m+1}(\eta) \frac{d\eta}{\eta} \right\|_{\mathbf{L}^2} \\
&+ 2 \left\| |\xi|^{-n} \int_0^{\xi} e^{\eta^2-\xi^2} \left(\mathcal{F}_{x\to\eta} \left(|S_m|^{\sigma} S_m \right) - \psi_m(\eta) \right) \frac{d\eta}{\eta} \right\|_{\mathbf{L}^2} \\
&\le \frac{4\lambda}{2n-1} \left\| |\xi|^{-n} w_{m+1} \right\|_{\mathbf{L}^2} \\
&+ \frac{2}{2n-1} \left\| |\xi|^{-n} \left(\mathcal{F}_{x\to\xi} \left(|S_m|^{\sigma} S_m \right) - \psi_m \right) \right\|_{\mathbf{L}^2}.
\end{aligned}$$

We can choose $1 \le p \le 2$ and $n > 2\lambda + \frac{1}{2}$ such that $\frac{p+2}{2pn+p-2} < \sigma$; hence we have $\left\| |x|^{\frac{n+1}{1+\sigma}} S_m \right\|_{\mathbf{L}^p} \le C \left\| \langle x \rangle^n S_m \right\|_{\mathbf{L}^2}$. Therefore by the condition $2n-1 > 4\lambda$ and by Lemma 5.37, we obtain

$$\begin{aligned}
&\left\| |\xi|^{-n} w_{m+1} \right\|_{\mathbf{L}^2} \le C \left\| |\xi|^{-n} \left(\mathcal{F}_{x\to\xi} \left(|S_m|^{\sigma} S_m \right) - \psi_m \right) \right\|_{\mathbf{L}^2} \\
&\le C \left\| x^{n+1} |S_m|^{\sigma} S_m \right\|_{\mathbf{L}^p} + C \left\| |S_m|^{\sigma} S_m \right\|_{\mathbf{L}^2} \\
&= C \left\| \left| |x|^{\frac{n+1}{1+\sigma}} S_m \right|^{\sigma} |x|^{\frac{n+1}{1+\sigma}} S_m \right\|_{\mathbf{L}^p} + C \left\| S_m \right\|_{\mathbf{L}^2} \left\| S_m \right\|_{\mathbf{L}^\infty}^{\sigma} \\
&\le C \left\| \langle x \rangle^n S_m \right\|_{\mathbf{L}^2} \left\| \langle x \rangle^n S_m \right\|_{\mathbf{L}^\infty}^{\sigma} \le C_1 \varepsilon^{1+\sigma}.
\end{aligned}$$

Thus estimate (5.117) is fulfilled with m replaced by $m+1$.

Now we prove estimate (5.118). In view of inequality $e^{\eta^2-\xi^2} \le e^{-(\xi-\eta)^2}$ for $0 \le \frac{\eta}{\xi} \le 1$ we have by the Hölder inequality

$$\begin{aligned}
&\left\| \langle \xi \rangle \int_0^{\xi} e^{\eta^2-\xi^2} \Phi(\eta) \frac{d\eta}{\eta} \right\|_{\mathbf{L}^2} \le C \left\| |\xi|^{-n} \Phi \right\|_{\mathbf{L}^2} + C \left\| \Phi \right\|_{\mathbf{L}^2} \\
&\le C \left\| |\xi|^{-n} \Phi \right\|_{\mathbf{L}^2} + C \left\| |\xi|^{-n} \Phi \right\|_{\mathbf{L}^2}^{\frac{1}{1+n}} \left\| \langle \xi \rangle \Phi \right\|_{\mathbf{L}^2}^{\frac{n}{1+n}}.
\end{aligned}$$

Then from equation (5.116) we find

$$\begin{aligned}
&\left\| \langle \xi \rangle w_{m+1} \right\|_{\mathbf{L}^2} \le 2\lambda \left\| \langle \xi \rangle \int_0^{\xi} e^{\eta^2-\xi^2} w_{m+1}(\eta) \frac{d\eta}{\eta} \right\|_{\mathbf{L}^2} \\
&+ \left\| \langle \xi \rangle \int_0^{\xi} e^{\eta^2-\xi^2} \left(2\mathcal{F}_{x\to\eta} \left(|S_m|^{\sigma} S_m \right) - \psi_m(\eta) \right) \frac{d\eta}{\eta} \right\|_{\mathbf{L}^2} \\
&\le \frac{1}{2} \left\| \langle \xi \rangle w_{m+1} \right\|_{\mathbf{L}^2} + C \left\| |\xi|^{-n} w_{m+1} \right\|_{\mathbf{L}^2} + C \left\| \langle x \rangle^n S_m \right\|_{\mathbf{L}^2} \left\| S_m \right\|_{\mathbf{L}^\infty}^{\sigma} \\
&\le \frac{1}{2} \left\| \langle \xi \rangle w_{m+1} \right\|_{\mathbf{L}^2} + C\varepsilon^{1+\sigma}.
\end{aligned}$$

Thus we have estimate (5.118) with m replaced by $m+1$.

Now we consider estimate (5.119). We differentiate n times equation (5.116) to find

$$\begin{aligned}
&\frac{d^n}{d\xi^n} w_{m+1}(\xi) + 2\frac{d^{n-1}}{d\xi^{n-1}} \left(\xi w_{m+1}(\xi) \right) \\
&= 2\lambda \frac{d^{n-1}}{d\xi^{n-1}} \left(\frac{1}{\xi} w_{m+1} \right) - \frac{d^{n-1}}{d\xi^{n-1}} \left(\frac{1}{\xi} \left(2\mathcal{F}_{x\to\xi} \left(|S_m|^{\sigma} S_m \right) - \psi_m \right) \right).
\end{aligned}$$

Multiplying the last equation by $\frac{d^n}{d\xi^n} w_{m+1}(\xi)$ and integrating the result over $\xi \in \mathbf{R}$ we get

$$\left\| \frac{d^n}{d\xi^n} w_{m+1} \right\|_{\mathbf{L}^2} \le C \left\| \frac{d^{n-2}}{d\xi^{n-2}} w_{m+1} \right\|_{\mathbf{L}^2}$$
$$+ 2\lambda \left\| \frac{d^{n-1}}{d\xi^{n-1}} \left(\frac{1}{\xi} w_{m+1} \right) \right\|_{\mathbf{L}^2}$$
$$+ \left\| \frac{d^{n-1}}{d\xi^{n-1}} \left(\frac{1}{\xi} \left(2\mathcal{F}_{x\to\xi} \left(|S_m|^\sigma S_m \right) - \psi_m \right) \right) \right\|_{\mathbf{L}^2}.$$

By interpolation we have

$$\left\| \frac{d^{n-2}}{d\xi^{n-2}} w_{m+1} \right\|_{\mathbf{L}^2} \le \frac{1}{3} \left\| \frac{d^n}{d\xi^n} w_{m+1} \right\|_{\mathbf{L}^2} + C \left\| w_{m+1} \right\|_{\mathbf{L}^2}$$

and

$$\left\| \frac{d^{n-1}}{d\xi^{n-1}} \left(\frac{1}{\xi} w_{m+1} \right) \right\|_{\mathbf{L}^2} \le \frac{1}{3} \left\| \frac{d^n}{d\xi^n} w_{m+1} \right\|_{\mathbf{L}^2}$$
$$+ C \left\| |\xi|^{-n} w_{m+1} \right\|_{\mathbf{L}^2} + C \left\| w_{m+1} \right\|_{\mathbf{L}^2};$$

hence, we obtain

$$\left\| \frac{d^n}{d\xi^n} w_{m+1} \right\|_{\mathbf{L}^2} \le C_3 \varepsilon^{1+\sigma}.$$

As above we get estimate

$$\left\| \xi \frac{d^n}{d\xi^n} w_{m+1} \right\|_{\mathbf{L}^2} \le C_3 \varepsilon^{1+\sigma}.$$

Thus estimate (5.119) with m replaced by $m+1$ is valid. The rest estimate (5.120) follows from the previous estimates via formulas

$$\sum_{j=0}^{n} \left| a_j^{(m+1)} \right| \le C \int_{\mathbf{R}} \langle x \rangle^n \left| S_{m+1}(x) \right|^{\sigma+1} dx$$
$$\le C \left\| \langle x \rangle^n S_{m+1} \right\|_{\mathbf{L}^2} \left\| \langle x \rangle^n S_{m+1} \right\|_{\mathbf{L}^\infty}^\sigma$$
$$\le C \left\| \langle \xi \rangle \langle \partial_\xi \rangle^n \left(\chi + \phi_m + w_{m+1} \right) \right\|_{\mathbf{L}^2}^{1+\sigma} \le C_3 \varepsilon^{1+\sigma}.$$

Therefore estimate (5.120) follows with m replaced by $m+1$. Thus by induction, estimates (5.117) - (5.120) are true for all m.

In the same manner we prove the estimate

$$\left\| w_{m+1} - w_m \right\|_{\mathbf{X}} \le \frac{1}{2} \left\| w_m - w_{m-1} \right\|_{\mathbf{X}},$$

where $\|\psi\|_{\mathbf{X}} \equiv \left\| |\xi|^{-n} \psi \right\|_{\mathbf{L}^2} + \left\| \langle \xi \rangle \psi \right\|_{\mathbf{L}^2} + \left\| \langle \xi \rangle \partial_\xi^n \psi \right\|_{\mathbf{L}^2}$. Therefore there exists a unique solution of equation (5.108) having the form (5.109). Lemma 5.38 is proved.

5.6.2 Proof of Theorem 5.36

The local existence of solutions for the Cauchy problem (5.107) can be obtained by the standard contraction mapping principle. In order to get a priori estimates of local solutions as in the papers Hayashi et al. [2000] and Hayashi et al. [2003b], we make a change of the dependent variable $u(t,x) = e^{-\varphi(t)} v(t,x)$, then we get

$$v_t - v_{xx} + e^{-\sigma\varphi} |v|^\sigma v - v\varphi' = 0. \tag{5.122}$$

We choose $\varphi(t)$ by the condition

$$\int_{\mathbf{R}} \left(e^{-\sigma\varphi} |v|^\sigma v - \varphi'(t) v \right) dx = 0,$$

where the initial value $\varphi(0) = 0$. Then the mean value of v satisfies the conservation law

$$\frac{d}{dt} \int_{\mathbf{R}} v(t,x)\, dx = 0,$$

so

$$\int_{\mathbf{R}} v(t,x)\, dx = \int_{\mathbf{R}} v(0,x)\, dx = \theta = \int_{\mathbf{R}} u_0(x)\, dx.$$

Thus we obtain from (5.122)

$$\begin{cases} v_t - v_{xx} = e^{-\sigma\varphi} \left(\frac{v}{\theta} \int_{\mathbf{R}} |v|^\sigma v dx - |v|^\sigma v \right), \\ \varphi'(t) = \frac{1}{\theta} e^{-\sigma\varphi} \int_{\mathbf{R}} |v|^\sigma v dx. \end{cases} \tag{5.123}$$

Now we substitute $v = (1+t)^\lambda f + w$, where f is a self-similar solution of equation (5.107) such that $f = (1+t)^{-\frac{1}{\sigma}} S\left(\frac{x}{\sqrt{1+t}}\right)$ and $\lambda = \frac{1}{\sigma} - \frac{1}{2}$. By equation (5.107) we see that $(1+t)^\lambda f$ satisfies

$$\begin{aligned} \left((1+t)^\lambda f \right)_t &= \left((1+t)^\lambda f \right)_{xx} + \lambda (1+t)^{\lambda-1} f \\ &\quad - (1+t)^{-\sigma\lambda} \left| (1+t)^\lambda f \right|^\sigma (1+t)^\lambda f, \end{aligned}$$

then we get for w

$$\begin{aligned} w_t &= w_{xx} + e^{-\sigma\varphi} \left(\frac{v}{\theta} \int_{\mathbf{R}} |v|^\sigma v dx - |v|^\sigma v \right) \\ &\quad - \lambda (1+t)^{\lambda-1} f + (1+t)^{-\sigma\lambda} \left| (1+t)^\lambda f \right|^\sigma (1+t)^\lambda f. \end{aligned}$$

Note that the mean value is conserved

$$\int_{\mathbf{R}} v(t,x)\, dx = \int_{\mathbf{R}} \frac{1}{\sqrt{1+t}} S\left(\frac{x}{\sqrt{1+t}} \right) dx + \int_{\mathbf{R}} w(t,x)\, dx = \theta.$$

If we choose the mean value θ, such that $\int_{\mathbf{R}} S(x)\,dx = \theta$, we obtain

$$\int_{\mathbf{R}} w(t,x)\,dx = 0.$$

Denote $h(t) = e^{\sigma\varphi(t)}$. We prove the existence of the solution $(w(t,x), h(t))$ by the successive approximations. Let $w_0 = 0$, $h_0 = (1+t)^{1-\frac{\sigma}{2}}$. We define the functions $(w_m(t,x), h_m(t))$ for $m = 1, 2, ...$ by the following equations

$$\partial_t w_m - \partial_x^2 w_m = h_{m-1}^{-1}(t)\left(\frac{v_m}{\theta}\int_{\mathbf{R}} |v_{m-1}|^{\sigma} v_{m-1}dx - |v_{m-1}|^{\sigma} v_{m-1}\right)$$
$$- \lambda(1+t)^{\lambda-1} f + (1+t)^{-\sigma\lambda}\left|(1+t)^{\lambda} f\right|^{\sigma}(1+t)^{\lambda} f \tag{5.124}$$

and

$$h_m'(t) = \frac{\sigma}{\theta}\int_{\mathbf{R}} |v_m|^{\sigma} v_m dx. \tag{5.125}$$

Applying the Fourier transformation to (5.124) and changing the variables $\widehat{w}_m(t,\xi) = z_m(t,\eta)$, $\eta = \xi\sqrt{t+1}$, we get

$$\partial_t z_m + \frac{\eta}{2(t+1)}\partial_\eta z_m + \frac{\eta^2}{t+1} z_m = \psi_{m-1} z_m - g_{m-1}, \tag{5.126}$$

where we denote

$$\psi_{m-1}(t) = \frac{1}{\theta h_{m-1}(t)}\int_{\mathbf{R}} |v_{m-1}|^{\sigma} v_{m-1}(t,x)\,dx,$$
$$g_{m-1}(t,\eta) = \frac{1}{h_{m-1}(t)}\mathcal{F}_{x\to\eta(1+t)^{-\frac{1}{2}}}\left(|v_{m-1}|^{\sigma} v_{m-1}\right)$$
$$- (1+t)^{-1}\mathcal{F}_{x\to\eta}\left(|S|^{\sigma} S\right)$$
$$+ \left(\psi_{m-1}(t) - \lambda(1+t)^{-1}\right)\widehat{S}.$$

Let us prove the estimates

$$\left\|\eta^{-1} z_m\right\|_{\mathbf{L}^2} + \|z_m\|_{\mathbf{H}^{1,1}} \le C\varepsilon^{1+\sigma}(1+t)^{-\gamma} \tag{5.127}$$

and

$$\left|h_{m-1}(t) - (1+t)^{1-\frac{\sigma}{2}}\right| \le C\varepsilon^{\sigma}(1+t)^{1-\frac{\sigma}{2}-\gamma}, \tag{5.128}$$

where $\gamma > 0$ is sufficiently small, and $\frac{1}{2} > \lambda + \varepsilon$, $\lambda = \frac{1}{\sigma} - \frac{1}{2}$. By induction we suppose that (5.127) - (5.128) are fulfilled for some m.

Then we have the estimate

$$\begin{aligned} h_m'(t) &= \frac{\sigma}{\theta}\int_{\mathbf{R}} |v_m|^\sigma v_m dx \\ &= \frac{\sigma}{\theta}\int_{\mathbf{R}} \left|(1+t)^\lambda f + w_m\right|^\sigma \left((1+t)^\lambda f + w_m\right) dx \\ &= \frac{\sigma}{\theta}(1+t)^{-\frac{\sigma}{2}} \int_{\mathbf{R}} |S|^\sigma S dx + O\left(\varepsilon^\sigma (1+t)^{-\frac{\sigma}{2}-\gamma}\right) \\ &= \lambda\sigma (1+t)^{-\frac{\sigma}{2}} + O\left(\varepsilon^\sigma (1+t)^{-\frac{\sigma}{2}-\gamma}\right) \end{aligned}$$

since $\int_{\mathbf{R}} |S|^\sigma S dx = \lambda\theta$. Integrating this estimate with respect to time we get estimate (5.128) with $m-1$ replaced by m. In the same manner we have

$$\begin{aligned} \psi_m(t) &= \frac{1}{\theta h_m(t)} \int_{\mathbf{R}} |v_m|^\sigma v_m dx \\ &= \lambda (1+t)^{-1}\left(1 + O\left(\varepsilon^\sigma (1+t)^{-\gamma}\right)\right), \end{aligned}$$

in particular

$$\psi_m(t) \le \frac{\lambda + \varepsilon^\sigma}{1+t}.$$

We multiply equation (5.126) by $\overline{z_{m+1}}\eta^{-2}$ and integrate the result over $\eta \in \mathbf{R}$ to find

$$\begin{aligned} &\frac{d}{dt}\left\|\eta^{-1} z_{m+1}\right\|_{\mathbf{L}^2}^2 + \frac{1}{2(t+1)}\int_{\mathbf{R}} \eta^{-1}\partial_\eta |z_{m+1}|^2 d\eta + \frac{2}{t+1}\|z_{m+1}\|_{\mathbf{L}^2}^2 \\ &\le \frac{2\lambda + 2\varepsilon^\sigma}{1+t}\left\|\eta^{-1} z_{m+1}\right\|_{\mathbf{L}^2}^2 - 2\,\mathrm{Re}\int_{\mathbf{R}} g_{m-1} |\eta|^{-2}\,\overline{z_{m+1}} d\eta. \end{aligned} \tag{5.129}$$

Integrating by parts we get

$$\int_{\mathbf{R}} \eta^{-1}\partial_\eta |z_{m+1}|^2 d\eta = \left\|\eta^{-1} z_{m+1}\right\|_{\mathbf{L}^2}^2.$$

Using Lemma 5.37 and taking into account estimates (5.127)-(5.128) we obtain

$$\left\|\eta^{-1} g_m\right\|_{\mathbf{L}^2} \le C\varepsilon^{1+\sigma}(1+t)^{-1-\gamma};$$

hence (5.129) yields

$$\frac{d}{dt}\left\|\eta^{-1} z_{m+1}\right\|_{\mathbf{L}^2} \le -\frac{\varepsilon^\sigma}{t+1}\left\|\eta^{-1} z_{m+1}\right\|_{\mathbf{L}^2} + C\varepsilon^{1+\sigma}(1+t)^{-1-\gamma} \tag{5.130}$$

since $2\lambda + 2\varepsilon^\sigma - \frac{1}{2} \le -\varepsilon^\sigma$ which follows from $\sigma > \frac{4}{3}$. The integration of (5.130) with respect to time yields estimate

$$\left\||\eta|^{-1} z_{m+1}\right\|_{\mathbf{L}^2} \le C\varepsilon^{1+\sigma}(1+t)^{-\gamma}. \tag{5.131}$$

Now let us prove the estimate $\|\langle\eta\rangle z_{m+1}\|_{\mathbf{L}^2} \le C\varepsilon^{1+\sigma}(1+t)^{-\gamma}$. We multiply (5.126) by $\langle\eta\rangle^2 \overline{z_{m+1}}$ and integrate over $\eta \in \mathbf{R}$ to find

$$\begin{aligned}
&\frac{d}{dt}\|\langle\eta\rangle z_{m+1}\|_{\mathbf{L}^2}^2 + \frac{1}{2(t+1)}\int_{\mathbf{R}} \eta\langle\eta\rangle^2 \partial_\eta |z_{m+1}|^2 d\eta \\
&+ \frac{2}{t+1}\|\eta\langle\eta\rangle z_{m+1}\|_{\mathbf{L}^2}^2 \\
&= 2\psi_m \|\langle\eta\rangle z_{m+1}\|_{\mathbf{L}^2}^2 - 2\operatorname{Re}\int_{\mathbf{R}} \langle\eta\rangle^2 g_m \overline{z_{m+1}} d\eta.
\end{aligned}$$

Integration by parts yields

$$-\int_{\mathbf{R}} \eta\langle\eta\rangle^2 \partial_\eta |z_{m+1}|^2 d\eta \le 3\|\langle\eta\rangle z_{m+1}\|_{\mathbf{L}^2}^2.$$

Therefore by using the estimate

$$\begin{aligned}
\|\langle\eta\rangle z_{m+1}\|_{\mathbf{L}^2}^2 &\le \|\eta\langle\eta\rangle z_{m+1}\|_{\mathbf{L}^2}^2 + C\left\|\eta^{-1} z_{m+1}\right\|_{\mathbf{L}^2}^2 \\
&\le \|\eta\langle\eta\rangle z_{m+1}\|_{\mathbf{L}^2}^2 + C\varepsilon^{2+2\sigma}(1+t)^{-2-2\gamma}
\end{aligned}$$

we find

$$\begin{aligned}
&\frac{d}{dt}\|\langle\eta\rangle z_{m+1}\|_{\mathbf{L}^2}^2 \le -\frac{2}{t+1}\|\eta\langle\eta\rangle z_{m+1}\|_{\mathbf{L}^2}^2 \\
&+ \frac{C}{t+1}\|\langle\eta\rangle z_{m+1}\|_{\mathbf{L}^2}^2 + C\varepsilon^{2+2\sigma}(1+t)^{-2-2\gamma} \\
&\le -\frac{\varepsilon^\sigma}{t+1}\|\langle\eta\rangle z_{m+1}\|_{\mathbf{L}^2}^2 + C\varepsilon^{2+2\sigma}(1+t)^{-2-2\gamma}. \qquad (5.132)
\end{aligned}$$

The integration of (5.132) with respect to time yields the estimate

$$\|\langle\eta\rangle z_{m+1}\|_{\mathbf{L}^2} \le C\varepsilon^{1+\sigma}(1+t)^{-1-\gamma}.$$

In the same way we obtain the estimate

$$\|\langle\eta\rangle \partial_\eta z_{m+1}\|_{\mathbf{L}^2} \le C\varepsilon^{1+\sigma}(1+t)^{-1-\gamma}.$$

Thus estimates (5.127) to (5.128) are true with m is replaced by $m+1$. Therefore by induction estimates (5.127) to (5.128) are fulfilled for all m.

In the same manner we prove the estimates

$$\|z_{m+1} - z_m\|_{\mathbf{X}} \le \frac{1}{2}\|z_m - z_{m-1}\|_{\mathbf{X}},$$

where $\|\psi\|_{\mathbf{X}} \equiv \left\||\xi|^{-1}\psi\right\|_{\mathbf{L}^2} + \|\langle\xi\rangle\psi\|_{\mathbf{H}^{1,1}}$. Therefore, there exists a unique solution of equations (5.124) to (5.126) which obeys the estimates (5.127) to (5.128).

Returning to the function w we find that

$$\left\| |\xi|^{-1} \widehat{w} \right\|_{\mathbf{L}^2} \leq C\varepsilon^{1+\sigma} (1+t)^{\frac{1}{4}-\gamma},$$
$$\|\widehat{w}\|_{\mathbf{L}^2} + \sqrt{t+1}\, \||\xi|\, \widehat{w}\|_{\mathbf{L}^2} \leq C\varepsilon^{1+\sigma} (1+t)^{-\frac{1}{4}-\gamma}.$$

The last estimate gives us

$$\|w\|_{\mathbf{L}^2} + \sqrt{t+1}\, \|w_x\|_{\mathbf{L}^2} \leq C\varepsilon^{1+\sigma} (1+t)^{-\frac{1}{4}-\gamma};$$

hence, by the Sobolev imbedding inequality we get

$$\|w\|_{\mathbf{L}^\infty} \leq C \|w\|_{\mathbf{L}^2}^{\frac{1}{2}} \|w_x\|_{\mathbf{L}^2}^{\frac{1}{2}} \leq C\varepsilon^{1+\sigma} (1+t)^{-\frac{1}{2}-\gamma}.$$

This implies the estimate of the theorem, and Theorem 5.36 is thus proved.

5.7 Comments

Section 5.1.

The large time asymptotic behavior of positive solutions to the nonlinear heat equation

$$u_t - \Delta u + u^{\sigma+1} = 0, \tag{5.133}$$

in the subcritical case of $\sigma \in \left(0, \frac{2}{n}\right)$ was studied in Escobedo and Kavian [1988], Escobedo et al. [1995], Gmira and Véron [1984], Kavian [1987]. In paper Gmira and Véron [1984] it was proved that if the initial data are nonnegative $u_0 \geq 0$, $u_0 \in \mathbf{L}^1(\mathbf{R}^n)$ and decay slowly at infinity as $\lim_{x\to\pm\infty} |x|^{\frac{2}{\sigma}} u_0(x) = +\infty$, then the solution of (5.133) has the asymptotic representation

$$u(t,x) = t^{-\frac{1}{\sigma}} \sigma^{-\frac{1}{\sigma}} + o\left(t^{-\frac{1}{\sigma}}\right)$$

as $t \to \infty$ uniformly in domains $\left\{x \in \mathbf{R}^n;\ |x| \leq C\sqrt{t}\right\}$ with any $C > 0$. In the paper Escobedo and Kavian [1988], the authors considered the nonnegative initial data decaying sufficiently rapidly at infinity, that is $0 \leq u_0(x) \leq Ce^{-b|x|^2}$ for all $x \in \mathbf{R}^n$, with some $b, C > 0$. Then it was shown that the main term of the asymptotic behavior of the solution has a self-similar character

$$u(t,x) = t^{-\frac{1}{\sigma}} w_0\left(\frac{x}{\sqrt{t}}\right) + o\left(t^{-\frac{1}{\sigma}}\right)$$

as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$, where $w_0(\xi)$ is a unique positive solution of the elliptic equation

$$-\Delta w - \frac{1}{2}\xi\nabla w + w^{1+\sigma} = \frac{1}{\sigma} w \tag{5.134}$$

which decays rapidly at infinity: $\lim_{|\xi|\to\infty} |\xi|^{\frac{2}{\sigma}} w_0(\xi) = 0$. This result was improved in paper Escobedo et al. [1995], where the intermediate case was considered: if the initial data are such that $u_0 \in \mathbf{L}^1$, $u_0 \neq 0$ and $\lim_{|x|\to\infty} |x|^{\frac{2}{\sigma}} u_0(x) = \kappa > 0$, then the solutions of the nonlinear heat equation have the asymptotic representation

$$u(t,x) = t^{-\frac{1}{\sigma}} w_\kappa\left(\frac{x}{\sqrt{t}}\right) + o\left(t^{-\frac{1}{\sigma}}\right)$$

as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$, where $w_\kappa(\xi)$ is a positive solution of equation (5.134) such that $\lim_{|\xi|\to\infty} |\xi|^{\frac{2}{\sigma}} w_\kappa(\xi) = \kappa$. We emphasize that in these papers there was no restriction on the size of initial data, except the positivity.

Section 5.2.

Equation (5.38) with $\rho = 2$ and $\sigma = \frac{2}{n}$ is known as the complex Landau-Ginzburg equation. Local in time existence of solutions to the Cauchy problem (5.38) with $\rho = 2$ was studied by many authors (see, e.g. Ginibre and Velo [1996], Ginibre and Velo [1997] and references cited therein). Nonlinear dissipative equations with a fractional power of the negative Laplacian in the principal part were studied extensively (see, e.g., Bardos et al. [1979], Biler et al. [1998], Komatsu [1984], Shlesinger et al. [1995], Taylor [1992], Zhang [2001] and references cited therein). Blow-up in finite time of positive solutions to the Cauchy problem

$$\partial_t u + (-\Delta)^{\frac{\rho}{2}} u - u^{1+\sigma} = 0,\ u(0,x) = u_0(x) > 0 \tag{5.135}$$

was proved in papers Fujita [1966], Weissler [1981] for the case of $0 < \sigma < \frac{2}{n}$, $\rho = 2$, in papers Hayakawa [1973], Kobayashi et al. [1977] for the case of $\sigma = \frac{2}{n}$, $\rho = 2$, and in paper Sugitani [1975] for the case of $0 < \rho \le 2$ and $0 < \sigma \le \frac{\rho}{n}$. Their proofs of blow-up results are based on the positivity of linear evolution operator $\overline{\mathcal{F}}_{\xi\to x} e^{-|\xi|^\rho}$, associated with equation (5.135) for $0 < \rho \le 2$ (see book Yosida [1995]), and do not work for the case of $\rho > 2$, since $\overline{\mathcal{F}}_{\xi\to x} e^{-|\xi|^\rho}$ is not necessarily positive. Large time behavior of positive solutions was studied extensively for a particular case of (5.38) with $\rho = 2$ with any $\sigma > 0$ (see paper Kamin and Peletier [1985] for the supercritical case of $\sigma > \frac{2}{n}$, Galaktionov et al. [1985] for the critical case of $\sigma = \frac{2}{n}$ and papers Escobedo and Kavian [1988], Escobedo et al. [1995], Gmira and Véron [1984], Kavian [1987] for the subcritical case $\sigma \in \left(0, \frac{2}{n}\right)$). The global in time existence of small solutions to (5.135) in the supercritical case $\sigma > \frac{2}{n}$, $\rho = 2$ was shown in Fujita [1966]. The results stated in this section for the Cauchy problem (5.38) with $\rho \neq 2$ is applicable, in particular, to the problem

$$\partial_t u + (-\Delta)^{\frac{\rho}{2}} u + \lambda u^{1+\sigma} = \mu u^{1+\kappa},\ u(0,x) = u_0(x) > 0, \tag{5.136}$$

with $0 < \sigma < \kappa \le \frac{\rho}{n}, \lambda, \mu > 0$. The solutions of (5.136) blow up in finite time, when $\lambda = 0$, $\mu > 0$ and $0 < \rho \le 2$ and exist globally in time, when $\lambda > 0$, $\mu = 0$ and $0 < \rho < \infty$. The result of Theorem 5.15 shows that the dissipation term $u^{1+\sigma}$ in equation (5.136) is stronger than the blow-up term $u^{1+\kappa}$. Note that the problem of asymptotic behavior of solutions to (5.136) is still open for the subcritical case of $0 < \kappa < \sigma \le \frac{\rho}{n}$ even if $\rho = 2$. Theorem 5.15 was proved in paper Hayashi et al. [2004a]. The case of large initial data was studied in Hayashi et al. [2006b].

Section 5.3.

Model equation (5.56) combines many well-known equations of modern mathematical physics which describe various wave processes in different media. For example, the potential Whitham equation (5.62) follows from (5.56) if we take $N(u) = (u_x)^2$, that is $a(t,\xi,y) = -(\xi - y)y$ and $Lu = \mu_1 |\partial_x|^\delta u + \mu_2 |\partial_x|^{\delta-1} u_x$, where $\mu_1 > 0,\ \mu_2 \in R.$ Here the value $\delta = 3$ is critical from the point of view of the large time behavior and global existence. Equation (5.62) comes from the Whitham Whitham [1999] equation

$$v_t + vv_x + \mathcal{L}v = 0,\ x \in \mathbf{R},\ t > 0, \tag{5.137}$$

if we introduce a potential $u = \int_{-\infty}^{x} u(t,x)\,dx$, which vanishes as $x \to \infty$ if we consider initial data $v(0,x)$ with zero total mass $\int_{\mathbf{R}} v(0,x)\,dx = 0$; therefore, $\int_{\mathbf{R}} v(t,x)\,dx = 0$ for all $t > 0$ in view of equation (5.137). The material of this section was taken from paper Hayashi et al. [2005b].

Section 5.4.

Recently much attention has been drawn to nonlinear wave equations with dissipative terms. The blow-up results were proved in Todorova and Yordanov [2001] for the case of nonlinearity $-|v|^{1+\sigma}$, with $\sigma < 2$, when the initial data are such that $\int_{\mathbf{R}} v_0(x)\,dx > 0,\ \int_{\mathbf{R}} v_1(x)\,dx > 0$. Blow-up results for the critical and subcritical cases of $\sigma \le 2$ were obtained in Li and Zhou [1995]. Paper Todorova and Yordanov [2001] proved the global existence and large time decay estimates of solutions to the Cauchy problem for the damped wave equation (5.66) with nonlinearities $\pm |v|^{1+\sigma}$ or $\pm |v|^{\sigma} v,$ for the supercritical case of $\sigma > 2,$ if the initial data are sufficiently small and have a compact support. Problem (5.66) was considered in Nishihara [2003] and Ono [2003], when the initial data are in the usual Sobolev space $u_0 \in \mathbf{W}_1^{1,0}(\mathbf{R}^n) \cap \mathbf{W}_\infty^1(\mathbf{R}^n),\ u_1 \in \mathbf{L}^1(\mathbf{R}^n) \cap \mathbf{L}^\infty(\mathbf{R}^n)$. Via the energy type estimates obtained in papers Matsumura [1976/77] and Kawashima et al. [1995] it was proved in Karch [2000b] that solutions of the nonlinear damped wave equation (5.66) in the supercritical cases of $\sigma > 3$ with arbitrary initial data $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{L}^1(\mathbf{R}),$ $u_1 \in \mathbf{L}^2(\mathbf{R}) \cap \mathbf{L}^1(\mathbf{R})$ (that is without smallness assumption on the initial data) have the same large time asymptotics as that for the linear heat equation $\partial_t - \partial_x^2$:

$$\|u(t) - \theta G_0(t)\|_{\mathbf{L}^p} = o\left(t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}\right)$$

as $t \to \infty$, where $2 \le p \le \infty$. Here $G_0(t,x) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{4t}}$ is the heat kernel, and θ is a constant. The material of this section concerning the case of small initial data was taken from paper Hayashi et al. [2004e].

Section 5.5.

The results of this section were published in paper Kaĭkina et al. [2005].

Section 5.6.

The blow-up phenomena for positive solutions to the semilinear parabolic equation $u_t - u_{xx} = u^{1+\sigma}$ were obtained in paper Fujita [1966] for $\sigma \in (0,2)$, in Hayakawa [1973] for $\sigma = 2,$ and in paper Kobayashi et al. [1977] for $\sigma = 2$ in the case of higher space dimensions. In the subcritical case of $\sigma \in (0,2)$ large time behavior of positive solutions was studied extensively (see Escobedo and Kavian [1988], Escobedo et al. [1995], Gmira and Véron [1984]). The material of this section was taken from paper Hayashi and Naumkin [2006a].

6

Subcritical Convective Equations

In this chapter we study the large time asymptotic behavior of solutions to dissipative equations with subcritical convective type nonlinearities, taking as a typical example the famous Burgers type equation $u_t + u^\sigma u_x - u_{xx} = 0$, when $\sigma \in (0,1)$. The large time behavior of solutions for convective equations we represent as the product of a rarefaction and shock waves.

6.1 Burgers type equations

We study the large time asymptotic behavior of positive solutions $u(t,x)$ to the Cauchy problem for the following Burgers type parabolic equations

$$\begin{cases} u_t + u^\sigma u_x - u_{xx} = 0, \ x \in \mathbf{R}, \, t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R} \end{cases} \tag{6.1}$$

in the subcritical case of $\sigma \in (0,1)$. Equation (6.1) is a simple model describing diffusive-convective phenomena in different physical applications (see Bear [1996], Whitham [1999]). Note that the subcritical case of $\sigma \in (0,1)$ naturally appears in some models of fluid mechanics (see Lister [1992], Philip [1969], Rosenau and Kamin [1983]).

In this section we are interested in the subcritical case of $\sigma \in (0,1)$, when the initial data $u_0(x)$ have a nonzero mean value

$$\int_{\mathbf{R}} u_0(x)\, dx = \theta > 0.$$

In the case of zero mean value $\theta = \int_{\mathbf{R}} u_0(x)\, dx = 0$, the nonlinearity $u^\sigma u_x$ is supercritical for the range $\sigma > \frac{1}{2}$, and the large time asymptotics of solutions can be obtained by methods of Chapter 2. Then the critical $\sigma = \frac{1}{2}$ and subcritical $\sigma \in \left(0, \frac{1}{2}\right)$ cases can be treated by the methods of Chapter 3 and Chapter 5, respectively.

Note that the mean value of the solution

$$\int_{\mathbf{R}} u(t,x)\,dx = \theta$$

is a conservation law according to equation (6.1) with zero boundary conditions at $x = \pm\infty$.

In this section we construct an asymptotic approximation which is close in the uniform norm to the solution. We represent the solution in the form $u(t,x) = \varphi(t,x)\,r(t,x)$, where $\varphi(t,x)$ is a rarefaction wave and $r(t,x)$ is a shock wave. In other words near the front of the wave $x_f(t)$ the solution $u(t,x)$ resembles the shock wave $r(t,x)$ and in the far region $|x - x_f(t)| >> 1$ the solution behaves as the rarefaction wave $\varphi(t,x)$.

We organize this section as follows. In Subsection 6.1.1 we consider solutions of the rarefaction wave form when the initial data are such that $u_0(x) \to 0$ as $x \to -\infty$, and $u_0(x) \to 1$ as $x \to +\infty$. In Subsection 6.1.2 we construct the shock wave solutions such that $u(t,x) \to 1$ as $x \to -\infty$ and $u(t,x) \to 0$ as $x \to +\infty$. We will show that solutions $u(t,x)$ tend as $t \to \infty$ to the self-similar solution $(1 + e^{\nu x})^{-\frac{1}{\sigma}}$, $\nu = \frac{\sigma}{1+\sigma}$. The most difficult and intriguing case of the zero boundary conditions $u(t,x) \to 0$ as $x \to \pm\infty$ is treated in Subsection 6.1.3, where we prove that solutions of the Cauchy problem (6.1) can be represented as the product of a rarefaction and a shock wave.

6.1.1 Rarefaction wave

First we investigate the case of the rarefaction wave. Consider the initial value problem for the Hopf equation

$$\begin{cases} \varphi_t + \varphi^{\sigma}\varphi_x = 0, \ x \in \mathbf{R}, \, t > 0, \\ \varphi(0,x) = \varphi_0(x), \ x \in \mathbf{R}, \end{cases} \tag{6.2}$$

where the initial data $\varphi_0(x) \in \mathbf{C}^2(\mathbf{R})$ are monotonically increasing $0 < \varphi_0'(x) < C$ for all $x \in \mathbf{R}$, $\varphi_0(x) \to 0$ as $x \to -\infty$, $\varphi_0(x) \to 1$ as $x \to +\infty$. The solution to problem (6.2) is given by $\varphi(t,\chi(t,\xi)) = \varphi_0(\xi)$, where the characteristics $\chi(t,\xi) = \xi + t\varphi_0^{\sigma}(\xi)$ for $\xi \in \mathbf{R}$, $t > 0$. Note that

$$\varphi_x(t,\chi(t,\xi)) = \frac{\varphi_0'(\xi)}{1 + t\sigma\varphi_0^{\sigma-1}(\xi)\,\varphi_0'(\xi)} > 0$$

and

$$\varphi_{xx}(t,\chi(t,\xi)) = \frac{\varphi_0''(\xi)}{\left(1 + t\sigma\varphi_0^{\sigma-1}(\xi)\,\varphi_0'(\xi)\right)^2} - \frac{\varphi_0'(\xi)\,t\sigma\left(\varphi_0^{\sigma-1}(\xi)\,\varphi_0'(\xi)\right)'}{\left(1 + t\sigma\varphi_0^{\sigma-1}(\xi)\,\varphi_0'(\xi)\right)^3}$$

for all $\xi \in \mathbf{R}$, $t > 0$. We assume also that $\varphi_0(x)$ is such that

$$\left\|\varphi_{xx}(t)\right\|_{\mathbf{L}^2} \leq Ct^{-1-\gamma}, \quad \int_t^{\infty} \left\|\varphi_{xx}(\tau)\right\|_{\mathbf{L}^\infty} d\tau \to 0 \tag{6.3}$$

as $t \to \infty$, where $\gamma > 0$. For example, condition (6.3) are fulfilled if we take the initial data $\varphi_0(\xi) \in \mathbf{C}^2(\mathbf{R})$ which have the asymptotics

$$\varphi_0(\xi) = \vartheta(\xi) + O\left(|\xi|^{-\beta}\right), \quad \frac{d^k}{d\xi^k}\varphi_0(\xi) = O\left(|\xi|^{-\beta-k}\right), \ k = 1, 2$$

as $\xi \to \pm\infty$, where $\beta > 0$ and $\vartheta(\xi) = 1$ for $\xi \geq 0$ and $\vartheta(\xi) = 0$ for $\xi < 0$.

First we give a sufficiently general result about convergence as $t \to \infty$ of solutions $u(t,x)$ of problem (6.1) to the rarefaction wave $\varphi(t,x)$. Some other related works can be found in papers Liu et al. [1998], Matsumura and Nishihara [1986], Matsumura and Nishihara [1994a].

Theorem 6.1. *Let $u_0 - \varphi_0 \in \mathbf{L}^2(\mathbf{R})$ and $u_0 > 0$. We assume that $\varphi_0(x) \in \mathbf{C}^2(\mathbf{R})$ is such that condition (6.3) is true. Then*

$$u(t,x) = \varphi(t,x) + o(1)$$

as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$.

Proof. First we note that by the maximum principle the solution $u(t,x)$ of the problem (6.1) is positive for all $x \in \mathbf{R}$, $t \geq 0$. For the difference $w = u - \varphi$ we get the Cauchy problem

$$\begin{cases} w_t + (w+\varphi)^{\sigma}(w+\varphi)_x - \varphi^{\sigma}\varphi_x - w_{xx} - \varphi_{xx} = 0, \\ w(x,0) = w_0(x), \end{cases} \tag{6.4}$$

where $w_0(x) = u_0(x) - \varphi_0(x) \in \mathbf{L}^2(\mathbf{R})$. By the method of Naumkin and Shishmarev [1994b] we can easily prove the existence of a unique solution

$$\begin{aligned} w(t,x) &\in \mathbf{C}\left([0,\infty); \mathbf{L}^2(\mathbf{R})\right) \cap \mathbf{C}\left((0,\infty); \mathbf{H}^2(\mathbf{R})\right) \\ &\cap \mathbf{C}^1\left((0,\infty); \mathbf{L}^\infty(\mathbf{R}) \cap \mathbf{L}^2(\mathbf{R})\right). \end{aligned}$$

Multiplying equation (6.4) by w and integrating with respect to x over $\mathbf{R}$ we get energy type a priori estimate

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w\|_{\mathbf{L}^2}^2 + \int_{\mathbf{R}} \left(w(w+\varphi)^{\sigma}(w+\varphi)_x - w\varphi^{\sigma}\varphi_x\right) dx \\ &= -\|w_x\|_{\mathbf{L}^2}^2 + \int_{\mathbf{R}} w\varphi_{xx} dx. \end{aligned}$$

Note that (see Matsumura and Nishihara [1986])

$$\begin{aligned} &\int_{\mathbf{R}} \left(w(w+\varphi)^{\sigma}(w+\varphi)_x - w\varphi^{\sigma}\varphi_x\right) dx \\ &= (1+\sigma)^{-1}\int_{\mathbf{R}} \left((w+\varphi)^{\sigma+1} - \varphi^{\sigma+1} - (1+\sigma)\varphi^{\sigma}w\right)\varphi_x dx \\ &= \sigma\int_{\mathbf{R}} dx w^2\varphi_x \int_0^1 (\varphi+\lambda w)^{\sigma-1}(1-\lambda)\, d\lambda \geq 0. \end{aligned}$$

Whence by the Cauchy inequality and estimate $\|\varphi_{xx}(t)\|_{\mathbf{L}^2} \le C\langle t\rangle^{-1-\gamma}$ we have

$$\frac{d}{dt}\|w\|_{\mathbf{L}^2}^2 + 2\|w_x\|_{\mathbf{L}^2}^2 \le C\langle t\rangle^{-1-\gamma}\|w\|_{\mathbf{L}^2}.$$

Integration with respect to time $t > 0$ yields

$$\|w(t)\|_{\mathbf{L}^2} \le C$$

and

$$\int_0^t \|w_x(\tau)\|_{\mathbf{L}^2}^2 d\tau \le C. \tag{6.5}$$

Hence via inequalities

$$\|w\|_{\mathbf{L}^\infty}^4 \le 2\|w\|_{\mathbf{L}^2}^2\|w_x\|_{\mathbf{L}^2}^2 \le C\|w_x\|_{\mathbf{L}^2}^2$$

and (6.5) we obtain

$$\int_0^\infty \|w(t)\|_{\mathbf{L}^\infty}^4 dt < C.$$

We see that there exists a sequence $t_k \to \infty$ such that $\|w(t_k)\|_{\mathbf{L}^\infty} \to 0$. In order to prove that $\|w(t)\|_{\mathbf{L}^\infty} \to 0$ as $t \to \infty$, let us estimate $\sup_{x\in\mathbf{R}} w(t,x)$ and $\inf_{x\in\mathbf{R}} w(t,x)$. Since $w \in \mathbf{C}\left((0,\infty);\mathbf{H}^1(\mathbf{R})\right)$ we see that $\lim_{|x|\to\infty} w(t,x) = 0$, hence we have $\sup_{x\in\mathbf{R}} w(t,x) \ge 0$ and $\inf_{x\in\mathbf{R}} w(t,x) \le 0$ for all $t \in (0,\infty)$. By the method of paper Constantin and Escher [1998] we have the following result.

Lemma 6.2. *Let $w \in \mathbf{C}^1((T_1,T_2);\mathbf{L}^\infty(\mathbf{R}))$ and $\widetilde{w}(t) = \sup_{x\in\mathbf{R}} w(t,x) > 0$ for all $t \in (T_1,T_2)$. Then there exists a point $\zeta(t) \in \mathbf{R}$ such that $\widetilde{w}(t) = w(t,\zeta(t))$, moreover $\widetilde{w}'(t) = w_t(t,\zeta(t))$ almost everywhere on (T_1,T_2).*

We now prove that $\widetilde{w}(t) \to 0$ as $t \to \infty$. Since $\|w(t_k)\|_{\mathbf{L}^\infty} \to 0$ for some sequence $t_k \to \infty$ we consider the time interval $T_2 > T_1 \ge t_k$ such that $\widetilde{w}(t) > 0$ for all $t \in (T_1,T_2)$. By virtue of Lemma 6.2 we get from equation (6.4)

$$\widetilde{w}' + ((\varphi+\widetilde{w})^\sigma - \varphi^\sigma)\varphi_x - w_{xx}(t,\zeta(t)) - \varphi_{xx}(t,\zeta(t)) = 0$$

almost for all $t \in (T_1,T_2)$, where we have used the fact that $w_x(t,\zeta(t)) = 0$. Whence integrating with respect to $t \in (T_1,T_2)$ via

$$w_{xx}(t,\zeta(t)) \le 0 \text{ and } ((\varphi+\widetilde{w})^\sigma - \varphi^\sigma)\varphi_x \ge 0$$

we have

$$\widetilde{w}(t) \le \widetilde{w}(T_1) + \int_{T_1}^t \varphi_{xx}(\tau,\zeta(\tau))d\tau.$$

Since $\widetilde{w}(T_1) = 0$ or $\widetilde{w}(T_1) = \widetilde{w}(t_k)$ and $\widetilde{w}(t_k) \to 0$ as $t_k \to \infty$, we have $\widetilde{w}(T_1) \to 0$ as $T_1 \to \infty$. Also by our assumption

$$\left|\int_{T_1}^{t} \varphi_{xx}(\tau,\zeta(\tau))\,d\tau\right| \le \int_{T_1}^{\infty} \|\varphi_{xx}(t)\|_{\mathbf{L}^\infty}\,dt = o(1)$$

as $T_1 \to \infty$. Therefore $\widetilde{w}(t) \to 0$ as $t \to \infty$. Similarly we prove that $\inf_{x\in\mathbf{R}} w(t,x) \to 0$ as $t \to \infty$. Hence $\|w(t)\|_{\mathbf{L}^\infty} \to 0$ as $t \to \infty$. Theorem 6.1 is proved.

We now suppose some more conditions to be fulfilled for the initial data $u_0(x)$ and compute more precisely the large time asymptotic behavior of solution $u(t,x)$ to the problem (6.1). We assume that initial data $u_0(x)$ monotonically increase and are slowly varying, so that the higher order derivatives are less comparing with the first one. More precisely we suppose that the initial data $u_0(x) \in \mathbf{C}^3(\mathbf{R})$ have the following estimates

$$\frac{d}{dx}u_0^\sigma(x) \le \varepsilon^{\frac{1}{1-\gamma}}\left(u_0^\sigma(x)\right)^{\frac{2}{1-\gamma}},$$
$$\frac{d^2}{dx^2}\left(u_0^\sigma(x)\right) \le \varepsilon\left(\frac{d}{dx}u_0^\sigma(x)\right)^{\frac{3+\gamma}{2}},\quad \frac{d^3}{dx^3}\left(u_0^\sigma(x)\right) \le \varepsilon\left(\frac{d}{dx}u_0^\sigma(x)\right)^{2+\gamma} \tag{6.6}$$

for all $x \in \mathbf{R}$, where $\gamma \in \left(0,\frac{1}{2}\right)$, $\varepsilon > 0$ is sufficiently small. For example, we can take the initial data of the form

$$u_0(x) = C\left(\int_{-\infty}^{x} \varepsilon^2\left(1+\varepsilon^4\xi^2\right)^{\gamma-1} d\xi\right)^{\frac{1}{\sigma}},$$

where $C > 0$ is such that $u_0(+\infty) = 1$. Note that this initial data decay like $u_0(x) = O\left(|x|^{\frac{2\gamma-1}{\sigma}}\right)$ as $x \to -\infty$.

By Theorem 6.1 we know that solutions of (6.1) are similar to those of the Hopf equation (6.2). Therefore the nonlinearity in equation (6.1) grows with time more rapidly than the term with the second derivative, hence the large time behavior of solutions should be determined by the first two terms in equation (6.1). That is why we try to solve equation (6.1) by the method of characteristics. We define characteristics $\chi(t,\xi)$ as the solutions to the Cauchy problem

$$\begin{cases} \chi_t = u^\sigma(t,\chi) - \frac{u_{\chi\chi}}{u_\chi},\ t>0, \xi\in\mathbf{R}, \\ \chi(0,\xi) = \xi,\ \xi \in \mathbf{R}. \end{cases}$$

Then from equation (6.1) we get a simple equation

$$\begin{aligned} w_t(t,\xi) &= u_t + u_\chi\chi_t = u_t + u_\chi\left(u^\sigma - \frac{u_{\chi\chi}}{u_\chi}\right) \\ &= u_t - u_{\chi\chi} + u^\sigma u_\chi = 0 \end{aligned}$$

for the new dependent variable $w(t,\xi) = u(t,\chi(t,\xi))$. Hence $w(t,\xi) = u_0(\xi)$ for all $t > 0$, $\xi \in \mathbf{R}$. By a straightforward calculation we have

$$\partial_\chi u = \frac{u_0'(\xi)}{\chi_\xi(t,\xi)},\ \partial_\chi^2 u = \frac{u_0''(\xi)}{\chi_\xi^2(t,\xi)} - \frac{u_0'(\xi)\,\chi_{\xi\xi}(t,\xi)}{\chi_\xi^3(t,\xi)},$$

whence

$$\chi_t = u_0^\sigma(\xi) - \frac{1}{u_0'(\xi)}\partial_\xi\left(\frac{u_0'(\xi)}{\chi_\xi(t,\xi)}\right).$$

We now change the independent variable $\eta = u_0^\sigma(\xi)$, then the real axis $\xi \in \mathbf{R}$ is transformed biuniquelly to a segment $(0,1)$ (in view of our assumption $u_0'(\xi) > 0$). Denote $m(\eta) = \frac{\partial\eta}{\partial\xi} = \sigma u_0^{\sigma-1}(\xi)\, u_0'(\xi)$ and $Z(\tau,\eta) = u_\chi(t,\chi) = \frac{m(\eta)}{\chi_\xi(t,\xi)}$. Then we have

$$\begin{aligned}\partial_t\chi_\xi(t,\xi) &= \partial_\xi\left(u_0^\sigma(\xi)\right) - \partial_\xi\left(\frac{1}{u_0'(\xi)}\partial_\xi\left(\frac{u_0'(\xi)}{\chi_\xi(t,\xi)}\right)\right)\\ &= m(\eta) - m(\eta)\,\partial_\eta\left(\eta^{1-\frac{1}{\sigma}}\partial_\eta\left(\eta^{\frac{1}{\sigma}-1}Z\right)\right)\\ &= m(\eta)\left(1 - \partial_\eta^2 Z - \frac{1-\sigma}{\sigma}\partial_\eta\left(\frac{Z}{\eta}\right)\right),\end{aligned}$$

whence for $Z(t,\eta)$ we get

$$\partial_t Z = -Z^2\frac{1}{m(\eta)}\partial_t\chi_\xi(t,\xi) = -Z^2(1-A),$$

where

$$A(t,\eta) = \partial_\eta^2 Z(t,\eta) + \kappa\partial_\eta\left(\frac{Z(t,\eta)}{\eta}\right)$$

and $\kappa = \frac{1-\sigma}{\sigma}$. Thus for $Z(t,\eta)$ we get the following initial-boundary value problem

$$\begin{cases} Z_t = -Z^2(1-A),\ t>0, \eta\in(0,1),\\ Z(0,\eta) = m(\eta),\ \eta\in(0,1),\\ Z^{k-\frac{3}{2}}\partial_\eta^k Z\Big|_{\eta=0,1} = 0,\ t>0,\ k=1,2,\end{cases} \tag{6.7}$$

since by virtue of (6.6) we suppose that $u_0(\xi)$ is such that $\partial_\eta^k m(\eta) = o\left(\eta^{2-k}\right)$ as $\eta \to 0$, for $k = 1,2$.

From the existence of a unique solution $u(t,x)$ to problem (6.1) it follows that there exists a unique global solution $Z(\tau,\eta) \in \mathbf{C}\left([0,\infty), \mathbf{C}^2(0,1)\right) \cap \mathbf{C}^1((0,\infty), \mathbf{C}(0,1))$ to the initial-boundary value problem (6.7). Integrating equation (6.7) with respect to time $t > 0$ we get the following representation

$$Z(t,\eta) = m(\eta)\left(1 + m(\eta)\left(t + \int_0^t A(\tau,\eta)\,d\tau\right)\right)^{-1}. \tag{6.8}$$

We prove the following result.

Theorem 6.3. *Let conditions (6.6) for the initial data $u_0(x)$ be fulfilled with sufficiently small $\varepsilon > 0$. Then the estimate*

$$\sup_{\eta\in(0,1)}\left|\int_0^t A(\tau,\eta)\,d\tau\right| \le C\varepsilon(1+t)^{1-\gamma}$$

is true for all $t \ge 0$, where $\gamma \in \left(0,\frac{1}{2}\right)$ is taken from conditions (6.6).

Remark 6.4. By virtue of representation (6.8) and estimate of Theorem 6.3 we get the asymptotics

$$Z(t,\eta)=\frac{m(\eta)}{1+m(\eta)t+O(t^{1-\gamma})}\text{ as } t\to\infty,$$

whence

$$u_\chi(t,\chi(t,\xi))=\frac{u_0'(\xi)}{1+\sigma u_0^{\sigma-1}(\xi)u_0'(\xi)t+O(t^{1-\gamma})}, \tag{6.9}$$

where

$$\chi(t,\xi)=\xi+tu_0^\sigma(\xi)-\frac{1}{u_0'(\xi)}\partial_\xi\int_0^t u_\chi(t',\chi(t',\xi))\,dt'. \tag{6.10}$$

Thus we see that the solution $u(t,x)$ to problem (6.1) asymptotically behaves as a solution of the Hopf equation (6.2). Note that via formulas (6.9) and (6.10) we can obtain a higher-order asymptotic expansion of the solution $u(t,x)$.

Proof of Theorem 6.3 : As in the proof of Theorem 6.1 we apply the maximum principle to equation (6.7)

$$Z_t=-Z^2-\frac{\kappa}{\eta^2}Z^3+Z^2Z_{\eta\eta}+\frac{\kappa}{\eta}Z^2Z_\eta.$$

By virtue of Lemma 6.2 we get for $\widetilde{Z}(t)=\max_{\eta\in(0,1)}Z(t,\eta)$

$$\frac{d}{dt}\widetilde{Z}\le-\widetilde{Z}^2-\frac{\kappa}{\eta^2}\widetilde{Z}^3\le-\widetilde{Z}^2,$$

whence integrating with respect to time $t>0$ we get the estimate

$$0<Z(t,\eta)\le\frac{m(\eta)}{1+m(\eta)t} \tag{6.11}$$

for all $\eta\in(0,1)\,,\,t>0$.

Denote $Y=Z^{\frac{\rho}{2}}$, $\rho=1-\gamma$, where $\gamma\in\left(0,\frac12\right)$ is taken from the condition (6.6), then from (6.7) we obtain

$$Y_t=-\frac{\rho}{2}Y^{\frac{2}{\rho}+1}-\frac{\kappa\rho}{2\eta^2}Y^{\frac{4}{\rho}+1}+\frac{\kappa}{\eta}Y^{\frac{4}{\rho}}Y_\eta+\left(\frac{2}{\rho}-1\right)Y^{\frac{4}{\rho}-1}Y_\eta^2+Y^{\frac{4}{\rho}}Y_{\eta\eta},$$

whence taking derivative with respect to η we have

$$\begin{aligned}\partial_tY_\eta=&-\frac{2+\rho}{2}Y^{\frac{2}{\rho}}Y_\eta-\frac{\kappa(6+\rho)}{2\eta^2}Y^{\frac{4}{\rho}}Y_\eta+\frac{\rho\kappa}{\eta^3}Y^{\frac{4}{\rho}+1}\\&+\frac{\frac{4}{\rho}\kappa}{\eta}Y^{\frac{4}{\rho}-1}Y_\eta^2+\left(\frac{2}{\rho}-1\right)\left(\frac{4}{\rho}-1\right)Y^{\frac{4}{\rho}-2}Y_\eta^3\\&+2\left(\frac{4}{\rho}-1\right)Y^{\frac{4}{\rho}-1}Y_\eta Y_{\eta\eta}+\frac{\kappa}{\eta}Y^{\frac{4}{\rho}}Y_{\eta\eta}+Y^{\frac{4}{\rho}}Y_{\eta\eta\eta}.\end{aligned} \tag{6.12}$$

We apply the maximum principle to equation (6.12) by virtue of Lemma 6.2 we get for $\widetilde{Y_\eta}(t) = \max_{\eta\in(0,1)} Y_\eta(t,\eta)$

$$\frac{d}{dt}\widetilde{Y_\eta} \le -\frac{2+\rho}{2}Z\widetilde{Y_\eta} - \frac{\kappa(6+\rho)}{2\eta^2}Z^2\widetilde{Y_\eta} + \frac{\rho\kappa}{\eta^3}Z^{2+\frac{\rho}{2}} + \frac{4\kappa}{\rho\eta}Z^{2-\frac{\rho}{2}}\widetilde{Y_\eta}^2 + \left(\frac{2}{\rho}-1\right)\left(\frac{4}{\rho}-1\right)Z^{2-\rho}\widetilde{Y_\eta}^3,$$

whence via (6.11) we find

$$\frac{d}{dt}\widetilde{Y_\eta} \le C\eta^{-3}m^{2+\frac{\rho}{2}}(1+mt)^{-2-\frac{\rho}{2}},$$

hence integration with respect to time yields

$$\widetilde{Y_\eta}(t) \le C\varepsilon + C\eta^{-3}m^{2+\frac{\rho}{2}}\int_0^t (1+mt)^{-2-\frac{\rho}{2}}\,dt \le C\varepsilon + C\eta^{-3}m^{1+\frac{\rho}{2}} \le C\varepsilon$$

for all $t>0$, since $m^\rho(\eta) \le C\varepsilon\eta^2$, hence $m^{1+\frac{\rho}{2}}(\eta) \le C\varepsilon\eta^3$ and $|Y_\eta(0)| = \frac{\rho}{2}m^{\frac{\rho}{2}-1}|m'| \le C\varepsilon$ by virtue of conditions (6.6). In the same way we estimate $\min_{\eta\in(0,1)} Y_\eta(t,\eta)$. Therefore we have

$$|Y_\eta(t,\eta)| \le C\varepsilon$$

for all $\eta\in(0,1)$, $t>0$. Note that $Y_\eta = \frac{\rho}{2}Z^{\frac{\rho}{2}-1}Z_\eta$, hence

$$|Z_\eta(t,\eta)| \le \frac{2}{\rho}Z^{1-\frac{\rho}{2}}|Y_\eta(t,\eta)| \le C\varepsilon Z^{1-\frac{\rho}{2}}. \tag{6.13}$$

Now we define $X = Z^\rho$, $\rho\in(0,1)$. We have

$$|X_\eta(t,\eta)| = \rho Z^{\rho-1}|Z_\eta| \le C\varepsilon Z^{\frac{\rho}{2}}.$$

We now take (6.12) with $\frac{\rho}{2}$ replaced by ρ and Y replaced by X, then differentiating twice with respect to η we get

$$\begin{aligned}\partial_t X_{\eta\eta} =& -(\rho+1)X^{\frac{1}{\rho}}X_{\eta\eta} - \frac{\kappa}{\eta^2}X^{\frac{2}{\rho}}X_{\eta\eta} - \frac{6\rho\kappa}{\eta^4}X^{\frac{2}{\rho}+1} - \frac{1+\rho}{\rho}X^{\frac{1}{\rho}-1}X_\eta^2\\ &+ \frac{2\kappa(5+2\rho)}{\eta^3}X^{\frac{2}{\rho}}X_\eta - \frac{2\kappa(4+\rho)}{\rho\eta^2}X^{\frac{2}{\rho}-1}X_\eta^2\\ &+ \frac{2\kappa(2-\rho)}{\rho^2\eta}X^{\frac{2}{\rho}-2}X_\eta^3 + 2\rho^{-3}(1-\rho)^2(2-\rho)X^{\frac{2}{\rho}-3}X_\eta^4\\ &- \frac{\kappa(3+\rho)}{\eta^2}X^{\frac{2}{\rho}}X_{\eta\eta} + \frac{6\kappa}{\rho\eta}X^{\frac{2}{\rho}-1}X_\eta X_{\eta\eta}\\ &+ \rho^{-2}(7-5\rho)(2-\rho)X^{\frac{2}{\rho}-2}X_\eta^2X_{\eta\eta} + \frac{2}{\rho}(2-\rho)X^{\frac{2}{\rho}-1}X_{\eta\eta}^2\\ &+ \frac{\kappa}{\eta}X^{\frac{2}{\rho}}X_{\eta\eta\eta} + \frac{2}{\rho}(3-\rho)X^{\frac{2}{\rho}-1}X_\eta X_{\eta\eta\eta} + X^{\frac{2}{\rho}}X_{\eta\eta\eta\eta}.\end{aligned} \tag{6.14}$$

We apply the maximum principle to equation (6.14) by virtue of Lemma 6.2 we get for $\widetilde{X_{\eta\eta}}(t) = \max_{\eta\in(0,1)} X_{\eta\eta}(t,\eta)$

$$\frac{d}{dt}\widetilde{X_{\eta\eta}} \leq CZ^{2+\rho}\left(\eta^{-4} + \varepsilon Z^{-2\rho}\right),$$

whence via (6.11) we find

$$\frac{d}{dt}\widetilde{X_{\eta\eta}} \leq C\eta^{-4}m^{2+\rho}(1+mt)^{-2-\rho} + C\varepsilon m^{2-\rho}(1+mt)^{-2+\rho},$$

hence integration with respect to time yields

$$\begin{aligned}\widetilde{X_{\eta\eta}}(t) &\leq C\varepsilon + C\eta^{-4}m^{2+\rho}\int_0^t (1+mt)^{-2-\rho}\,dt \\ &+ Cm^{2-\rho}\int_0^t (1+mt)^{-2+\rho}\,dt \\ &\leq C\varepsilon + C\eta^{-4}m^{1+\rho} + Cm^{1-\rho} \leq C\varepsilon\end{aligned}$$

for all $t > 0$, since $m^{1+\rho}(\eta) \leq \varepsilon\eta^4$ and

$$|X_{\eta\eta}(0)| \leq \rho m^{\rho-1}|m''| + \rho(\rho-1)m^{\rho-2}m'^2 \leq C\varepsilon$$

by virtue of conditions (6.6). In the same manner we estimate the function $\min_{\eta\in(0,1)} X_{\eta\eta}(t,\eta)$. Thus we see that

$$|X_{\eta\eta}(t,\eta)| \leq C\varepsilon$$

for all $\eta \in (0,1)$, $t > 0$. Note that $X_{\eta\eta} = \rho Z^{\rho-1}Z_{\eta\eta} + \rho(\rho-1)Z^{\rho-2}Z_\eta^2$, hence

$$|Z_{\eta\eta}(t,\eta)| \leq \frac{1}{\rho}Z^{1-\rho}|X_{\eta\eta}(t,\eta)| + (\rho-1)Z^{-1}Z_\eta^2(t,\eta) \leq C\varepsilon Z^{1-\rho}. \quad (6.15)$$

Now by estimates (6.11), (6.13) and (6.15) we get

$$\begin{aligned}&\sup_{\eta\in(0,1)}\left|\int_0^t A(\tau,\eta)\,d\tau\right| \leq C\sup_{\eta\in(0,1)}\int_0^t |Z_{\eta\eta}(\tau,\eta)|\,d\tau \\ &+ C\sup_{\eta\in(0,1)}\eta^{-1}\int_0^t |Z_\eta(\tau,\eta)|\,d\tau + C\sup_{\eta\in(0,1)}\eta^{-2}\int_0^t Z(\tau,\eta)\,d\tau \\ &\leq C\varepsilon\sup_{\eta\in(0,1)}\int_0^t Z^{1-\rho}d\tau \leq C\varepsilon(t+1)^{1-\gamma}\end{aligned}$$

for all $t \geq 0$. Theorem 6.3 is proved.

6.1.2 Shock wave

Here we consider another type of the boundary conditions, corresponding to the shock wave solutions. We study the Cauchy problem

$$\begin{cases} u_t + B(1+t)^{-\nu} u^\sigma u_x - u_{xx} = 0, \ x \in \mathbf{R}, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}, \end{cases} \tag{6.16}$$

with initial data satisfying shock-wave type boundary conditions $u_0(x) \to 1$ for $x \to -\infty$ and $u_0(x) \to 0$ for $x \to +\infty$, here $\nu \in \left[0, \frac{1}{2}\right)$, $\sigma \in (0,1)$, $B > 0$. Changing

$$u(t,x) = w(t,y), \ x = y(1+t)^\nu + \int_0^t a(\tau)(1+\tau)^{-\nu} d\tau,$$

the function $a(t)$ will be defined later, we get

$$\begin{cases} w_t - \frac{\nu}{1+t} y w_y + (1+t)^{-2\nu} (B w^\sigma w_y - a(t) w_y - w_{yy}) = 0, \ y \in \mathbf{R}, \ t > 0, \\ w(0,y) = u_0(y), \ y \in \mathbf{R}. \end{cases} \tag{6.17}$$

We introduce the approximate solution (better approximation it is for bigger t)

$$W(t,y) = \sum_{k=0}^{n} (1+t)^{-\alpha k} \Phi_k(y),$$
$$a(t) = \sum_{k=0}^{n+1} (1+t)^{-\alpha k} a_k,$$

where $n \geq 1$, $\alpha = 1 - 2\nu > 0$, the functions $\Phi_k(y)$ and coefficients a_k for $0 \leq k \leq n$ we define recurrently below. We substitute W and a into (6.17) to get

$$\begin{aligned} &\sum_{k=1}^{n+1} (1+t)^{-\alpha k} \left(\alpha(k-1)\Phi_{k-1} + \nu y \Phi'_{k-1}\right) \\ &= \frac{B}{1+\sigma} \frac{d}{dy} \left(\sum_{k=0}^{n} (1+t)^{-\alpha k} \Phi_k(y) \right)^{\sigma+1} \\ &- \sum_{l=0}^{n+1} \sum_{m=0}^{n} (1+t)^{-\alpha(l+m)} a_l \Phi'_m - \sum_{k=0}^{n} (1+t)^{-\alpha k} \Phi''_k. \end{aligned} \tag{6.18}$$

We now collect the terms with the same power of $1+t$. Then for Φ_0 we obtain

$$B\Phi_0^\sigma \Phi'_0 - a_0 \Phi'_0 - \Phi''_0 = 0$$

with boundary conditions $\Phi_0(y) \to 1$ as $y \to -\infty$, $\Phi_0(y) \to 0$ as $y \to +\infty$. This boundary conditions imply that $a_0 = \frac{B}{\sigma+1}$. Therefore integration yields

$$B\left(\Phi_0^\sigma - 1\right)\Phi_0 = (\sigma+1)\,\Phi_0',$$

hence

$$\Phi_0(y) = \left(1+e^{hy}\right)^{-\frac{1}{\sigma}},$$

where $h = \sigma a_0 = \frac{B\sigma}{1+\sigma}$. Define $b_0 = 0$ and let b_{k-1} for $k \geq 2$ be the coefficient of the Taylor expansion with respect to $\tau = (1+t)^{-\alpha}$ of the function $a_0\left(\sum_{j=0}^{k-1}(1+t)^{-\alpha j}\,\Phi_j(y)\right)^{\sigma+1}$

$$b_{k-1} \equiv \frac{a_0}{k!}\,\frac{d^k}{d\tau^k}\left(\sum_{j=0}^{k-1}\tau^j\Phi_j\right)^{1+\sigma}\Bigg|_{\tau=0}.$$

Note that b_{k-1} depends only on $\Phi_0, \ldots, \Phi_{k-1}$. Thus we have the following expansion

$$a_0\left(\sum_{j=0}^{n}(1+t)^{-\alpha j}\,\Phi_j(y)\right)^{\sigma+1}$$
$$=\sum_{k=1}^{n}(1+t)^{-\alpha k}\left(b_{k-1}+B\Phi_0^\sigma\Phi_k\right)+r_n\,(1+t)^{-\alpha n-\alpha},$$

where

$$r_n = \frac{a_0}{n!}\,\frac{d^{n+1}}{d\tau^{n+1}}\left(\sum_{j=0}^{n}\tau^j\Phi_j\right)^{1+\sigma}.$$

Then for the functions $\Phi_k(y)$, $1 \leq k \leq n$ we get from (6.18)

$$\Phi_k'' + a_0\Phi_k' - B\frac{d}{dy}\left(\Phi_0^\sigma\Phi_k\right)$$
$$= -\alpha\,(k-1)\,\Phi_{k-1} - \nu y\Phi_{k-1}' - \sum_{l=1}^{k} a_l\Phi_{k-l}' + \frac{d}{dy}b_{k-1} \tag{6.19}$$

with boundary conditions $\Phi_k(y) \to 0$ for $y \to \pm\infty$, $k \geq 1$. Integrating (6.19) with respect to y over (y,∞) we obtain a linear differential equation for Φ_k

$$\Phi_k' = \Phi_k\left(B\Phi_0^\sigma - a_0\right) + b_{k-1} - \sum_{l=1}^{k} a_l\Phi_{k-l}$$
$$+\,\nu\int_y^\infty \eta\Phi_{k-1}'(\eta)\,d\eta + \alpha\,(k-1)\int_y^\infty \Phi_{k-1}(\eta)\,d\eta.$$

Multiplying both sides of the above equation by $e^{-hy}\left(1+e^{hy}\right)^{\frac{1}{h}}$, and then integrating with respect to y we have

$$\Phi_k(y) = \int_{-\infty}^{y} \frac{e^{hz}\left(1+e^{hy}\right)^{\frac{1}{h}}}{e^{hy}\left(1+e^{hz}\right)^{\frac{1}{h}}} \left(\alpha(k-1)\int_z^{\infty}\Phi_{k-1}(\eta)\,d\eta \right.$$
$$\left. +\nu\int_z^{\infty}\eta\Phi_{k-1}'(\eta)\,d\eta + b_{k-1}(z) - \sum_{l=1}^{k} a_l\Phi_{k-l}(z)\right) dz.$$

We define a_k for all $1 \le k \le n+1$ by the conditions

$$a_k = \nu\int_{\mathbf{R}}\eta\Phi_{k-1}'(\eta)\,d\eta + \alpha(k-1)\int_{\mathbf{R}}\Phi_{k-1}(\eta)\,d\eta \tag{6.20}$$

which guarantee that $\Phi_k(y)$ decay exponentially. Indeed, we have

$$a_1 = \nu\int_{\mathbf{R}}\eta\Phi_0'(\eta)\,d\eta,$$

therefore

$$\nu\int_z^{\infty}\eta\Phi_0'(\eta)\,d\eta - a_1\Phi_0(z)$$
$$= \begin{cases} -(\nu z + a_1)\Phi_0(z) - \nu\int_z^{\infty}\Phi_0(\eta)\,d\eta \text{ for } z>0, \\ a_1(1-\Phi_0(z)) - \nu\int_{-\infty}^{z}\eta\Phi_0'(\eta)\,d\eta \text{ for } z<0. \end{cases}$$

Then using estimates

$$\Phi_0(y) \le e^{-a_0 y}, |\Phi_0'(y)| \le Che^{-a_0 y}$$

for $y>0$ and

$$1-\Phi_0(y) \le e^{-h|y|}, |\Phi_0'(y)| \le Che^{-h|y|}$$

for $y<0$; $|a_1| \le \frac{C}{h}$, we obtain

$$|\Phi_1(y)|$$
$$= e^{-hy}\left(1+e^{hy}\right)^{\frac{1}{h}}\left|\int_{-\infty}^{y} e^{hz}\left(1+e^{hz}\right)^{-\frac{1}{h}}\left(\nu\int_z^{\infty}\eta\Phi_0'(\eta)\,d\eta - a_1\Phi_0(z)\right)dz\right|$$
$$\le Ce^{-hy}\left(1+e^{hy}\right)^{\frac{1}{h}}\int_{-\infty}^{y} e^{hz}\left(1+e^{hz}\right)^{-\frac{1}{h}} e^{-2L(z)}(1+|z|)\,dz$$
$$\le \frac{C}{h}(1+|y|)\,e^{-h|y|},$$

where $L(z) = \frac{1}{2}a_0|z|$ for $z>0$, and $L(z) = \frac{1}{2}h|z|$ for $z<0$. Then by induction we can see that the estimates are true

$$h|\Phi_k(y)| + |\Phi_k'(y)| \le Ch^{1-k}(1+|y|)\,e^{-h|y|} \tag{6.21}$$

for all $y \in \mathbf{R}$, $1 \le k \le n$. Thus in view of (6.18) and (6.19) we see that the approximate solution W satisfy the following equation

$$W_t - \frac{\nu}{1+t} y W_y + (1+t)^{-2\nu} \left(BW^{\sigma} W_y - a(t) W_y - W_{yy}\right) = R(t,y), \quad (6.22)$$

where the remainder term

$$R(t,y)$$
$$= -(1+t)^{-\alpha n-1} \left(\alpha n \Phi_n - \nu y \Phi_n' - \partial_y r_n + \sum_{k=0}^{n} \Phi_k' \sum_{l=0}^{k} (1+t)^{-\alpha l} a_{n+1+l-k} \right).$$

By virtue of estimates (6.21) we get

$$h\left|r_n(t,y)\right| + \left|\partial_y r_n(t,y)\right| \le C h^{1-n} (1+|y|) e^{-h|y|}$$

and

$$|R(t,y)| \le C h^{1-n} (1+t)^{-\alpha n-1} \left(1+y^2\right) e^{-h|y|}. \quad (6.23)$$

Moreover since we defined a_{n+1} via (6.20), then the integrated remainder $R_1(t,y) = \int_{-\infty}^{y} R(t,z)\,dz$ also decay at infinity and we have the estimate

$$|R_1(t,y)| \le C h^{-n} (1+t)^{-\alpha n-1} \left(1+y^2\right) e^{-h|y|}. \quad (6.24)$$

Note that the remainder terms $R(t,y)$ and $R_1(t,y)$ are small uniformly with respect to $t>0$ and $y \in \mathbf{R}$, if the value $h = \frac{B\sigma}{1+\sigma}$ is sufficiently large.

We now suppose that the initial data $u_0(x)$ for the problem (6.16) are close to the approximate shock wave $W(0,x)$, so that

$$e^{L(x)} \int_{-\infty}^{x} \left(u_0(\xi) - W(0,\xi)\right) d\xi \in \mathbf{L}^{\infty},$$

where

$$L(x) = \begin{cases} \frac{1}{2} a_0 |x|, & \text{for } x > 0, \\ \frac{1}{2} h |x|, & \text{for } x < 0, \end{cases}$$

where $h = \sigma a_0$, $a_0 = \frac{B}{1+\sigma}$. We prove the asymptotic stability of the approximate solution $W(t,y)$. Some other results on the asymptotic stability of shock waves can be found in Il′in and Oleĭnik [1960], Liu [1985], Matano [1982], Matsumura and Nishihara [1994b].

Theorem 6.5. *Let the initial data $u_0(x)$ be close to the shock wave $W(0,x)$, that is*

$$\left\| e^{L(x)} \int_{-\infty}^{x} \left(u_0(\xi) - W(0,\xi)\right) d\xi \right\|_{\mathbf{L}^{\infty}} + \left\| e^{L(x)} \left(u_0(x) - W(0,x)\right) \right\|_{\mathbf{L}^{\infty}} \le \varepsilon,$$

where $\varepsilon > 0$ is sufficiently small. Suppose that the coefficient B is sufficiently large $B \ge \frac{C}{\varepsilon}$. Then there exists a unique solution $w(t,y)$ to the initial-boundary value problem (6.17) such that the estimate

$$\|w(t) - W(t)\|_{\mathbf{L}^{\infty}} \le C(1+t)^{-n\alpha}$$

is true for all $t > 0$, where $\alpha = 1 - 2\nu$.

Remark 6.6. Thus the solution $u(t,x)$ of the initial-boundary value problem (6.16) tends to the shock wave $W(t,y)$ as $t \to \infty$. We choose the large parameter B to be able to estimate uniformly the remainder term R for the approximate shock wave solution $W(t,y)$. The smallness assumption for the initial data in fact follows from the method of the proof via the contraction mapping principle.

Proof. By virtue of (6.17) and (6.22) we find for the difference $v(t,y) = w(t,y) - W(t,y)$

$$
\begin{aligned}
&v_t - \frac{\nu}{1+t} y v_y + (1+t)^{-2\nu} a_0 \partial_y \left((W+v)^{1+\sigma} - W^{1+\sigma} \right) \\
&- (1+t)^{-2\nu} a(t) v_y - (1+t)^{-2\nu} v_{yy} + R = 0,
\end{aligned}
\tag{6.25}
$$

Integration of equation (6.25) with respect to y yields the Cauchy problem

$$
\begin{cases}
V_t - \frac{\nu}{1+t} y V_y + \frac{\nu}{1+t} V + (1+t)^{-2\nu} a_0 \left((W+V_y)^{1+\sigma} - W^{1+\sigma} \right) \\
- (1+t)^{-2\nu} a(t) V_y - (1+t)^{-2\nu} V_{yy} + R_1 = 0, \ y \in \mathbf{R}, \ t > 0, \\
V(0,y) = V_0(y), \ y \in \mathbf{R},
\end{cases}
\tag{6.26}
$$

where

$$
V(t,y) = \int_{-\infty}^{y} v(t,y')\, dy', \ R_1(t,y) = \int_{-\infty}^{y} R(t,z)\, dz
$$

and

$$
V_0(x) = \int_{-\infty}^{x} (u_0(\xi) - W(0,\xi))\, d\xi.
$$

Note that $V_0(x) \to 0$ as $x \to \pm\infty$.

We change the dependent variable $g(t,y) = V(t,y) e^{\theta(y)}$, where $\theta(y)$ we define such that

$$
\theta'(y) = \frac{1}{2} \left(a_0 - B \Phi_0^{\sigma}(y) \right),
$$

that is

$$
\begin{aligned}
\theta(y) &= \frac{1}{2} \int \left(a_0 - B \left(1 + e^{hy} \right)^{-1} \right) dy = -\frac{1}{2} h y + \frac{B}{2h} \ln \left(1 + e^{hy} \right) \\
&= L(y) + \frac{B}{2h} \ln \left(1 + e^{-h|y|} \right),
\end{aligned}
$$

where

$$
L(y) = \begin{cases} \frac{1}{2} a_0 |y|, \ \text{for } y > 0, \\ \frac{1}{2} h |y|, \ \text{for } y < 0, \end{cases}
$$

$a_0 = \frac{B}{1+\sigma}$, $h = \sigma a_0$. By virtue of equation (6.26), changing

$$
V_y e^{\theta} = g_y - \theta' g
$$

and

$$V_{yy}e^{\theta} = g_{yy} - 2\theta' g_y - \left(\theta'' - \theta'^2\right) g$$

we have

$$\begin{aligned} & g_t + (1+t)^{-2\nu} a_0 e^{\theta} \left(\left(W + (g_y - \theta' g) e^{-\theta}\right)^{1+\sigma} - W^{1+\sigma} \right) \\ & - (1+t)^{-2\nu} g_{yy} - \phi g_y + \psi g + R_1 e^{\theta} = 0, \end{aligned} \tag{6.27}$$

where the coefficients

$$\begin{aligned} \phi(t,y) &= (1+t)^{-2\nu} \left(a(t) - 2\theta'(y)\right) + \frac{\nu y}{1+t}, \\ \psi(t,y) &= (1+t)^{-2\nu} \left(a(t)\theta'(y) - \theta'^2(y) + \theta''(y)\right) + \frac{\nu}{1+t}\left(y\theta'(y) + 1\right). \end{aligned}$$

As above we apply the maximum principle. We prove that

$$\sup_{y \in \mathbf{R}} B\,|g(t,y)| + \sup_{y \in \mathbf{R}} |g_y(t,y)| < C\varepsilon (1+t)^{-\alpha n}$$

for all $t > 0$. By the contradiction we can find a maximal time interval $T > 0$ such that

$$\sup_{y \in \mathbf{R}} B\,|g(t,y)| + \sup_{y \in \mathbf{R}} |g_y(t,y)| \le C\varepsilon (1+t)^{-\alpha n} \tag{6.28}$$

for all $t \in [0,T]$. Denote $\widetilde{g}(t) = \sup_{y\in\mathbf{R}} g(t,y)$, then via Lemma 6.2 we get from (6.27)

$$\frac{d}{dt}\widetilde{g} \le -(1+t)^{-2\nu} a_0 e^{\theta} \left(\left(W - \theta' \widetilde{g} e^{-\theta}\right)^{1+\sigma} - W^{1+\sigma} \right) - \psi \widetilde{g} - R_1 e^{\theta}.$$

Since $1 - (1-z)^{1+\sigma} \le (1+\sigma) z$ for all $z < 1$, we have

$$\begin{aligned} & -(1+t)^{-2\nu} a_0 e^{\theta} \left(\left(W - \theta' \widetilde{g} e^{-\theta}\right)^{1+\sigma} - W^{1+\sigma} \right) \\ & = W^{1+\sigma} e^{\theta} (1+t)^{-2\nu} a_0 \left(1 - \left(1 - \theta' \frac{\widetilde{g}}{W} e^{-\theta}\right)^{1+\sigma} \right) \le (1+t)^{-2\nu} B\theta' W^{\sigma} \widetilde{g}. \end{aligned}$$

By the definition of the approximate solution and estimates (6.21)

$$W^{\sigma}(t,y) = \Phi_0^{\sigma}(y) + O\left(h^{-1}(1+t)^{-\alpha\sigma}\right),$$

with $\Phi_0^{\sigma}(y) = \left(1 + e^{hy}\right)^{-1}$, and we defined θ such that

$$\theta'(y) = \frac{1}{2}\left(a_0 - B\Phi_0^{\sigma}(y)\right),$$

we write

$$\begin{aligned} & \psi - (1+t)^{-2\nu} B\theta' W^{\sigma} \ge (1+t)^{-2\nu} \left(\left(a_0 - B\Phi_0^{\sigma}\right)\theta' - \theta'^2 + \theta'' \right) \\ & + \frac{\nu}{(1+t)}\left(y\theta' + 1\right) + O\left(B(1+t)^{-2\nu-\alpha\sigma}\right) \\ & \ge (1+t)^{-2\nu}\left(\theta'^2 + \theta''\right) - CB(1+t)^{-2\nu-\alpha\sigma} \ge CB^2 (1+t)^{-2\nu} \end{aligned}$$

for all $y \in \mathbf{R}$, since

$$\left(\theta'(y)\right)^2 + \theta''(y) \geq CB^2 > 0.$$

Thus we find in view of (6.24)

$$\frac{d}{dt}\widetilde{g} \leq -CB^2 (1+t)^{-2\nu} \widetilde{g} + CB^{-1} (1+t)^{-\alpha n - 1}, \tag{6.29}$$

whence integrating with respect to time we get

$$B\widetilde{g}(t) < C\varepsilon (1+t)^{-\alpha n} \tag{6.30}$$

for all $t \geq 0$. In the same way we obtain estimate for $\widetilde{\widetilde{g}}(t) = \inf_{y\in\mathbf{R}} g(t,y)$

$$B\widetilde{\widetilde{g}}(t) > -C\varepsilon (1+t)^{-\alpha n} \tag{6.31}$$

Now we estimate the derivative g_y. Applying the maximum principle via Lemma 6.2 we get for $\widetilde{g_y}(t) = \sup_{y\in\mathbf{R}} g_y(t,y)$

$$\begin{aligned}
&\frac{d}{dt}\widetilde{g_y} \leq -(1+t)^{-2\nu} a_0 e^{\theta}\theta' \Big(\left(W + (\widetilde{g_y} - \theta' g) e^{-\theta}\right)^{1+\sigma} - W^{1+\sigma} \\
&- (1+\sigma) W^{\sigma} (\widetilde{g_y} - \theta' g) e^{-\theta} \Big) \\
&- (1+t)^{-2\nu} BW_y e^{\theta} \Big(\left(W + (\widetilde{g_y} - \theta' g) e^{-\theta}\right)^{\sigma} - W^{\sigma} - \sigma W^{\sigma-1} (\widetilde{g_y} - \theta' g) e^{-\theta} \Big) \\
&+ (1+t)^{-2\nu} B \Big(\left(W + (\widetilde{g_y} - \theta' g) e^{-\theta}\right)^{\sigma} - W^{\sigma} \Big) \left(2\theta'\widetilde{g_y} - \left(\theta'^2 - \theta''\right) g\right) \\
&- \Big(\psi - \phi_y - (1+t)^{-2\nu} BW^{\sigma}\theta' + (1+t)^{-2\nu} \sigma BW^{\sigma-1} W_y\Big) \widetilde{g_y} \\
&- \Big(\psi_y - (1+t)^{-2\nu} BW^{\sigma}\theta'^2 - (1+t)^{-2\nu} \sigma BW^{\sigma-1} W_y \theta'\Big) g - \left(R_1 e^{\theta}\right)_y .
\end{aligned}$$

Since

$$1 - (1-z)^{1+\sigma} - (1+\sigma) z = O\left(|z|^{1+\sigma}\right)$$

for all $|z| < 1$, we have by virtue of (6.28)

$$\begin{aligned}
&- (1+t)^{-2\nu} a_0 e^{\theta}\theta' \Big(\left(W + (\widetilde{g_y} - \theta' g) e^{-\theta}\right)^{1+\sigma} - W^{1+\sigma} \\
&- (1+\sigma) W^{\sigma} (\widetilde{g_y} - \theta' g) e^{-\theta} \Big) = O\left(B (1+t)^{-2\nu - \alpha n(1+\sigma)}\right),
\end{aligned}$$

$$\begin{aligned}
&- (1+t)^{-2\nu} BW_y e^{\theta} \Big(\left(W + (\widetilde{g_y} - \theta' g) e^{-\theta}\right)^{\sigma} - W^{\sigma} - \sigma W^{\sigma-1} (\widetilde{g_y} - \theta' g) e^{-\theta} \Big) \\
&= O\left(B (1+t)^{-2\nu - \alpha n(1+\sigma)}\right)
\end{aligned}$$

and

$$\begin{aligned}
&(1+t)^{-2\nu} B \Big(\left(W + (\widetilde{g_y} - \theta' g) e^{-\theta}\right)^{\sigma} - W^{\sigma} \Big) \left(2\theta'\widetilde{g_y} - \left(\theta'^2 - \theta''\right) g\right) \\
&= O\left((1+t)^{-2\nu - \alpha n(1+\sigma)}\right).
\end{aligned}$$

Note that

$$\sigma BW^{\sigma-1}W_y = -2\theta'' + O\left(B^2 (1+t)^{-\alpha}\right),$$

therefore

$$\begin{aligned}
&\psi - \phi_y - (1+t)^{-2\nu} BW^{\sigma}\theta' + (1+t)^{-2\nu} \sigma BW^{\sigma-1}W_y \\
&= (1+t)^{-2\nu} \left((a_0 - B\Phi_0^{\sigma})\,\theta' - \theta'^2 + \theta''\right) + \frac{\nu}{1+t} y\theta' + O\left((1+t)^{-2\nu-\alpha}\right) \\
&+ 2(1+t)^{-2\nu}\theta'' + (1+t)^{-2\nu}\sigma BW^{\sigma-1}W_y \geq CB^2 (1+t)^{-2\nu}
\end{aligned}$$

for all $y \in \mathbf{R}$. Also we have the estimates

$$\begin{aligned}
&\psi_y - (1+t)^{-2\nu} BW^{\sigma}\theta'^2 - (1+t)^{-2\nu}\sigma BW^{\sigma-1}W_y\theta' \\
&= O\left(B^3 (1+t)^{-2\nu}\right), \\
&\left(R_1 e^{\theta}\right)_y = O\left(B^{1-n} (1+t)^{-\alpha n-1}\right).
\end{aligned}$$

Thus we find

$$\begin{aligned}
\frac{d}{dt}\widetilde{g_y} \leq& -CB^2 (1+t)^{-2\nu}\,\widetilde{g_y} + CB^3 (1+t)^{-2\nu}\left(\widetilde{g} - \widetilde{\widetilde{g}}_y\right) \\
&+ CB^{1-n} (1+t)^{-\alpha n-1}.
\end{aligned}$$

Combining this estimate with (6.29) we get

$$\begin{aligned}
&\frac{d}{dt}\left(\widetilde{g_y} + B\widetilde{g} - B\widetilde{\widetilde{g}}\right) \\
&\leq -CB^2 (1+t)^{-2\nu}\left(\widetilde{g_y} + B\widetilde{g} - B\widetilde{\widetilde{g}}\right) + CB^{1-n} (1+t)^{-\alpha n-1},
\end{aligned}$$

whence integrating with respect to time $t > 0$ we obtain

$$\widetilde{g_y}(t) + B\widetilde{g}(t) - B\widetilde{\widetilde{g}}(t) < C\varepsilon (1+t)^{-\alpha n}$$

for all $t \geq 0$. Via (6.30) and (6.31) we get

$$\widetilde{g_y}(t) < C\varepsilon (1+t)^{-\alpha n}$$

for all $t \geq 0$. The value $\inf_{y\in\mathbf{R}} g_y(t,y)$ is estimated in the same way. Whence the result of the theorem follows. Theorem 6.5 is proved.

6.1.3 Zero boundary conditions

In this subsection we study the most difficult and intriguing case when the initial data decay at infinity. So we consider the Cauchy problem

$$\begin{cases} u_t + u^{\sigma}u_x - u_{xx} = 0, \ x \in \mathbf{R}, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R} \end{cases} \tag{6.32}$$

with the initial data $u_0(x) \to 0$ as $x \to \pm\infty$. We assume that the initial data $u_0(x)$ have a non zero mean value

$$\int_{\mathbf{R}} u_0(x)\,dx = M > 0.$$

Note that the mean value of the solution

$$\int_{\mathbf{R}} u(t,x)\,dx = M \tag{6.33}$$

is a conservation law according to equation (6.32).

The results of Escobedo et al. [1993a] say that the solution $u(t,x)$ approaches the viscosity solution of the Hopf equation $u_t + u^\sigma u_x = 0$. However the viscosity solution is too rough for the uniform norm $\mathbf{L}^\infty(\mathbf{R})$ since the solution of the Cauchy problem (6.32) is smooth, but the viscosity solution has a discontinuity. In this section we construct another asymptotic approximation of the solution which is close in the uniform norm $\mathbf{L}^\infty(\mathbf{R})$.

Define $\varphi(t,x)$ as a rarefaction wave constructed in Section 6.1.1

$$\begin{cases} \varphi_t + \varphi^\sigma \varphi_x - \varphi_{xx} = 0, \ x \in \mathbf{R}, \ t > 0, \\ \varphi(0,x) = \varphi_0(x), \ x \in \mathbf{R}, \end{cases}$$

where the initial data $\varphi_0(x)$ are monotonically increasing $\varphi_0'(x) > 0$ for all $x \in \mathbf{R}$, such that $\varphi_0(x) \to 0$ as $x \to -\infty$ and $\varphi_0(x) \to 1$ as $x \to +\infty$.

Now we define the shock wave solution $r(t,x)$ as the solution of the problem

$$\begin{cases} r_t + \varphi^\sigma r^\sigma r_x + \varphi^{\sigma-1}\varphi_x (r^\sigma - 1) r - 2\frac{\varphi_x}{\varphi} r_x - r_{xx} = 0, \ x \in \mathbf{R}, \ t > 0, \\ r(0,x) = r_0(x), \ x \in \mathbf{R}, \end{cases} \tag{6.34}$$

where the initial data $r_0(x) = \frac{u_0(x)}{\varphi_0(x)}$ satisfy boundary conditions $r_0(-\infty) = 1$, $r_0(+\infty) = 0$, i.e. $\varphi_0(x) \approx u_0(x)$ as $x \to -\infty$. Then the function $u = \varphi r$ satisfies the Cauchy problem (6.32).

We suppose that the initial data $\varphi_0(\xi)$ monotonically increase $\varphi_0'(\xi) > 0$, $0 < \varphi_0(\xi) < 1$ for all $\xi \in \mathbf{R}$ and $\varphi_0(\xi)$ satisfies the conditions (6.6) of Theorem 6.3. Using the method of characteristics from Section 6.1.1 (see formulas (6.8) and (6.10)) we derive asymptotic expansions for the solution $\varphi(x,t)$. We have

$$\varphi(t,\chi(t,\xi)) = \varphi_0(\xi), \varphi_\chi(t,\chi(t,\xi)) = \varphi_0'(\xi)(\chi_\xi(t,\xi))^{-1}$$

and

$$\chi(t,\xi) = \xi + \varphi_0^\sigma(\xi) t - \frac{1}{\varphi_0'(\xi)} \partial_\xi \int_0^t \varphi_\chi(t', \chi(t',\xi))\,dt'.$$

In the first approximation we can take

$$\chi(t,\xi) = \xi + \varphi_0^\sigma(\xi) t + O\left(t^{\frac{1}{2}-\gamma}\right) \tag{6.35}$$

for large time $t \to \infty$, where $\gamma > 0$ is small.

As in Section 6.1.2 we make changes

$$r(t,x) = w(t,y), \; x = y(1+t)^{\nu} + \int_0^t a(\tau)(1+\tau)^{-\nu} d\tau,$$

($a(t)$ will be defined later) to get

$$\begin{cases} w_t - \frac{\nu}{1+t} y w_y + (1+t)^{-2\nu} \left(B w^{\sigma} w_y - \left(a(t) + 2\frac{\varphi_x}{\varphi}(1+t)^{\nu} \right) w_y - w_{yy} \right) \\ \quad + (\varphi^{\sigma}(1+t)^{\nu} - B)(1+t)^{-2\nu} w^{\sigma} w_y \\ \quad + \varphi^{\sigma-1}\varphi_x (w^{\sigma} - 1) w = 0, \; y \in \mathbf{R}, \; t > 0, \\ w(0,y) = r_0(y), \; y \in \mathbf{R}. \end{cases} \tag{6.36}$$

We construct the first approximation $W(t,y)$ in the form

$$W(t,y) = \Phi_0(y) + (1+t)^{-\alpha} \Phi_1(t,y),$$
$$a(t) = a_0 + a_1(1+t)^{-\alpha} + a_2(t)(1+t)^{-2\alpha},$$

where $\alpha = 1 - 2\nu > 0$. As in Section 6.1.2, the function $\Phi_0(y)$ is defined via equation

$$B\Phi_0^{\sigma}\Phi_0' - a_0\Phi_0' - \Phi_0'' = 0$$

with boundary conditions $\Phi_0(y) \to 1$ as $y \to -\infty$, $\Phi_0(y) \to 0$ as $y \to +\infty$. Hence we get $a_0 = \frac{B}{\sigma+1}$ and

$$\Phi_0(y) = \left(1 + e^{hy}\right)^{-\frac{1}{\sigma}},$$

with $h = \frac{B\sigma}{1+\sigma}$. For $\Phi_1(y)$ we get

$$\Phi_1'' + a_0'\Phi_1 - \partial_y(\Phi_0^{\sigma}\Phi_1) = -\nu y \Phi_0' - a_1\Phi_0' \tag{6.37}$$

with boundary conditions $\Phi_1(y) \to 0$ for $y \to \pm\infty$. Integrating (6.37) with respect to y over (y,∞) we obtain a linear differential equation for φ_1

$$\Phi_1' = \Phi_1(\Phi_0^{\sigma} - a_0) - a_1\Phi_0 + \nu \int_y^{\infty} \eta \Phi_0'(\eta)\, d\eta.$$

Multiplying both sides of the above equation by $e^{-hy}\left(1+e^{hy}\right)^{\frac{1}{h}}$, and then integrating with respect to y, we have

$$\Phi_1(y) = \int_{-\infty}^{y} \frac{e^{hz}\left(1+e^{hy}\right)^{\frac{1}{h}}}{e^{hy}\left(1+e^{hz}\right)^{\frac{1}{h}}} \left(\nu \int_z^{\infty} \eta \Phi_0'(\eta)\, d\eta - a_1 \Phi_0(z) \right) dz.$$

We define a_1 by the condition

$$a_1 = \nu \int_{\mathbf{R}} \eta \Phi_0'(\eta)\, d\eta,$$

which guarantees that $\Phi_1(y)$ decay exponentially

$$|\Phi_1(y)| \le \frac{C}{h}(1+|y|)\, e^{-h|y|}$$

for all $y \in \mathbf{R}$.

The curve $y = 0$, i.e. $x_f(t) = \int_0^t a(\tau)(1+\tau)^{-\nu}\, d\tau$, we call the front of the wave. We have

$$x_f(t) = \frac{a_0}{1-\nu}(1+t)^{1-\nu} + O\left((1+t)^{1-\nu-\alpha}\right) \tag{6.38}$$

as $t \to \infty$. Define $\xi_0(t)$ such that $\chi(t, \xi_0(t)) = x_f(t)$, then

$$\varphi(t, x_f(t)) = \varphi_0(\xi_0(t))$$

and for $\xi_0(t)$ we have by (6.35) and (6.38)

$$\xi_0 + \varphi_0^\sigma(\xi_0)\, t + O\left(t^{\frac{1}{2}-\gamma}\right) = \frac{a_0}{1-\nu} t^{1-\nu} + O\left(t^{1-\nu-\alpha}\right)$$

for $t \to \infty$, so

$$t^\nu \varphi^\sigma(t, x_f(t)) - \frac{a_0}{1-\nu} = O\left(t^{-\alpha}\right) + O\left(t^{\nu-\frac{1}{2}-\gamma}\right) + O\left(t^{\nu-1}\xi_0\right)$$

for $t \to \infty$. By the condition $\varphi^\sigma(t, x_f(t)) = Bt^{-\nu} + o(t^{-\nu})$ we get the following equation

$$B = \frac{a_0}{1-\nu},$$

whence we obtain $\nu = \frac{\sigma}{1+\sigma}$.

If we choose the asymptotics of

$$\varphi_0(\xi) = |\xi|^{-\beta} + o\left(|\xi|^{-\beta}\right)$$

as $\xi \to -\infty$, where $\beta \in \left(0, \frac{1}{\sigma}\right)$, then we have

$$\xi_0(t) = O\left((1+t)^{\frac{1}{\beta(1+\sigma)}}\right).$$

Hence the asymptotics follows

$$|\xi_0(t)| = B^{-\frac{1}{\nu\beta(1+\sigma)}}(1+t)^{\frac{1}{\beta(1+\sigma)}} + O\left((1+t)^{\frac{1}{\beta(1+\sigma)}-\alpha}\right),$$

$$x_f(t) = \xi_0 + (1+t)|\xi_0|^{-\sigma\beta} = B(1+t)^{\frac{1}{1+\sigma}} + O\left((1+t)^{\frac{1}{1+\sigma}-\alpha}\right);$$

and then

$$\varphi^\sigma(t, \chi(t, \xi)) = \varphi_0^\sigma(\xi) = B(1+t)^{-\nu} + O\left((1+t)^{-\nu-\alpha}\right).$$

Here $\nu = \frac{\sigma}{1+\sigma} \in \left(0, \frac{1}{2}\right)$. Now by using the estimates of Theorem 6.3 we obtain in the first approximation

$$\chi(t,\xi) = |\xi|^{-\sigma\beta}(1+t)$$

for $\xi \to -\infty$. Therefore the asymptotic expansion

$$\varphi_x(t, x_f(t)) = O\left(\frac{1}{(1+t)} u_0^{1-\sigma}(\xi)\right) = O\left((1+t)^{-1-\alpha}\right),$$

is valid for $t \to \infty$. Then by virtue of the Taylor formula

$$\varphi(t,x) = \varphi(t, x_f(t)) + (x - x_f(t))\, \varphi_x(t, \widetilde{x})$$

and $\varphi^\sigma(t,x) = B(1+t)^{-\nu} + O\left(|x - x_f(t)|(1+t)^{-2\nu-2\alpha}\right)$, we get

$$\varphi^\sigma(1+t)^\nu - B = O\left(|y|(1+t)^{-\nu-2\alpha}\right),$$
$$\varphi_x(t,x) = O\left((1+t)^{-1-\alpha}\right)$$

and

$$\frac{\varphi_x}{\varphi} = O\left((1+t)^{-\nu-2\alpha}\right) \tag{6.39}$$

for $t \to \infty$, where $y = (x - x_f(t))(1+t)^{-\nu}$.

We now prove that solutions $w(t,y)$ of (6.36) converge to the approximate solution $W(t,y)$ for large time $t \to \infty$. By virtue of (6.36) and (6.37) we find for the difference $v(t,y) = w(t,y) - W(t,y)$

$$\begin{cases} v_t - \frac{\nu}{1+t} y v_y + (1+t)^{-2\nu}\left(\frac{B}{1+\sigma}\partial_y\left((v+W)^{1+\sigma} - W^{1+\sigma}\right) - a(t)\, v_y - v_{yy}\right) \\ \quad + (\varphi^\sigma(1+t)^\nu - B)\frac{(1+t)^{-2\nu}}{1+\sigma}\partial_y(W+v)^{1+\sigma} \\ \quad + \varphi^{\sigma-1}\varphi_x\left((W+v)^\sigma - 1\right)(W+v) - \frac{2\varphi_x}{\varphi}(1+t)^{-\nu} v_y + R = 0, \end{cases} \tag{6.40}$$

where

$$\begin{aligned} R(t,y) &= W_t - \frac{\nu}{1+t} y W_y + (1+t)^{-2\nu}\left(B W^\sigma W_y - a(t)\, W_y - W_{yy}\right) \\ &- \frac{2\varphi_x}{\varphi}(1+t)^{-\nu} W_y \\ &= \sum_{k=0}^{1} (1+t)^{-\alpha k-1}\left(-\alpha k \Phi_k - \nu y \Phi_k' - a(t)(1+t)^\alpha \Phi_k'\right) \\ &- \sum_{k=0}^{1} (1+t)^{-\alpha k+\alpha-1}\Phi_k'' + \sum_{k=0}^{1}(1+t)^{-\alpha k+\alpha-1}\partial_y\left(\Phi_0^\sigma \Phi_k\right) \\ &+ \partial_y r_1 - \frac{2\varphi_x}{\varphi}(1+t)^{-\nu} W_y \end{aligned}$$

since

$$a_0 (1+t)^{\alpha-1} \left(\sum_{k=0}^{1} (1+t)^{-\alpha k} \Phi_k \right)^{1+\sigma}$$
$$= B \sum_{k=0}^{1} (1+t)^{-\alpha k+\alpha-1} \Phi_0^\sigma \Phi_k + r_1,$$

where

$$h |r_1 (t,y)| + |\partial_y r_1 (t,y)| \le C (1+t)^{-1-\alpha} (1+|y|) e^{-h|y|}.$$

By virtue of equation (6.37) we get

$$\begin{aligned} |R(t,y)| = \Bigg| &(1+t)^{-\alpha-1} \left(-\alpha\Phi_1 - \nu y \Phi_1' - \sum_{l=0}^{2} a_l \Phi_{2-l}' \right) \\ &- (1+t)^{-2\alpha-1} \sum_{l=0}^{2} a_l \Phi_{3-l}' + \partial_y r_1 - 2(1+t)^{-\nu} \frac{\varphi_x}{\varphi} W_y \Bigg| \\ &\le C (1+t)^{-1-\alpha} (1+|y|) e^{-h|y|}. \end{aligned}$$

Moreover, since

$$a_2 (t) = \nu \int_{\mathbf{R}} \eta \Phi_1' (\eta) \, d\eta + \alpha \int_{\mathbf{R}} \Phi_1 (\eta) \, d\eta - 2 (1+t)^{-\nu} \int_{\mathbf{R}} \frac{\varphi_x}{\varphi} W_\eta d\eta$$

for

$$R_1 (t,y) = \int_{-\infty}^{y} R(t,z) \, dz = - \int_{y}^{\infty} R(t,z) \, dz,$$

we also have the estimate

$$|R_1 (t,y)| \le C h^{-1} (1+t)^{-1-\alpha} \left(1+y^2\right) e^{-h|y|}.$$

The integration of equation (6.40) with respect to y yields the Cauchy problem

$$\begin{cases} V_t - \frac{\nu}{1+t} y V_y + \frac{\nu}{1+t} V + (1+t)^{-2\nu} a_0 \left((W+V_y)^{1+\sigma} - W^{1+\sigma} \right) \\ - (1+t)^{-2\nu} a(t) V_y - (1+t)^{-2\nu} V_{yy} + F = 0, \ y \in \mathbf{R}, \ t > 0, \\ V(0,x) = V_0 (x), \ x \in \mathbf{R}, \end{cases} \tag{6.41}$$

where

$$\begin{aligned} F = &\int_{-\infty}^{y} R(t,z) \, dz + \frac{(1+t)^{-2\nu}}{1+\sigma} \int_{-\infty}^{y} \left(\varphi^\sigma (1+t)^\nu - B\right) \partial_y (W+v)^{1+\sigma} dy \\ &+ \int_{-\infty}^{y} \varphi^{\sigma-1} \varphi_x \left((W+v)^\sigma - 1\right)(W+v) \, dy - 2 (1+t)^{-\nu} \int_{-\infty}^{y} \frac{\varphi_x}{\varphi} v_y dy, \end{aligned}$$

$$V(t,y) = \int_{-\infty}^{y} v(t,y')\,dy',\ R_1(t,y) = \int_{-\infty}^{y} R(t,z)\,dz$$

and $V_0(x) = \int_{-\infty}^{x} (w_0(x) - W(1,x))\,dx \to 0$ as $x \to \pm\infty$.

We now suppose that the initial data $w_0(x)$ for the problem (6.36) are close to the approximate shock wave $W(0,x)$ so that

$$e^{L(x)} \int_{-\infty}^{x} (w_0(\xi) - W(0,\xi))\,d\xi \in \mathbf{L}^{\infty},$$

where

$$L(x) = \begin{cases} \frac{1}{2}a_0|x|, & \text{for } x > 0, \\ \frac{1}{2}h|x|, & \text{for } x < 0, \end{cases}$$

and $h = \sigma a_0$, $a_0 = \frac{B}{1+\sigma}$.

Denote $\nu = \frac{\sigma}{1+\sigma}$, $\alpha = 1-2\nu = \frac{1-\sigma}{1+\sigma}$, $\sigma \in (0,1)$. We now prove the following result.

Theorem 6.7. *Let the function $\varphi_0(x)$ be such that $\varphi_0'(x) > 0$ for all $x \in \mathbf{R}$, and satisfy estimates (6.6) and $\varphi_0(x) = u_0(x) + o(1)$ as $x \to -\infty$. Let the initial data $u_0(x)$ be close to the shock wave $W(0,x)$, so that the norm*

$$\left\| e^{L(x)} \int_{-\infty}^{x} \left(\frac{u_0(\xi)}{\varphi_0(\xi)} - W(0,\xi) \right) d\xi \right\|_{\mathbf{L}^{\infty}} + \left\| e^{L(x)} \left(\frac{u_0(x)}{\varphi_0(x)} - W(0,x) \right) \right\|_{\mathbf{L}^{\infty}} \le \varepsilon,$$

where $\varepsilon > 0$ is sufficiently small. Suppose that B is sufficiently large $B \ge \frac{C}{\varepsilon}$. Then a unique solution $u(t,x)$ to the Cauchy problem (6.1) has asymptotic representation

$$u(t,x) = \varphi(t,x)\,\Phi_0(y) + O\left((1+t)^{-\nu-\alpha}\right) \tag{6.42}$$

for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$.

Remark 6.8. Thus we see that the solution $u(t,x)$ of the Cauchy problem (6.1) tends to the product of the rarefaction wave $\varphi(t,x)$ and the shock wave $W(t,y)$ as $t \to \infty$. We assume that the value B is large to guarantee that the function $\Phi_0(y)$ gives us a good uniform approximation of the shock wave solution $W(t,y)$. The smallness assumption for the initial data comes from the method of the proof via the contraction mapping principle, and probably it is not necessary.

Remark 6.9. Note that the rate of decay $\alpha = \frac{1-\sigma}{1+\sigma}$ for the remainder term in asymptotic representation (6.42) vanishes in the limiting case $\sigma = 1$ and ν tends to $\frac{1}{2}$. This means that the representation (6.42) does not work in the case $\sigma \to 1$. When $\sigma = 1$, equation (6.1) is the classical Burgers equation for which an explicit formula for the solution is known (see Hopf [1950]). The large time asymptotics (6.45) has a self-similar structure and cannot be considered as the product of the rarefaction wave and shock wave solutions.

Proof. We follow the method of the proof of Theorem 6.5. We change the dependent variable $g(t,y) = V(t,y)\,e^{\theta(y)}$. The function $\theta(y)$ we define such that

$$\theta'(y) = \frac{1}{2}\left(a_0 - B\Phi_0^{\sigma}(y)\right),$$

then

$$\begin{aligned}\theta(y) &= \frac{1}{2}\int\left(a_0 - B\left(1+e^{hy}\right)^{-1}\right)dy = -\frac{1}{2}hy + \frac{B}{2h}\ln\left(1+e^{hy}\right)\\ &= L(y) + \frac{B}{2h}\ln\left(1+e^{-h|y|}\right),\end{aligned}$$

where

$$L(y) = \begin{cases}\frac{1}{2}a_0|y|, & \text{for } y>0,\\ \frac{1}{2}h|y|, & \text{for } y<0,\end{cases}$$

$a_0 = \frac{B}{1+\sigma}$, $h = \sigma a_0$. By virtue of equation (6.41) we have

$$\begin{aligned}&g_t + (1+t)^{-2\nu} e^{\theta} a_0\left(\left(W + (g_y - \theta' g)\,e^{-\theta}\right)^{1+\sigma} - W^{1+\sigma}\right)\\ &- (1+t)^{-2\nu} g_{yy} - \phi g_y + \psi g + F = 0,\end{aligned}$$

where

$$\begin{aligned}\phi(t,y) &= (1+t)^{-2\nu}\left(a(t) - 2\theta'(y)\right) + \frac{\nu}{(1+t)}y + 2(1+t)^{-\nu}\frac{\varphi_x(t,x)}{\varphi(t,x)},\\ \psi(t,y) &= (1+t)^{-2\nu}\left(a\theta'(y) - \theta'^2(y) + \theta''(y)\right)\\ &+ \frac{\nu}{(1+t)}\left(y\theta'(y)+1\right) + 2(1+t)^{-\nu}\frac{\varphi_x(t,x)}{\varphi(t,x)}\theta'(y)\end{aligned}$$

and

$$\begin{aligned}F &= \frac{(1+t)^{-2\nu}}{1+\sigma}e^{\theta}\int_{-\infty}^{y}\left(\varphi^{\sigma}(1+t)^{\nu} - B\right)\partial_y\left(W + (g_y - \theta' g)\,e^{-\theta}\right)^{1+\sigma}dy\\ &+ 2(1+t)^{-\nu}e^{\theta}\int_{-\infty}^{y}(g_y - \theta' g)\,e^{-\theta}\partial_y\left(\frac{\varphi_x}{\varphi}\right)dy\\ &+ e^{\theta}\int_{-\infty}^{y}\varphi^{\sigma-1}\varphi_x\left(\left(W + (g_y-\theta' g)\,e^{-\theta}\right)^{\sigma} - 1\right)\\ &\times\left(W + (g_y - \theta' g)\,e^{-\theta}\right)dy + Re^{\theta}.\end{aligned}$$

Let us prove that

$$B\sup_{y\in\mathbf{R}}|g(t,y)| + \sup_{y\in\mathbf{R}}|g_y(t,y)| < C(1+t)^{-\alpha}$$

for all $t>0$. By the contradiction we can find a maximal time interval $T>0$ such that

$$B\sup_{y\in\mathbf{R}}|g(t,y)|+\sup_{y\in\mathbf{R}}|g_y(t,y)|\le C(1+t)^{-\alpha} \tag{6.43}$$

for all $t\in[0,T]$.

Denote $\widetilde{g}(t)=\sup_{y\in\mathbf{R}}g(t,y)$, then by Lemma 6.2 and by applying the maximum principle we get

$$\frac{d}{dt}\widetilde{g}\le-(1+t)^{-2\nu}e^{\theta}a_0\left(\left(W-\theta'\widetilde{g}e^{-\theta}\right)^{1+\sigma}-W^{1+\sigma}\right)-\psi\widetilde{g}+\sup_{y\in\mathbf{R}}|F(t,y)|.$$

As in the proof of Theorem 6.5 we use estimates

$$\begin{aligned}&-(1+t)^{-2\nu}e^{\theta}a_0\left(\left(W-\theta'\widetilde{g}e^{-\theta}\right)^{1+\sigma}-W^{1+\sigma}\right)\\&=(1+t)^{-2\nu}B\theta'W^{\sigma}\widetilde{g}+O\left(\widetilde{g}(1+t)^{-2\nu-\sigma\alpha}\right),\end{aligned}$$

$$W^{\sigma}(t,y)=\Phi_0^{\sigma}(y)+O\left((1+t)^{-\alpha}\right)$$

and

$$\psi-(1+t)^{-2\nu}\theta'BW^{\sigma}\ge CB^2(1+t)^{-2\nu}$$

for all $y\in\mathbf{R}$, since

$$\Phi_0^{\sigma}(y)=\left(1+e^{hy}\right)^{-1}$$

and $\theta(y)$ is chosen such that

$$\theta'(y)=\frac{1}{2}\left(a_0-B\varphi_0^{\sigma}(y)\right).$$

We have by virtue of (6.39) and (6.43)

$$\sup_{y\in\mathbf{R}}|F(t,y)|\le C(1+t)^{-\alpha-1}.$$

Thus we find

$$\frac{d}{dt}\widetilde{g}\le-CB^2(1+t)^{-2\nu}\widetilde{g}+C(1+t)^{-\alpha-1}; \tag{6.44}$$

hence, integration with respect to time yields

$$B\widetilde{g}(t)\le C\varepsilon(1+t)^{-\alpha}$$

for all $t\ge0$. For the value $\widetilde{\widetilde{g}}(t)=\inf_{y\in\mathbf{R}}g(t,y)$ similarly we obtain

$$B\widetilde{\widetilde{g}}(t)\ge-C\varepsilon(1+t)^{-\alpha}$$

for all $t\ge0$.

Now we estimate the derivative $g_y(t,y)$

$$g_{yt}+\partial_y\left((1+t)^{-2\nu}e^{\theta}a_0\left(\left(W+(g_y-\theta' g)e^{-\theta}\right)^{1+\sigma}-W^{1+\sigma}\right)\right)$$
$$+\psi_y g+(\psi-\phi_y)g_y-\phi g_{yy}-(1+t)^{-2\nu}g_{yyy}+\partial_y F=0.$$

Applying the maximum principle via Lemma 6.2 we get for the function $\widetilde{g_y}(t)=\sup_{y\in\mathbf{R}}g_y(t,y)$

$$\begin{aligned}
\frac{d}{dt}\widetilde{g_y} &\le -(1+t)^{-2\nu}e^{\theta}\theta' a_0\Big(\left(W+(\widetilde{g_y}-\theta' g)e^{-\theta}\right)^{1+\sigma}-W^{1+\sigma}\\
&-(1+\sigma)W^{\sigma}(\widetilde{g_y}-\theta' g)e^{-\theta}\Big)\\
&-(1+t)^{-2\nu}W_y e^{\theta}B\Big(\left(W+(\widetilde{g_y}-\theta' g)e^{-\theta}\right)^{\sigma}\\
&-W^{\sigma}-\sigma W^{\sigma-1}(\widetilde{g_y}-\theta' g)e^{-\theta}\Big)\\
&+(1+t)^{-2\nu}B\left(\left(W+(\widetilde{g_y}-\theta' g)e^{-\theta}\right)^{\sigma}-W^{\sigma}\right)\left(2\theta'\widetilde{g_y}-\left(\theta'^2-\theta''\right)g\right)\\
&-\left(\psi-\phi_y-(1+t)^{-2\nu}BW^{\sigma}\theta'+(1+t)^{-2\nu}\sigma BW^{\sigma-1}W_y\right)\widetilde{g_y}\\
&-\left(\psi_y-(1+t)^{-2\nu}BW^{\sigma}\theta'^2-(1+t)^{-2\nu}\sigma BW^{\sigma-1}W_y\theta'\right)g-\partial_y F(t,y).
\end{aligned}$$

As in the proof of Theorem 6.5 we have

$$-(1+t)^{-2\nu}e^{\theta}\theta' a_0\Big(\left(W+(\widetilde{g_y}-\theta' g)e^{-\theta}\right)^{1+\sigma}-W^{1+\sigma}$$
$$-(1+\sigma)W^{\sigma}(\widetilde{g_y}-\theta' g)e^{-\theta}\Big)=O\left(B(1+t)^{-2\nu-\alpha-\alpha\sigma}\right),$$

$$-(1+t)^{-2\nu}W_yBe^{\theta}\left(\left(W+(\widetilde{g_y}-\theta' g)e^{-\theta}\right)^{\sigma}-W^{\sigma}-\sigma W^{\sigma-1}(\widetilde{g_y}-\theta' g)e^{-\theta}\right)$$
$$=O\left(B(1+t)^{-2\nu-\alpha-\alpha\sigma}\right),$$

$$(1+t)^{-2\nu}B\left(\left(W+(\widetilde{g_y}-\theta' g)e^{-\theta}\right)^{\sigma}-W^{\sigma}\right)\left(2\theta'\widetilde{g_y}-\left(\theta'^2-\theta''\right)g\right)$$
$$=O\left((1+t)^{-2\nu-\alpha-\alpha\sigma}\right).$$

and

$$\psi-\phi_y-(1+t)^{-2\nu}BW^{\sigma}\theta'+(1+t)^{-2\nu}\sigma BW^{\sigma-1}W_y\ge CB^2(1+t)^{-2\nu}$$

for all $y\in\mathbf{R}$. Also we have the estimates

$$\psi_y-(1+t)^{-2\nu}BW^{\sigma}\theta'^2-(1+t)^{-2\nu}\sigma BW^{\sigma-1}W_y\theta'=O\left(B^3(1+t)^{-2\nu}\right)$$

and

$$\left(Fe^{\theta}\right)_y=O\left((1+t)^{-\alpha-1}\right).$$

Thus we find

$$\frac{d}{dt}\widetilde{g_y} \leq -(1+t)^{-2\nu} CB^2 \widetilde{g_y} + CB^3 (1+t)^{-2\nu} \left(\widetilde{g} - \widetilde{\widetilde{g}}\right) + C(1+t)^{-\alpha-1}.$$

Combining this estimate with (6.29) we get

$$\frac{d}{dt}\left(\widetilde{g_y} + B\widetilde{g} - B\widetilde{\widetilde{g}}\right) \leq -CB^2 (1+t)^{-2\nu} \left(\widetilde{g_y} + B\widetilde{g} - B\widetilde{\widetilde{g}}\right) + C(1+t)^{-\alpha-1};$$

hence by integrating with respect to time $t \geq 0$ we obtain

$$\widetilde{g_y}(t) + B\widetilde{g}(t) - B\widetilde{\widetilde{g}}(t) \leq C\varepsilon (1+t)^{-\alpha}$$

for all $t \geq 0$. The value $\inf_{y\in\mathbf{R}} g_y(t,y)$ is estimated in the same way. The result of the theorem follows now from (6.39)

$$\begin{aligned}
&|u(t,x) - \varphi(t,x)\Phi_0(y)| \\
&\leq |u(t,x) - \varphi(t,x)W(t,y)| + |\varphi(t,x)(W(t,y) - \Phi_0(y))| \\
&\leq |\varphi(t,x)|\,|V_y(t,y)| + C(1+t)^{-\nu-\alpha} \\
&\leq |\varphi|\left|(g_y - \theta' g)e^{-\theta}\right| + C(1+t)^{-\nu-\alpha} \leq C(1+t)^{-\nu-\alpha}.
\end{aligned}$$

Theorem 6.7 is proved.

6.2 Comments

Section 6.1.

The asymptotic behavior of solutions to the Cauchy problem (6.1) for the supercritical case of $\sigma > 1$ was studied (see, e.g., Escobedo and Kavian [1988], Escobedo and Zuazua [1991], Escobedo and Zuazua [1997]). In this case the large time asymptotics of the solution has the same form as that of the linear heat equation

$$u(t,x) = \frac{\theta}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} + O\left(t^{-\frac{1}{2}-\gamma}\right),$$

where $\gamma > 0$, $\theta = \int_{\mathbf{R}} u_0(x)\,dx$ is a total mass of the initial data. Thus the effect of the convection term $u^\sigma u_x$ completely disappears from the main term of the asymptotics in the supercritical case of $\sigma > 1$.

In the critical case of $\sigma = 1$ equation (6.1) is known as the Burgers equation Burgers [1948] and can be solved via the Hopf-Cole transformation Hopf [1950] $u = -2\partial_x \log(\phi)$, where

$$\phi(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}} e^{-\frac{(x-y)^2}{4t}} \exp\left(-\frac{1}{2}\int_{-\infty}^{y} u_0(z)\,dz\right) dy$$

solves the heat equation $\phi_t = \phi_{xx}$. Thus the large time asymptotics can be computed explicitly (see , for example, Amick et al. [1989], Karch [1999b], Naumkin and Shishmarev [1989])

$$u(t,x) = \frac{1}{\sqrt{t}} \Psi \left(\frac{x}{\sqrt{t}} \right) + O\left(t^{-\frac{1}{2}-\gamma} \right), \tag{6.45}$$

where $\gamma > 0$,

$$\Psi(x) = -2\partial_x \log \left(\cosh \frac{\theta}{4} - \sinh \left(\frac{\theta}{4} \right) \operatorname{Erf} \left(\frac{x}{2} \right) \right)$$

and $\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$ is the error function.

In the subcritical case of $0 < \sigma < 1$ it was proved in paper Escobedo et al. [1993a] that any solution $u(t,x)$ of the Cauchy problem (6.1) with integrable initial data $u_0 \in \mathbf{L}^1(\mathbf{R})$ approaches for large time $t \to \infty$ the viscosity solution

$$U(t,x) = \begin{cases} \left(\frac{x}{t}\right)^{\frac{1}{\sigma}} & \text{if } 0 < x < x_f(t), \\ 0 & \text{otherwise} \end{cases}$$

with

$$x_f(t) = \nu^{-\nu}\theta^{\nu} t^{\frac{1}{1+\sigma}}, \ \theta = \int_{\mathbf{R}} u_0(x)\, dx > 0, \ \nu = \frac{\sigma}{1+\sigma}$$

which satisfies (see Ladyzhenskaya [1963], Lax [1957]) the Hopf equation

$$u_t + u^{\sigma} u_x = 0.$$

More precisely it was proved in Escobedo et al. [1993a] that

$$\left\| u(t,x) - U(t,x) \right\|_{\mathbf{L}^p} = o\left(t^{-\frac{1}{1+\sigma}\left(1-\frac{1}{p}\right)} \right)$$

as $t \to \infty$, where $1 \le p < \infty$. Note that the convergence $u(t,x) \to U(t,x)$ cannot be uniform with respect to $x \in \mathbf{R}$ since the entropy solution $U(t,x)$ is discontinuous in the front $x_f(t)$ of the shock wave, but the solutions $u(t,x)$ of (6.1) are smooth for all $t > 0$. Hence the viscosity solution gives us a rough approximation when working in the uniform norm $\mathbf{L}^{\infty}$. In this section we construct another asymptotic approximation which is close to the solution in the uniform norm. We follow the idea of paper Hayashi and Naumkin [2003] and represent the solution as a product of a rarefaction wave and a shock wave. Thus, near the front of the wave $x_f(t)$, the solution resembles a shock wave and in the far region the solution behaves as a rarefaction wave. In this section we considered the case of the flux function $\frac{u^{\sigma+1}}{\sigma+1}$ for simplicity. The same technique could be applied for a general $\mathbf{C}^2$ flux function; however, it seems that the convexity of the flux function plays a fundamental role.